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Controllability Matrix Analysis of Structured Networks: A Tight Lower Bound on the Dimension of Controllable Subspace

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***Abstract*— Network controllability in structured networks, characterized by edge weights as either zero or non-zero, is an emerging research area. This field has grappled with determining the dimension of the controllable subspace. From a graph-theoretical perspective, our study offers an intuitive analysis of the controllability matrix for structured networks. We categorize our analysis based on networks with single and multiple leaders and propose graph-theoretical conditions to determine the tight lower bounds of the controllable subspace. Our results provide a solid foundation for analyzing and designing complex networked systems.**

I. Introduction

Network controllability of a structured network is a research topic that has recently attracted attention. Here, a network that divides edge weights into zero or non-zero is called a *structured network*. The problem of network controllability of a structured network has been studied under the name *structural controllability*, which was first introduced by Lin [1]. A structured network is called structurally controllable if the network is controllable for almost all choices of edge weights. It follows that the structurally controllable network may become uncontrollable for a particular choice of network parameters, i.e., non-zero edge weights [2]. This is called *generic property* [3] of structural controllability, which can lead to a situation where the entire network becomes uncontrollable from an attack that manipulates the edge weights [4]. For this reason, the network controllability that considers all choices of edge weights is being studied under the name of *strong structural controllability* [5], [6].

When a structured network is uncontrollable, the network can be analyzed based on the dimension of the *controllable subspace*. The dimensions of the controllable subspace within a structured network can vary within a defined boundary depending on the network parameters, implying that the dimension of the controllable subspace for such a network is not uniquely determined. The upper bound of this dimension is recognized as the *dimension of structurally controllable subspace (SCS)* or the *generic dimension of controllable subspace*, as highlighted in [7], [8]. Conversely, efforts to determine the lower bound of the dimension of controllable subspace have been studied under the name of *dimension of strongly structurally controllable subspace (SSCS)* with various approaches, such as the derived set [9], [10], pseudo monotonically increasing (PMI) sequences [11], [12], dedicated nodes [13], [14], and maximum disjoint stems [15], have been utilized. However, while the dimension of SCS boasts a clearly defined upper bound, the exact dimension for SSCS remains an open problem. Most existing research focuses on providing a tight lower bound, which provides values that are nearly accurate but smaller than the true lower bound, Nevertheless, determining this tight lower bound is crucial for understanding the controllability of structured networks.

The analysis of the controllability matrix, which is constructed from network parameters is necessary for determining the dimension of the controllable subspace in structured networks. The pioneering work by the authors in [16] approached the elements of the controllability matrix from a graph-theoretical perspective, focusing specifically on networks with a single leader (a node to which input is connected). However, their analysis was limited to specific cases and did not encompass networks with multiple leaders or explore the determination of the dimension of the controllable subspace. Building upon this foundational work, our study aims to rigorously analyze the structure of the controllability matrix for structured networks. We partition our analysis into the cases of single and multiple leaders, providing insights from a graph-theoretical standpoint. Leveraging these insights, we present graph-theoretical conditions that determine the tight lower bounds of the dimension of SSCS for both single and multiple leader cases. Our contributions not only offer a deep understanding of the controllability matrix in structured networks but also pave the way for designing and analyzing complex networked systems with precise control objectives.

II. Preliminaries

Let us consider a network of states x_i with inputs u_i:

$$\dot{x}_i = \sum_{j\in\mathcal{N}_i} a_{ij}x_j + u_i, \tag{1}$$

where $\mathcal{N}_i$ is a set of indices of the state x_j such that $a_{ij} \neq 0$, and a_{ij} represents the weight of the directed connection from states x_j to x_i. If no such connection exists, then $a_{ij} = 0$. If u_i is non-zero, it indicates that an external input is injected into the state x_i. The networks given by (1) can be represented by an adjacency matrix $A \in \mathbb{R}^{n\times n}$, where the (i, j)-th element of A is $[A]_{i,j} = a_{ij}$. This paper assumes that there are no diagonal elements in A, i.e., $a_{ii} = 0$ for $i \in \{1, ..., n\}$. This implies that no state has a direct influence

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on itself through its own connection weight. Let us define a family set $\mathcal{Q}(A)$ that includes all matrices that have the same non-zero/zero pattern as A:

$$\mathcal{Q}(A) = \{A' \in \mathbb{R}^{n\times n} : [A']_{i,j} \neq 0 \Leftrightarrow [A]_{i,j} \neq 0\}, \quad (2)$$

for all $i, j \in \{1, ..., n\}$. Then, the network (1) can be represented by the following structured network:

$$\dot{x} = A_\Lambda x + Bu, \quad (3)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $A_\Lambda \in \mathcal{Q}(A)$. The input matrix $B \in \mathbb{R}^{n\times m}$ is constructed as $B = [b_{i_1}, b_{i_2}, ..., b_{i_m}]$, where each $b_{i_k} \in \mathbb{R}^n$ is a vector that has only one non-zero entry at its i_k-th element. It implies the unique input-connection between n-states and m-inputs. This structure ensures that an input can only be connected to one state, i.e., a one-to-one correspondence. Now, we denote the controllability matrix of the pair (A_Λ, B) of (3) as follows:

$$\mathcal{C} = [B, A_\Lambda B, A_\Lambda^2 B, ..., A_\Lambda^{n-1} B] \in \mathbb{R}^{n\times nm}. \quad (4)$$

The pair (A_Λ, B) is controllable if the rank of $\mathcal{C}$ is n. To facilitate further analysis, we define the sub-controllability matrices $\mathcal{C}_{\mathcal{L}_i} = [b_{\mathcal{L}_i}, A_\Lambda b_{\mathcal{L}_i}, ..., A_\Lambda^{n-1} b_{\mathcal{L}_i}] \in \mathbb{R}^{n\times n}$ corresponding to each input $u_i, i \in \{1, ..., m\}$. With these definitions, the controllability matrix in (4) can be represented through an appropriate column permutation as:

$$\bar{\mathcal{C}} = [\mathcal{C}_{\mathcal{L}_1}, \mathcal{C}_{\mathcal{L}_2}, ..., \mathcal{C}_{\mathcal{L}_m}] \in \mathbb{R}^{n\times nm}. \quad (5)$$

This matrix has important properties, such as its rank and column space, that are preserved under column permutation. These properties will be utilized in our subsequent analysis.

A structured network, as represented by (3), is *strongly structurally controllable* if the controllability matrix of the pair (A', B) has a full rank for all $A' \in \mathcal{Q}(A)$. In contrast, the structured network is *structurally controllable* if the controllability matrix of the pair (A', B) has a full rank for "almost all" $A' \in \mathcal{Q}(A)$. In this context, the term "almost all" is associated with the *generic property* [3] of structural controllability. This indicates that within the algebraic variety of network parameters for a structurally controllable network, the network parameters for a full rank controllability matrix are generic (major), whereas those that do not result in a full rank are comparatively specific (minor) [17]. However, this allows for the possibility of a structurally controllable network becoming uncontrollable under certain network parameters. Therefore, from the perspective of network robustness, determining a strongly structurally controllable network is also of critical importance. In control theory, the structural and strong structural controllability of a structured network can be determined by the rank of the controllability matrix, which is commonly referred to as the *dimension of the controllable subspace*. The controllable subspace is essentially the column space of the controllability matrix and represents the set of states reachable by the system. Given a structured network as in (3), the dimension of controllable subspace can vary depending on the non-zero elements in A_Λ, which is referred to as *network parameters*. Accordingly, the dimension of the controllable subspace for (3) has the following boundary condition depending on the network parameters [18]:

$$\alpha \leq \text{rank}(\bar{\mathcal{C}}) \leq \beta \leq n, \quad (6)$$

where the lower bound α is the dimensions of *strongly structurally controllable subspace (SSCS)*, whereas the upper bound β is the dimensions of the *structurally controllable subspace (SCS)*. Therefore, the conditions for strong structural controllability and structural controllability of the structured network presented by (3) are $\alpha = n$ and $\beta = n$, respectively. The following example should further clarify this concept.

Example 1: Let us consider A_Λ and B as:

$$A_\Lambda = \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \quad (7)$$

From (4), the controllability matrix of (7) is given by:

$$\bar{\mathcal{C}} = \begin{bmatrix} 1 & 0 & a_{12}a_{21} + a_{13}a_{31} \\ 0 & a_{21} & a_{23}a_{31} \\ 0 & a_{31} & a_{21}a_{32} \end{bmatrix}. \quad (8)$$

The rank of $\bar{\mathcal{C}}$ is either 2 (when $a_{23}a_{31}^2 = a_{21}^2 a_{32}$) or 3 (when $a_{23}a_{31}^2 \neq a_{21}^2 a_{32}$), which is dependent on the conditions of the network parameters. According to (8), the dimensions of the SSCS and SCS, represented by α and β in (6), are 2 and 3, respectively. Therefore, the given structured network is structurally controllable but not strongly structurally controllable. This implies that a structurally controllable network may become uncontrollable under specific network parameter conditions.

We will approach the aforementioned concepts from a graph-theoretical perspective. The structured network in (3) can be represented as a digraph, denoted as $\mathcal{G}$, composed of a set of nodes $\mathcal{V}$ and a set of directed edges $\mathcal{E}$ as:

$$\mathcal{G}(\mathcal{V}, \mathcal{E}), \quad (9)$$

where $\mathcal{V}$ is the set of nodes satisfying $|\mathcal{V}| = n$ and $\mathcal{E}$ is the set of edges. A subset, $\mathcal{V}_\mathcal{L} \subset \mathcal{V}$, is designated as the set of leaders. Each leader $\mathcal{L}_i \in \mathcal{V}_\mathcal{L}$ is associated with an external input u through an edge $(u, \mathcal{L}_i) \in \mathcal{E}$. Furthermore, the set of edges $\mathcal{E}$ is used to describe the directed connections between the nodes in $\mathcal{V}$. For a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, let $\mathcal{N}_i$ be denoted as the set of out-neighbors of node i. Since we have assumed that all diagonal elements in A_Λ are zero, there is no self-loop, i.e., $i \notin \mathcal{N}_i$ for all $i \in \{1, ..., n\}$. If there exists an edge $(j, i) \in \mathcal{E}$, then the corresponding element a_{ij} in the adjacency matrix A_Λ would be non-zero. For a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, a *directed path* is a sequence of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{q-1}, i_q)$ such that $i_k \in \mathcal{V}$ and $(i_k, i_{k+1}) \in \mathcal{E}$ for $k \in \{1, \ldots, q-1\}$, and all i_k are distinct. This directed path can also be denoted as $(i_1 \to \ldots \to i_q)$. A directed path is called a *directed cycle* if $i_1 = i_q$. Building upon this, if the directed path starts from a leader, it is termed a *directed stem*. Moreover, a *shortest path* from i

to j is a directed path with the minimum number of edges. Similarly, a *shortest stem* from i to j is the shortest path from a leader node $i \in \mathcal{V}_{\mathcal{L}}$ to node j with the minimum number of edges. A digraph is considered *input-connected* if every node can be reached as the end node of a stem, an assumption that is maintained throughout this paper. From [7, *Theorem 1*], the following theorem provides the condition for the dimension of SCS.

Proposition 1: [7] For a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, the dimension of SCS is the maximum number of nodes that can be included in the disjoint set of directed stems and cycles.

However, unlike the dimension of SCS, the exact dimension of SSCS has not been well-established in the literature. Existing studies primarily focus on defining lower bounds for this dimension, employing concepts such as zero forcing sets [19] and graph distance [12], a detailed comparison of these methods can be found in [10]. In the following sections, we delve into this subject further, exploring the dimension of SSCS through an analysis of the controllability matrix in structured networks.

III. Controllability Matrix of Structured Networks

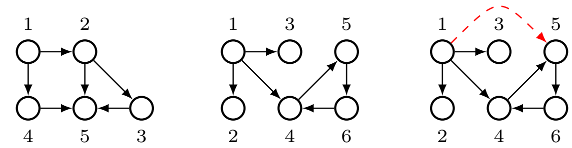
In the previous section, we introduced the dimension of the controllable subspace in structured networks. In this section, we delve further into this topic by interpreting the controllability matrix of structured networks from a graph-theoretical perspective. To facilitate this analysis, we define several terminologies. Firstly, let us consider a directed path from w to l that consists of k-edges, referred to as k-steps, and we will denote this as $\mathcal{P}_{k,p}^{w,l}$, where p is the number of distinct directed paths. Let us denote the weight of i-th edge in the sequence of $\mathcal{P}_{k,j}^{w,l}$ as e_i^j where $i \in \{1, ..., k\}$ and $j \in \{1, ..., p\}$. Then, the *weight product* of $\mathcal{P}_{k,j}^{w,l}$ is:

$$g(\mathcal{P}_{k,j}^{w,l}) = \prod_{i=1}^{k} e_i^j, \text{ for } j \in \{1, ..., p\}, \tag{10}$$

which reflects the cumulative product of the edge weights along the j-th directed path. Furthermore, we define the *sum of weight products* of $\mathcal{P}_{k,p}^{w,l}$ as:

$$\mathcal{W}_k^{w,l} = \sum_{i=1}^{p} g(\mathcal{P}_{k,i}^{w,l}), \tag{11}$$

where $k \in \{1, ..., n-1\}$. For example, let us consider the digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ depicted in Fig. 1(a). From this graph, we identify two distinct directed paths from node 1 to 5 with 2-steps, i.e., $(1 \to 4 \to 5)$ and $(1 \to 2 \to 5)$. It follows that the weight product of each directed path is $a_{54}a_{41}$ and $a_{52}a_{21}$, respectively. Thus, the sum of weight products is obtained as $\mathcal{W}_{|2|}^{1,4} = a_{54}a_{41} + a_{52}a_{21}$. With the notion of weight product, we can now interpret the elements of the k-th power of the adjacency matrix A_Λ^k. Specifically, the (w, l)-th element of A_Λ^k, denoted as $[A_\Lambda^k]_{w,l}$, represents the sum of weight products for all directed paths from node w to node l consisting of k edges. In other words, $[A_\Lambda^k]_{w,l}$ equals $\mathcal{W}_k^{w,l}$. However, when we focus on the sub-controllability matrices, especially $\mathcal{C}_{\mathcal{L}_w}$, our interest narrows down to those directed paths that specifically originate from leaders. In the context of our previous definitions, such directed paths are termed as *directed stems*. Now, let us consider a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with m-leaders. From (5), we have m-sub-controllability matrices $\mathcal{C}_{\mathcal{L}_w}$ corresponding to each leader $\mathcal{L}_w \in \mathcal{V}_{\mathcal{L}}$ for $w \in \{1, ..., m\}$. The k-th column of each $\mathcal{C}_{\mathcal{L}_w}$ consists of the $\mathcal{L}_w$-th column of A_Λ^k for all $k \in \{1, ..., n-1\}$. Note that when $k = 0$, the first column of $\mathcal{C}_{\mathcal{L}_w}$ always has the $\mathcal{L}_w$-th element as a non-zero entry in the standard column basis. Hence, the $(l, k+1)$-th entry of $\mathcal{C}_{\mathcal{L}_w}$ implies the sum of weight products for p-distinct directed stems from a leader $\mathcal{L}_w \in \mathcal{V}_{\mathcal{L}}$ to l with k-steps as:



(a) (b) (c)

Fig. 1. Examples

$$[\mathcal{C}_{\mathcal{L}_w}]_{l,k+1} = \mathcal{W}_k^{\mathcal{L}_w,l} = \sum_{i=1}^{p} g(\mathcal{P}_{k,i}^{\mathcal{L}_w,l}), \tag{12}$$

where $\mathcal{L}_w \in \mathcal{V}_{\mathcal{L}}$ and $w \in \{1, ..., m\}$ and $k \in \{1, ..., n-1\}$ and $l \in \mathcal{V}$. This relationship provides a bridge connecting our graph-theoretic constructs to the adjacency matrix representation of structured networks.

Example 2: Let us consider the controllability matrix corresponding to the graph shown in Fig. 1(a) with two leaders $1, 3 \in \mathcal{V}_{\mathcal{L}}$. Then, the controllability matrix $\bar{\mathcal{C}}$ is composed of two sub-controllability matrices, $\mathcal{C}_{\mathcal{L}_1}$ and $\mathcal{C}_{\mathcal{L}_2}$, as given in (13). In $\mathcal{C}_{\mathcal{L}_1}$, the non-zero entry in the first column represents leader $1 \in \mathcal{V}_{\mathcal{L}}$. The nodes that can be reached from leader 1 within 1-step are 2 and 4, and the corresponding sum of weight products for the directed stems are a_{21} and a_{41}, respectively, which are represented in the second column. Similarly, the nodes that can be reached from leader 1 within 2-steps are 3 and 5. The sum of weight products for the directed stems to these nodes are $a_{32}a_{21}$ and $a_{52}a_{21} + a_{54}a_{41}$, respectively, and are represented in the third column. The only node that can be reached from leader 1 within 3-steps is 5 and its corresponding sum of weight products, $a_{53}a_{32}a_{21}$, is represented in the fourth column. Since there are no nodes that can be reached from leader 1 within 4-steps, the fourth column is a zero vector. In $\mathcal{C}_{\mathcal{L}_2}$, except for the first column which corresponds to the leader node, the only directed stem begins at the leader $3 \in \mathcal{V}_{\mathcal{L}}$ and ends at node 5. This path is represented by the second column with the sum of weight products a_{53}. Thus, all remaining columns are zero vectors.

Now, let us consider a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with m leaders. From (12), we observe that an element $[\mathcal{C}_{\mathcal{L}_w}]_{l,k+1} \in \mathbb{R}^{n \times n}$

$$\bar{\mathcal{C}} = [\mathcal{C}_{\mathcal{L}_1}, \mathcal{C}_{\mathcal{L}_2}],$$

$$\text{where} \quad \mathcal{C}_{\mathcal{L}_1} = \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a_{21} & 0 & 0 & 0 \\ 0 & 0 & a_{32}a_{21} & 0 & 0 \\ 0 & a_{41} & 0 & 0 & 0 \\ 0 & 0 & a_{52}a_{21} + a_{54}a_{41} & a_{53}a_{32}a_{21} & 0 \end{bmatrix}}^{\text{k-steps}}, \quad \mathcal{C}_{\mathcal{L}_2} = \overbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & a_{53} & 0 & 0 & 0 \end{bmatrix}}^{\text{k-steps}} \tag{13}$$

in (5) is zero if and only if there is no directed stem from a leader node $\mathcal{L}_w$ to node $l \in \mathcal{V}$ with k-steps, for any $w \in \{1, ..., m\}$. For structured networks, when determining the dimension of the SSC from the controllability matrix, it is not just the zero elements that are of interest; we must also consider elements with a potential to be zero. To clarify this notion further, we introduce the *single-term* and *multi-terms*.

Definition 1: For the controllability matrix in (5):

- An element is called a *single-term* if it is a product of non-zero edge weights or a single non-zero edge weight.
- An element is called a *multi-terms* if it consists of the sum or difference of two or more *single-term*.

The above definition underscores a fundamental observation: The elements with *single-term* in the controllability matrix are always non-zero, whereas *multi-term* elements have the potential to be zero. For example, consider the element $[\mathcal{C}_{\mathcal{L}_1}]_{5,3}$ in (13). This element might be zero if the condition $a_{52}a_{21} = -a_{54}a_{41}$ is satisfied. From (10) and (12), the subsequent proposition provides the graph-theoretical conditions of *single-term* and *multi-terms*.

Proposition 2: For the digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with m leaders, an element $[\mathcal{C}_{\mathcal{L}_w}]_{l,k+1} \in \mathbb{R}^{n \times n}$ in (5) is:

- A *single-term* if there exists only one directed stem from leader $\mathcal{L}_w$ to node $l \in \mathcal{V}$ with k-steps.
- A *multi-terms* if there are multiple directed stems from leader $\mathcal{L}_w$ to node $l \in \mathcal{V}$ with k-steps.

This proposition clearly differentiates between *single-term* and *multi-terms*. One of the key determinants of the rank in the controllability matrix for structured networks is the pivot element, which is the first non-zero entry in each row. For a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, let us consider the sub-controllability matrix $\mathcal{C}_{\mathcal{L}_w}$ corresponding to a leader $\mathcal{L}_w \in \mathcal{V}_{\mathcal{L}}$. The i-th pivot element in $\mathcal{C}_{\mathcal{L}_w}$ represents the sum of weight products for the shortest stem from the leader $\mathcal{L}_w$ to node i for $i \in \{1, ..., n\}$. This implies that for an i-th pivot element to be classified as a *single-term*, its corresponding shortest stem must be unique. For example, let us consider the digraph depicted in Fig. 1(a) with a leader $1 \in \mathcal{V}_{\mathcal{L}}$. Then, there exist two distinct shortest stems from the leader 1 to node 5, both spanning two steps: the directed stems $(1 \to 2 \to 5)$ and $(1 \to 4 \to 5)$. In this case, the 5-th pivot element of $\mathcal{C}_{\mathcal{L}_1}$ would be characterized as *multi-terms*, represented by $a_{52}a_{21} + a_{54}a_{41}$. However, given that *multi-terms* can be zero, for the sake of simplification in our analysis, this paper adopts the following assumption:

Assumption 1: For every node $l \in \mathcal{V}$ in $\mathcal{G}(\mathcal{V}, \mathcal{E})$, there exists either no directed stem or a unique directed stem from a leader to node l with k-steps, where $k \in \{1, ..., n-1\}$.

For a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, the above assumption implies that there exists only one directed stem from a leader $\mathcal{L}_w \in \mathcal{V}_{\mathcal{L}}$ to l with the same k-steps. While multiple directed stems from a leader to a node can exist, they must have different lengths of steps. From (12), *Assumption 1* ensures that each element in the sub-controllability matrix is either a *single-term* or zero. Under the *Assumption 1*, this paper emphasizes the analysis of the controllability matrix from a graph perspective.

Remark 1: In a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, it follows from [20] that a unique directed stem from a leader to each node typically indicates a tree graph. However, *Assumption 1* permits multiple directed stems from a leader to a node as long as they have distinct step lengths, thus allowing for diverse network structures, including cycles.

IV. The Dimension of Strongly Structurally Controllable Subspace

In the previous section, we explored the intuitive meaning of the controllability matrix of a structured network from a graph-theoretical perspective. Building upon that analysis, in this section, we present a tight lower bound for the dimension of SSCS, leveraging the concept of stems. Let us first consider a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with a single leader. From the insights on the controllability matrix of structured networks presented in the previous section, we can obtain the following theorem.

Theorem 1: For a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with a single leader, the dimension of SSCS is lower bounded by the maximum number of nodes that can be included in the shortest stem.

Proof: Let us consider a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with a single leader $i_1 \in \mathcal{V}_{\mathcal{L}}$. Then, the controllability matrix is given by:

$$\bar{\mathcal{C}} = [b_{i_1}, A_\Lambda b_{i_1}, A_\Lambda^2 b_{i_1}, \ldots, A_\Lambda^{n-1} b_{i_1}] \in \mathbb{R}^{n \times n}. \tag{14}$$

Under the *Assumption 1*, it follows that the elements within (14) are either a single-term or are zero. Now, consider the shortest stem with the maximum number of nodes, described by the sequence $(i_1, i_2, \ldots, i_q)$. The standard bases corresponding to this sequence can be denoted as $e_{i_1}, e_{i_2}, \ldots, e_{i_q}$, where each e_v represents the standard basis with only its v-th element being non-zero. With these standard bases, we can formulate a row-swapping permutation matrix as:

$$P = [e_{i_1}, \ldots, e_{i_q} | e_{j_1}, \ldots, e_{j_{n-q}}] \in \mathbb{R}^{n \times n}, \tag{15}$$

where $j_1, j_2, ..., j_{n-q}$ represent the indices of the remaining nodes that are not included in the sequence $(i_1, i_2, \ldots, i_q)$. Using this permutation matrix, we can obtain the row-swapped controllability matrix $\bar{\mathcal{C}}' = P^T \bar{\mathcal{C}}$ as:

$$\bar{\mathcal{C}}' = \begin{bmatrix} \bar{\mathcal{C}}_{i_1} & * \\ * & * \end{bmatrix} \in \mathbb{R}^{n \times n}, \tag{16}$$

where the symbol $*$ denotes arbitrary entries, which can either be a single-term or zero. The submatrix $\bar{\mathcal{C}}_{i_1}$ typically has dimensions $q \times q$, corresponding to the sequence of the directed stem. However, in special cases where $q = n$, $\bar{\mathcal{C}}_{i_1}$ can expand to an $n \times n$ matrix, simplifying to $\bar{\mathcal{C}}' = \bar{\mathcal{C}}_{i_1}$. The comprehensive structure of $\bar{\mathcal{C}}_{i_1}$ is as follows:

$$\bar{\mathcal{C}}_{i_1} = \begin{bmatrix} \bullet & * & \cdots & * \\ 0 & \bullet & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \bullet \end{bmatrix} \in \mathbb{R}^{q \times q}, \tag{17}$$

where the symbol $\bullet$ represents a non-zero single-term entry and the first column in $\bar{\mathcal{C}}_{i_1}$ corresponds to b_{i_1}. Furthermore, for each $i \in \{1, ..., q\}$, the diagonal entries, denoted by $[\bar{\mathcal{C}}_{i_1}]_{i,i}$, signify the sum of weight products for the shortest stem from i_1 to each node in the sequence $(i_1, i_2, \ldots, i_q)$. Since we assumed that the directed stem from the leader i_1 to i_q with sequence $(i_1, i_2, \ldots, i_q)$ is the shortest stem, we can deduce that every directed stem from the leader i_1 to i_k for $k \in \{2, ..., q\}$ is also the shortest stem, and their sequences are always included within $(i_1, i_2, \ldots, i_q)$. With this observation, let us consider an element located in the lower triangular part in (17), specifically $[\bar{\mathcal{C}}_{i_1}]_{w,k}$ where $w < k$. For this element to be non-zero, there must exist a directed stem from i_1 to $i_w \in \mathcal{V}$ having fewer steps than the shortest stem with the sequence $(i_1, i_2, \ldots, i_q)$. However, this would contradict the definition of the shortest stem. As a result, all elements in the lower triangular part of $\bar{\mathcal{C}}_{i_1}$ in (17) are zero. That is, $[\bar{\mathcal{C}}_{i_1}]_{w,k} = 0$ whenever $w < k$. This observation confirms that $\bar{\mathcal{C}}_{i_1}$ forms a lower triangular matrix with non-zero diagonal elements, Consequently, the rank of $\bar{\mathcal{C}}_{i_1}$ is q, leading to the conclusion that the rank of $\bar{\mathcal{C}}'$ is at least q. Given that the ranks of $\bar{\mathcal{C}}'$ and $\bar{\mathcal{C}}$ are the same, we can conclude that the dimension of the SSCS of $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is lower bounded by q. ■

The following example aims to clarify and facilitate the understanding of the aforementioned proof.

Example 3: Consider the digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ illustrated in Fig. 1(b) with the single leader $1 \in \mathcal{V}_{\mathcal{L}}$. The sequence representing the shortest stem with the maximum number of nodes in $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is given by $(1, 4, 5, 6)$. Thus, the permutation matrix can be expressed as $P = [e_1, e_4, e_5, e_6 | e_2, e_3]$. Utilizing the row-swapped controllability matrix from (16), the submatrix $\bar{\mathcal{C}}_1$ associated with this shortest stem is:

$$\bar{\mathcal{C}}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{41} & 0 & 0 \\ 0 & 0 & a_{54}\, a_{41} & 0 \\ 0 & 0 & 0 & a_{65}\, a_{54}\, a_{41} \end{bmatrix} \tag{18}$$

In the above submatrix, each diagonal element represents the sum of weight products for the shortest stems from the leader 1 to each node in the sequence $(1, 4, 5, 6)$. From *Theorem 1*, it follows that the lower bound of the dimension of the SSCS for the digraph shown in Fig. 1(b) is 4. Now, suppose that the element $[\bar{\mathcal{C}}_1]_{3,2}$ in the lower triangular part from (17) is non-zero. This indicates the presence of a directed stem from leader 1 to node 5 in just one step. To achieve this, suppose that there exists an edge $(1, 5)$ is required as shown in Fig. 1(c). In this case, the existing sequence $(1, 4, 5, 6)$ no longer represents the shortest stem. Consequently, the shortest stem with the maximum number of nodes in the digraph in Fig. 1(c) becomes $(1, 5, 6)$, leading to a new permutation matrix $P = [e_1, e_5, e_6 | e_2, e_3, e_4]$. From (17), the submatrix $\bar{\mathcal{C}}_1$ associated with this shortest stem is:

$$\bar{\mathcal{C}}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{51} & a_{54} a_{41} \\ 0 & 0 & a_{65} a_{51} \end{bmatrix}. \tag{19}$$

In this case, since the maximum number of nodes in the shortest stems is 3, the lower bound of the dimension of the SSCS for the digraph shown in Fig. 1(c) is also 3.

Let us generalize *Theorem 1* for the case of multiple leaders. For a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with multiple leaders, define $\mathcal{M}(\mathcal{G})$ as the set of nodes in m-disjoint shortest stems containing the maximum possible number of nodes, and let $|\mathcal{M}(\mathcal{G})|$ represent the number of nodes in $\mathcal{M}(\mathcal{G})$. The below theorem provides a lower bound for the dimension of SSCS.

Theorem 2: For a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with m-leaders, the dimension of SSCS is lower bounded by $|\mathcal{M}(\mathcal{G})|$.

Proof: This proof extends the approach from the proof of *Theorem 1*. For a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with m-leaders, the controllability matrix is described in (5). Suppose that $\mathcal{M}(\mathcal{G})$ is the set of m-disjoint shortest stems with the maximum number of nodes. For each $k \in \{1, ..., m\}$, the sequence of nodes in the k-th shortest stem is $(i_{1_k}, i_{2_k}, \ldots, i_{q_k})$, where q_k signifies the number of nodes of the k-th shortest stem, ensuring that the total nodes across all disjoint stems, $|\mathcal{M}(\mathcal{G})| = \sum_{k=1}^{m} q_k \leq n$, where n is a total number of nodes. The standard bases associated with the sequence of the k-th shortest stem are presented as $e_{1_k}, e_{2_k}, \ldots, e_{q_k}$. Given the disjoint nature of m-shortest stems, each basis $e_{1_k}, e_{2_k}, \ldots, e_{q_k}$ is distinct. From these bases, a row-swapping permutation matrix is formulated as:

$$P = [E_1, \ldots, E_m, e_{j_1}, \ldots, e_{j_{n-|\mathcal{M}(\mathcal{G})|}}] \in \mathbb{R}^{n \times n}, \tag{20}$$

where $E_k = [e_{1_k}, \ldots, e_{q_k}]$ corresponds to the k-th shortest stem for $k \in \{1, ..., m\}$. The indices $j_1, j_2, \ldots, j_{n-|\mathcal{M}(\mathcal{G})|}$ are those of nodes not included in any of the shortest stems. Using this permutation matrix, we can obtain the row-swapped controllability matrix $\bar{\mathcal{C}}' = P^T \bar{\mathcal{C}}$ as:

$$\bar{\mathcal{C}}' = \left[\begin{array}{cc|cc|c|cc} \bar{\mathcal{C}}_{\mathcal{L}_1} & * & * & * & \cdots & * & * \\ * & * & \bar{\mathcal{C}}_{\mathcal{L}_2} & * & \cdots & \vdots & \vdots \\ * & * & * & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \cdots & \bar{\mathcal{C}}_{\mathcal{L}_m} & * \\ * & * & * & * & \cdots & * & * \end{array}\right] \in \mathbb{R}^{n \times nm}, \tag{21}$$

where the symbol $*$ denotes arbitrary entries, which can either be a single-term or zero. From (17), each $\bar{\mathcal{C}}_{\mathcal{L}_k} \in \mathbb{R}^{q_k \times q_k}$ is a lower triangular matrix with non-zero diagonal elements, representing the k-th shortest stem. Due to the properties of a lower triangular matrix, each $\bar{\mathcal{C}}_{\mathcal{L}_k}$ has full rank, specifically a rank of q_k for $k \in \{1, ..., m\}$. Given that each $\bar{\mathcal{C}}_{\mathcal{L}_k}$ does not share any rows or columns with each other, they are linearly independent. As such, the rank of (21) is at least the sum of their individual ranks, which is no less than $|\mathcal{M}(\mathcal{G})|$. Therefore, we can conclude that the dimension of the SSCS of $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is at least $|\mathcal{M}(\mathcal{G})|$. ■

Note that the above theorem is also applicable for the case of a single leader with $m = 1$ and $\mathcal{M}(\mathcal{G})$ may not be unique. We provide the following example to further illustrate the implications of the aforementioned theorem.

Example 4: Consider the digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ illustrated in Fig. 1(a) with leaders $2, 4 \in \mathcal{V}_{\mathcal{L}}$. The sequences representing two disjoint shortest stems with the maximum number of nodes in $\mathcal{G}(\mathcal{V}, \mathcal{E})$ can be $(2, 3)$ and $(4, 5)$. From (20), the permutation matrix can be expressed as $P = [e_2, e_3, e_4, e_5 | e_1]$. Utilizing the row-swapped controllability matrix from (21), the submatrices $\bar{\mathcal{C}}_2$ and $\bar{\mathcal{C}}_4$ are given by:

$$\bar{\mathcal{C}}_2 = \begin{bmatrix} 1 & 0 \\ 0 & a_{23} \end{bmatrix}, \ \bar{\mathcal{C}}_4 = \begin{bmatrix} 1 & 0 \\ 0 & a_{45} \end{bmatrix}. \qquad (22)$$

From *Theorem 2*, it follows that the lower bound of the dimension of the SSCS for the digraph in Fig. 1(b) is 4.

The theories introduced in this section provide insights into the dimension of SSCS within structured networks. These understandings arise from examining the roles each component of the controllability matrix plays from a graph-theoretical perspective. Compared to other methods for determining the lower bounds of the dimension of SSCS, such as zero forcing sets [19] and graph distance [12], our approach, based on the shortest stem, provides a more intuitive understanding of these lower bounds.

V. Conclusion

In this paper, we have investigated the tight lower bound of the dimension of SSCS for structured networks. Our study focused on analyzing the controllability matrix of structured networks from a graph-theoretic perspective. From this foundation, our work has highlighted the significance of the controllability matrix of structured networks in understanding control mechanisms. With the concept of the shortest stem, our *Theorem 1* and *Theorem 2* intuitively elucidate these lower bounds. The insights derived from our research are anticipated to not only enhance the understanding of controllability within structured networks but also serve as a foundational step towards determining the exact dimension of SSCS in future studies.

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