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Block Backstepping for Isotachic Hyperbolic PIDE-ODE systems

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***Abstract*— This work extends the theory of backstepping control of $(m+n)$ hyperbolic PIDEs and m ODEs to blocks of isotachic states (i.e. where some states have the same transport speed). This particular yet physical and interesting case has not received much attention beyond a few remarks in the early hyperbolic design, and leads to a block backstepping design. Our motivation is the rapid stabilization of N-layer Timoshenko composite beams with anti-damping and anti-stiffness at the uncontrolled boundaries. The problem of stabilization for a two-layer composite beam has been previously studied by transforming the model into a 1-D hyperbolic PIDE-ODE form and then applying backstepping to this new system. In principle this approach is generalizable to any number of layers. However, when some of the layers have the same physical properties (as e.g. in lamination of repeated layers), the approach leads to isotachic hyperbolic PDEs. We use a Riemann transformation to transform the states of N-layer Timoshenko beams into a 1-D hyperbolic PIDE-ODE system. The block backstepping method is then applied to this model, obtaining closed-loop stability of the origin in the L^2 sense. An arbitrarily rapid convergence rate can be obtained by adjusting control parameters. Finally, numerical simulations are presented corroborating the theoretical developments.**

I. Introduction

N-layer composite beams have been widely used in various fields, such as aeronautics [17], mechanism design [4], civil engineering [6] or electronics [30]. Several reasons justify their application, including weight reduction, higher overall stiffness, enhanced properties (with respect to fracture, fatigue or corrosion) or cost reduction. However, the coupling between layers can lead to vibration problems and, more critically, tip boundary conditions with anti-damping or anti-stiffness can cause divergence of the displacements and a consequent delamination of the beam into its unbonded constitutive layers. Therefore, it is necessary to design feedback controllers able to stabilize the equilibrium of the system.

Although designing controller for N-layer beams is rather challenging due to the involved complex mathematical models, there exists plenty of literature on stabilization. For example, [24] studied the stabilization of a laminated beam with interfacial slip, with an adhesive of small thickness bonding the two layers and creating damping. In [25], a viscoelastic laminated beam model is considered without additional control, and explicit energy decay formulae are established, giving the optimal decay rates by using minimal conditions on a relaxation function. The works [2], [3] investigated a one-dimensional laminated Timoshenko beam with a single nonlinear structural damping due to interfacial slip, and established an explicit and general decay result by adopting a multiplier method exploiting some properties of convex functions. In [18] the well-posedness and stability of structures with interfacial slip were researched; a large class of control kernels are considered and the system is proven to have a unique solution satisfying certain regularity properties. Apart from damping control, boundary control are frequently applied to laminated Timoshenko beams, for instance, [8] considered the stability of the closed loop system composed of laminated beams with boundary feedback controls, and a simple test method was used to verify exponential stability. Some researchers have adopted the simultaneous use of boundary and interfacial damping control to obtain exponential stability [1], [28]. Considering the time delay, [16] studied the long-time dynamics of laminated Timoshenko beams and established the existence of smooth finite-dimensional global attractors for the corresponding solution semigroup.

Most of these works achieve stability or even exponential stability, but not *rapid* stabilization (being able to set an arbitrarily fast decay rate in the closed loop), much less in the presence of destabilizing boundary conditions. This goal was achieved for beams by the use of backstepping technique in two pioneering works [22], [26], in the first case for an undamped shear beam, and in the second for an Euler-Bernoulli beam. More recently, backstepping was extended to obtain rapid stabilization for one-layer [10] and two-layer [9] Timoshenko beams (with potentially destabilizing boundary conditions) by using a Riemann transformation to cast the system as a 1-D hyperbolic PIDE-ODE system. This allows the use of the backstepping method, which has provided many designs for hyperbolic systems over the years; starting from a single 1-D hyperbolic partial integro-differential equation [21], the method was to extended next to 2×2 systems [12], [29], to $n+1\times n+1$ systems (n states convecting in one direction with one counter-convecting state that is controlled) in [13], and finally to the general case of $n+m\times n+m$ systems (n states convecting in one direction and m controlled states convecting in the opposite), both in the linear [19] and quasi-linear [20] cases. A later refinement allowed to obtain minimum-time convergence [5]. These results opened the door to PIDE-ODE designs such as [14], which was the key in our latest Timoshenko beam designs.

In principle the approach of applying Riemann transformation and then designing a backstepping control law for the resulting PIDE-ODE hyperbolic system is generalizable to any number of layers. However, when some of the layers have the same physical properties (as e.g. in lamination of repeated layers), the approach leads to *isotachic*

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hyperbolic PDEs (i.e. where some states have the same transport speeds). This particular yet physical and interesting case requires a modification of the general design, but has not received much attention beyond a few remarks in the earliest $(m+n)$ design [19]. Thus, driven by the multi-layer Timoshenko beam application, we extend the theory of backstepping control of $(m+n)$ hyperbolic PIDEs and m ODEs to *blocks* of isotachic states, leading to a block backstepping design based on a diagonalizing transformation. This is applied to the model of an N-layer Timoshenko beam by using a Riemann transformation to write the plant as a 1-D hyperbolic PIDE-ODE system. After that, we apply the proposed block backstepping theory to design a control law, obtaining exponential stabilitiy with an arbitrary rate of convergence.

This conference paper is a reduced version of our full journal version [11], which contains the full details of our results and its application to multilayer Timoshenko beams.

The paper is organized as follows: Section II presents the family of systems under study, namely a 1-D $(m+n)$ hyperbolic PIDE system coupled with m ODEs, involving some isotachic PDE states that are written as blocks. Section III gives the block design of the boundary controller for this case, and the main result. Then, Section IV analyzes the resulting controller. Section V studies the closed-loop stability. Section VI applies the proposed method to the N-layer Timoshenko beams (details are left out, see [11]). Section VII shows a numerical simulation corroborating the theoretical results. Finally, Section VIII closes the paper with some concluding remarks.

II. Problem Statement

Before addressing the problem of a N-layer Timoshenko Composite Beam, we need to investigate the following general system of $(m+n)$ hyperbolic PIDEs with m ODEs constant-coefficient model

$$Z_t = -\Sigma^+ Z_x + \Lambda^{++}Z + \Lambda^{+-}Y + \Pi^+ X + \int_0^x \left[F^{++}Z(y,t) + F^{+-}Y(y,t)\right] dy, \tag{1}$$

$$Y_t = \Sigma^- Y_x + \Lambda^{--}Y + \Lambda^{-+}Z + \Pi^- X + \int_0^x \left[F^{-+}Z(y,t) + F^{--}Y(y,t)\right] dy, \tag{2}$$

$$\dot{X} = AX + BY(0,t), \quad (x,t) \in [0,1]\times[0,\infty). \tag{3}$$

with boundary conditions

$$Y(1,t) = U, \tag{4}$$

$$Z(0,t) = CY(0,t) + DX. \tag{5}$$

where

$$Z = [z_1\ z_2\ \cdots\ z_m]^T, Y = [y_1\ y_2\ \cdots\ y_n]^T, \tag{6}$$

$$\Sigma^+ = \begin{bmatrix} \sigma_1^+ & & 0 \\ & \ddots & \\ 0 & & \sigma_m^+ \end{bmatrix} \in \mathbb{R}^{m\times m}, \tag{7}$$

$$\Sigma^- = \begin{bmatrix} \Sigma_1^- & & 0 \\ & \ddots & \\ 0 & & \Sigma_\kappa^- \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{8}$$

$$\Sigma_j^- = \sigma_j^- I_{n_j}, \sum_{j=1}^{\kappa} n_j = n,\ n_j \in \mathbb{N}^+, \tag{9}$$

$$\Lambda^{++}, \Pi^+, F^{++}, A, B, C, D \in \mathbb{R}^{m\times m}, \tag{10}$$

$$\Lambda^{+-}, F^{+-} \in \mathbb{R}^{m\times n}, \tag{11}$$

$$F^{--}, \Lambda^{--} \in \mathbb{R}^{n\times n}, \Pi^-, \Lambda^{-+}, F^{-+} \in \mathbb{R}^{n\times m} \tag{12}$$

with speeds

$$-\sigma_1^- < \cdots < -\sigma_\kappa^- < 0 < \sigma_1^+ \le \cdots \le \sigma_m^+ \tag{13}$$

In (9), I_{n_j} represents the n_j-sized identity matrix. Σ_j^- are the blocks that assemble to Σ^-. Thus, the Y-system has κ different transport speeds. When $\kappa = n$, that is $n_1 = \cdots = n_\kappa = 1$, all the states of $Y-$system have different transport speeds (non-isotachic case) and the classical backstepping design [19] (or its multiple variations) can be directly applied. When $\kappa < n$, there are at least two states have an identical transport speed (isotachic case). The direct coupling terms among the states with the same transport speed produce singularities in the kernel equations of [19]. To guarantee the kernel equations are solvable, we expand Remark 6 of [19] to this PIDE-ODE case and introduce an invertible transformation $\mathcal{A}(x)$ to transform the original system into an intermediate system where the isotachic stateshave no coupling between them. The details of transformation $\mathcal{A}(x)$ are presented in Section III-A.

In addition, the following assumption is essential to achieve the arbitrarily rapid stabilization of the coupled hyperbolic PIDE-ODE system.

Assumption 2.1: The coupled matrix pair (A,B) is controllable.

III. Controller design and main result

A. Block transformation for isotachic states

For ease of derivation, the coupling matrix Λ^{--} can be rewritten using blocks as follows

$$\Lambda^{--} = \begin{bmatrix} \Lambda_1^{--} & & \cdots \\ & \ddots & \\ \cdots & & \Lambda_\kappa^{--} \end{bmatrix} \tag{14}$$

where Λ_i^{--} refers to the coupling only between states belonging to the i-th block of isotachic states. To eliminate these, we introduce a transformation

$$\bar{Y}(x,t) = \mathcal{A}(x)Y(t) \tag{15}$$

$$\mathcal{A}(x) = \mathrm{diag}\{\mathcal{A}_1(x), \mathcal{A}_2(x), \cdots, \mathcal{A}_\kappa(x)\} \tag{16}$$

$$\frac{d\mathcal{A}_j(x)}{dx} = \frac{1}{\sigma_j^-}\mathcal{A}_j(x)\Lambda_j^{--},\ \mathcal{A}_j(0) = I_{n_j} \tag{17}$$

The matrices $\mathcal{A}_j(x)$ are all diagonal and invertible, with their inverses $\tilde{\mathcal{A}}_j(x) = (\mathcal{A}_j(x))^{-1}$ verifying

$$\frac{d\tilde{\mathcal{A}}_j(x)}{dx} = -\frac{1}{\sigma_j^-}\Lambda_j^{--}\tilde{\mathcal{A}}_j(x),\ \tilde{\mathcal{A}}_j(0) = I_{n_j} \tag{18}$$

It is easy to see $\tilde{\mathcal{A}}_j(x)$ is the inverse transformation of $\mathcal{A}_j(x)$ since $\mathcal{A}_j(0)\tilde{\mathcal{A}}_j(0) = I_{n_j}$ and $\frac{d\mathcal{A}_j(x)\tilde{\mathcal{A}}_j(x)}{dx} = 0$.

Use the transformation (15)–(18) and define $\bar{\Lambda}^{+-}(x) = \Lambda^{+-}\tilde{\mathcal{A}}(x), \bar{F}^{+-}(y) = F^{+-}\tilde{\mathcal{A}}(y), \bar{\Lambda}^{--}(x) =$

$\mathcal{A}(x)\left[-\Sigma^{-}\tilde{\mathcal{A}}(x)\frac{d\mathcal{A}}{dx}(x)+\Lambda^{--}(x)\right]\tilde{\mathcal{A}}(x)$, $\bar{\Lambda}^{-+}(x) = \mathcal{A}(x)\Lambda^{-+}, \bar{\Pi}^{-}(x) = \mathcal{A}(x)\Pi^{-}, \bar{F}^{-+}(x) = \mathcal{A}(x)F^{-+}$, $\bar{F}^{--}(x,y) = \mathcal{A}(x)F^{--}\tilde{\mathcal{A}}(y), \bar{U} = \mathcal{A}(1)U$, one has the $(Z, \bar{Y}, X)$ system

$$\begin{aligned} Z_t &= -\Sigma^{+}Z_x + \Lambda^{++}Z(t) + \bar{\Lambda}^{+-}(x)\bar{Y} + \Pi^{+}X \\ &\quad + \int_0^x \left[F^{++}Z(y,t) + \bar{F}^{+-}(y)\bar{Y}(y,t)\right]dy \end{aligned} \tag{19}$$

$$\begin{aligned} \bar{Y}_t &= \Sigma^{-}\bar{Y}_x + \bar{\Lambda}^{--}(x)\bar{Y} + \bar{\Lambda}^{-+}(x)Z \\ &\quad + \bar{\Pi}^{-}(x)X + \int_0^x \bar{F}^{-+}(x)Z(y,t)dy \\ &\quad + \int_0^x \bar{F}^{--}(x,y)\bar{Y}(y,t)dy \end{aligned} \tag{20}$$

$$\dot{X} = AX + B\bar{Y}(0,t) \tag{21}$$

with boundary conditions

$$\bar{Y}(1,t) = \bar{U}, \tag{22}$$

$$Z(0,t) = C\bar{Y}(0,t) + DX. \tag{23}$$

B. Stabilizing control law and main result

For system (1)–(5), the following control law is obtained in Section IV.

$$\begin{aligned} U &= \int_0^1 \tilde{\mathcal{A}}(1)K(1,y)\mathcal{A}(y)Y(y,t)dy \\ &\quad + \int_0^1 \tilde{\mathcal{A}}(1)L(1,y)Z(y,t)dy + \tilde{\mathcal{A}}(1)\Phi(1)X(t), \end{aligned} \tag{24}$$

whose gain kernels are the particular values of the matrices

$$K(x,y), \Phi(x) \in \mathbb{R}^{m\times m},\ L(x,y) \in \mathbb{R}^{m\times n} \tag{25}$$

evaluated at $x = 1$. These matrices will be defined in Section IV-B. Finding $\Phi(x)$ in particular requires setting boundary condition $\Phi(0)$. Define

$$E_1 = A + B\Phi(0) \tag{26}$$

We can obtain rapid stabilization of Timoshenko beam by choosing $\Phi(0)$ to adequately set the eigenvalues of the E_1 matrix, which is always possible due to Assumption 2.1 [31], thus obtaining the following result.

Theorem 1: Consider system (1)–(5), with initial conditions $Z_0, Y_0 \in L^2(0,1)$, $X_0 \in L^2$ under the control law (24). For all $C_2 > 0$ there exists gains $K(1,y)$, $L(1,y)$ and $\Phi(1)$ such that (1)–(5) has a solution $Y(\cdot,t)$, $Z(\cdot,t) \in L^2(0,1)$ for $t > 0$, and the following inequality is verified for some $C_1 > 0$:

$$\begin{aligned} &\|Z(\cdot,t)\|_{L^2}^2 + \|Y(\cdot,t)\|_{L^2}^2 + \|X(t)\|_{L^2}^2 \\ \le\ & C_1 \mathrm{e}^{-C_2 t}\Big(\|Z_0\|_{L^2}^2 + \|Y_0\|_{L^2}^2 + \|X_0\|_{L^2}^2\Big). \end{aligned} \tag{27}$$

The proof of Theorem 1 is given in Section V.

IV. Controller Analysis

This section presents the steps leading to (24). The backstepping method is used: first, the target system is presented in Section IV-A; next, the backstepping transformation (of Volterra type) is introduced in Section IV-B. The well-posedness of the kernel equations is stated in Theorem 2.

A. Target system

Inspired by [5], we design a target system as follows

$$\sigma_t = \Sigma^{-}\sigma_x + \Omega(x)\sigma, \tag{28}$$

$$Z_t = -\Sigma^{+}Z_x + \Lambda^{++}Z + \Lambda^{+-}\tilde{\mathcal{A}}(x)\sigma + \Xi_1(x)X \tag{29}$$

$$+ \int_0^x \Xi_2(x,y)\sigma(y,t)dy + \int_0^x \Xi_3(x,y)Z(y,t)dy, \tag{30}$$

$$\dot{X} = E_1 X + E_2\sigma(0,t). \tag{31}$$

with boundary conditions

$$\sigma(1,t) = 0,\ Z(0,t) = E_3 X + C\sigma(0,t) \tag{32}$$

where

$$\sigma = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_m \end{bmatrix}, E_2 = B,\ E_3 = C\Phi(0) + D. \tag{33}$$

and where the values of $\Xi_1(x,y)$, $\Xi_2(x,y)$, and $\Xi_3(x,y)$ are obtained in terms of the inverse backstepping transformation, in Section IV-B. The actual value of $\Omega(x)$ is also given in that section. The stability of this target system is shown in Section V.

B. Backstepping transformation

Firstly, inspired by [27], we introduce for $\bar{Y}$ the following backstepping transformation of Volterra type

$$\begin{aligned} \sigma &= \bar{Y} - \int_0^x K(x,y)\bar{Y}(y,t)dy \\ &\quad - \int_0^x L(x,y)Z(y,t)dy - \Phi(x)X(t). \end{aligned} \tag{34}$$

The kernel equations are deduced as usual, by a tedious but straightforward procedure of taking derivatives in the transformation, replacing the original and target equations, and integrating by parts. The details are skipped for brevity. The kernel equations have the following expressions:

$$\begin{aligned} \Sigma^{-}K_x + K_y\Sigma^{-} &= K\bar{\Lambda}^{--}(y) - L\bar{\Lambda}^{+-}(y) \\ &\quad - \Omega(x)K - \bar{F}^{--}(x,y) \\ &\quad + \int_y^x K(x,s)\bar{F}^{--}(s,y)ds, \\ &\quad + \int_y^x L(x,s)\bar{F}^{+-}(y)ds, \end{aligned} \tag{35}$$

$$\begin{aligned} \Sigma^{-}L_x - L_y\Sigma^{+} &= K(x,y)\bar{\Lambda}^{-+}(y) + L(x,y)\Lambda^{++} \\ &\quad - \Omega(x)L - \bar{F}^{-+}(x) \\ &\quad + \int_y^x K(x,s)\bar{F}^{-+}(s)ds \\ &\quad + \int_y^x L(x,s)F^{++}ds \end{aligned} \tag{36}$$

$$\begin{aligned} \Sigma^{-}\Phi_x &= \Phi A - \bar{\Pi}^{-}(x) \\ &\quad - \Omega(x)\Phi - L(x,0)\Sigma^{+}D \\ &\quad + \int_0^x K\bar{\Pi}^{-}(y) + L(x,y)\Pi^{+}dy \end{aligned} \tag{37}$$

with boundary conditions for K and L,

$$\Sigma^- L(x,x) + L(x,x)\Sigma^+ = -\bar{\Lambda}^{-+}(x), \tag{38}$$
$$\Sigma^- K(x,x) - K(x,x)\Sigma^- = -\bar{\Lambda}^{--}(x) + \Omega(x), \tag{39}$$
$$K(x,0)\Sigma^- + L(x,0)\Sigma^+ C = \Phi(x)B. \tag{40}$$

with

$$\Omega(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \omega_{2,1} & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 \\ \omega_{m,1} & \cdots & \omega_{m,m-1} & 0 \end{bmatrix}. \tag{41}$$

where $\omega_{i,j}(x,t) = (\Sigma^-_{i,i} - \Sigma^-_{j,j})K_{i,j}(x,x) + \bar{\Lambda}^{--}_{i,j}(x),\ i > j,\ i = 2, \cdots, m$. Notice that for i, j belonging to the same block, $\Sigma^-_{i,i} = \Sigma^-_{j,j}$ but also $\bar{\Lambda}^{--}_{i,j} = 0$, thus resulting in $\omega_{i,j} = 0$. Therefore there are no singularities in (39) for i, j such that $\Sigma^-_{i,i} = \Sigma^-_{j,j}$.

The structure of the kernel equations is similar to [5]. For $m \geq i \geq 2$, the kernel equations for K_{ij}, L_{ij} and Φ_{ij} seem to be nonlinear. However, one can start by solving K_{1j}, L_{1j} and Φ_{1j}, which are linear and can be proven solvable. Then, they become known coefficients of the equations verified K_{2j}, K_{2j} and Φ_{2j}. Thus K_{2j}, L_{2j} and Φ_{2j} become also linear and solvable. In the same recursive manner (in the spirit of [5]), we can obtain the solution of each kernel equation. Regarding the well-posedness of $K(x,y), L(x,y)$, the following result holds.

Theorem 2: There exists a unique bounded solution to the kernel equations (35)–(40), namely $K_{ij}(x,y)$, $L_{ij}(x,y)$, $\Phi_{ij}(x)$ for $i = 1, 2, \cdots, m; j = 1, 2, \cdots, n$; in particular, there exists positive numbers $\mathcal{N}, \mathcal{M}$ such that

$$\|K_{ij}(x,y)\|_\infty, \|L_{ij}(x,y)\|_\infty, \|\Phi_{ij}(x)\|_\infty \leq \mathcal{N} \mathrm{e}^{\mathcal{M}x} \tag{42}$$

The proof follows along the lines of [14] and is skipped; it is based on using the method of characteristics to write (35)–(40) in the form of integral equations and then posing a solution in terms of a successive approximation series, whose convergence is proven recursively. It is clear that the derivations of [14] can be easily adapted to the presence of the blocks, the integral terms and the differences in the boundary conditions without much effort.

Since the kernels appearing in (34) are bounded, the transformation is invertible from the theory of Volterra integral equation. Thus one can define

$$\bar{Y} = \sigma + \int_0^x \breve{K}(x,y)\,\sigma(y,t)dy + \int_0^x \breve{L}(x,y)\,Z(y,t)dy + \breve{\Phi}(x)\,X, \tag{43}$$

with bounded kernels. Both the transformation and its inverse map L^2 functions into L^2 functions (see e.g. [20]).

From the inverse transformation, the kernels $\Xi_1(x)$, $\Xi_2(x,y)$, $\Xi_3(x,y)$ appearing in (30) are

$$\Xi_1(x) = \Lambda^{+-}\tilde{\mathcal{A}}(x)\breve{\Phi}(x) + \Pi^+ \tag{44}$$
$$+ \int_0^x F^{+-}\tilde{\mathcal{A}}(y)\breve{\Phi}(y)dy, \tag{45}$$
$$\Xi_2(x,y) = \Lambda^{+-}\tilde{\mathcal{A}}(x)\breve{K}(x,y) + F^{+-}\tilde{\mathcal{A}}(y) \tag{46}$$
$$+ \int_0^x F^{+-}\tilde{\mathcal{A}}(y)\breve{K}(s,y)ds, \tag{47}$$
$$\Xi_3(x,y) = \Lambda^{+-}\tilde{\mathcal{A}}(x)\breve{L}(x,y) + F^{++} \tag{48}$$
$$+ \int_0^x F^{+-}\tilde{\mathcal{A}}(y)\breve{L}(s,y)ds. \tag{49}$$

from which it can be deduced that they are bounded kernels.

V. Stability and Analysis of Closed Loop

This section proves Theorem 1. First, in Section V-A, the solution of (28)–(32) is studied with the method of characteristics. This helps to find stability conditions in Section V-B. Then, a Lyapunov analysis in Section V-C shows exponential stability.

A. A semi-explicit solution for the target system

We start solving (28)–(32) with the method of characteristics. It can be shown (see [11] for details) that $\sigma(x,t)$ converges to zero in finite time $\frac{1}{\Sigma^-_{m,m}}$. For $t > \frac{1}{\Sigma^-_{m,m}}$,

$$\begin{aligned} Z_t(x,t) =& -\Sigma^+ Z_x(x,t) + \Lambda^{++}Z(x,t) + \Xi_1(x)X \\ &+ \int_0^x \Xi_3(x,y)Z(y,t)dy, \end{aligned} \tag{50}$$
$$\dot{X} = E_1 X, \tag{51}$$
$$Z(0,t) = E_3 X. \tag{52}$$

Solving for X we get $X(t) = X(0)\mathrm{e}^{E_1 t}$, where we have used the matrix exponential. Then

$$\begin{aligned} Z_t(x,t) =& -\Sigma^+ Z_x(x,t) + \Lambda^{++}Z(x,t) + \Xi_1(x)X(0)\mathrm{e}^{E_1 t} \\ &+ \int_0^x \Xi_3(x,y)Z(y,t)dy, \end{aligned} \tag{53}$$
$$Z(0,t) = E_3 X(0)\mathrm{e}^{E_1 t}. \tag{54}$$

Applying the method of characteristics, Volterra-type integral equations can be found for the components of Z. The details are skipped, but one can always find a unique L^2 solution for Z.

B. Stability conditions

The only requirement for stability is that E_1 is Hurwitz as then the origin of the state is exponentially stable for (50). Nevertheless, for rapid arbitrary stabilization, the eigenvalues of $E_1 = A + B\Phi(0)$ need to be set (e.g. by pole placement). Thus, if we choose the boundary conditions $\Phi(0)$ such that

$$E_1 + E_1^T < -2cI, \tag{55}$$

then the zero equilibrium of X in (51) is exponentially stable with a convergence rate of at least c. This is always possible to achieve by Assumption 2.1.

C. Lyapunov-based stability analysis of target system

Next, we use a Lyapunov functional for the stability analysis of target system, to show exponential stability of the origin with a fixed convergence rate. Define

$$V = \zeta_1 X^T X + \zeta_2 \int_0^1 \mathrm{e}^{\delta x} \sigma^T(x,t)(\Sigma^-)^{-1}\sigma(x,t)dx + \int_0^1 \mathrm{e}^{-\delta x} Z^T(x,t)(\Sigma^+)^{-1} Z(x,t)dx \quad (56)$$

It can be shown (see [11] for details) that, choosing $c = (c'+1)/2$ with $c' > 0$, and adequately setting $\Phi(0)$ to verify (55), one reaches $\dot{V} \leq -c'V$, achieving exponential stability with an arbitrary convergence rate.

VI. Rapid Stabilization of a N-layer Timoshenko Composite Beam

A. N-layer Timoshenko Composite Beam Model

The N-layer Timoshenko Composite Beam model can be expressed as [23] (see also [11] for more details)

$$\beta_i v_{i,tt} = \eta_i \left(v_{i,xx} + \theta_{i,x}\right), \quad (57)$$

$$+ C_1(i) k_n^{i-1} s_n^{i-1} - C_2(i) k_n^i s_n^i \quad (58)$$

$$\zeta_i \theta_{i,tt} = \alpha_i \theta_{i,xx} - \eta_i \left(v_{i,x} + \theta_i\right) \quad (59)$$

$$+ C_1(i) h_2^{i-1} k_t^{i-1} s_t^{i-1} + C_2(i) h_1^i k_t^i s_t^i, \quad (60)$$

$$s_t^{i-1} = -h_1^{i-1}\theta_{i-1} - h_2^{i-1}\theta_i, \quad (61)$$

$$s_n^{i-1} = v_{i-1} - v_i, \quad i = 1, \cdots, N \quad (62)$$

where

$$C_1(i) = \begin{cases} 1 & i > 1 \\ 0 & i = 1 \end{cases}, C_2(i) = \begin{cases} 1 & i < N \\ 0 & i = N \end{cases} \quad (63)$$

where the sub-index i makes reference to each of the layers, v_i are the transversal displacements, θ_i the rotational angles of the cross-sections, η_i the shear stiffnesses, ζ_i the rotational inertia, h_i the interface-centroids distances, k_t^i and k_n^i the tangential and normal interface stiffnesses, α_i and β_i the ratios of two layer beams with respect to normal stiffnesses and the moments of inertia of the cross-section, s_t^i and s_n^i the tangential and normal displacements in the interface between two beams, with boundary conditions

$$v_{i,x}(0,t) = \theta_i(0,t) - \xi_{2i-1} v_{i,t}(0,t) - \xi_{2i} v_i(0,t),$$
$$v_{i,x}(1,t) = U_{2i-1}(t),$$
$$\theta_{i,x}(0,t) = 0, \quad \theta_{i,x}(1,t) = U_{2i}(t), \quad i = 1, \cdots, N \quad (64)$$

where ξ_{2i-1} are the anti-damping of each beam, and ξ_{2N} the anti-stiffness, with $U_1(t), U_2(t), U_3(t), \cdots, U_{2i}(t)$ being the actuation variables that are designed next. All quantities in the model are dimensionless. It must note that the $2N$ actuators are independent and they will not de-laminate the adjacent layers due to the adhesives existing between them.

B. Transformation to a system of 1-D hyperbolic PDEs coupled with ODEs

Assumption 6.1: The anti-damping coefficients ξ_{2i-1} appearing in (64) verify $\xi_{2i-1} \neq \sqrt{\beta_i}/\sqrt{\eta_i}, \ i = 1, 2, \cdots, N$.

As a first step, the Timoshenko beam is maped into a first-order hyperbolic integro-differential system coupled with ODEs. The system becomes a $(2N + 2N) \times (2N + 2N)$ system of hyperbolic PIDEs , coupled with $2N$ ODEs, by using the following Riemann-like transformations:

$$p_i = \sqrt{\eta_i} v_{i,x} + \sqrt{\beta_i} v_{i,t}, \ r_i = \sqrt{\eta_i} v_{i,x} - \sqrt{\beta_i} v_{i,t} \quad (65)$$

$$q_i = \sqrt{\alpha_i}\theta_{i,x} + \sqrt{\zeta_i}\theta_{i,t}, \ s_i = \sqrt{\alpha_i}\theta_{i,x} - \sqrt{\zeta_i}\theta_{i,t} \quad (66)$$

$$0 = x_{2i-1} - v_i(0,t) \quad (67)$$

$$0 = x_{2i} - \theta_i(0,t), \ i = 1, 2, \cdots, N \quad (68)$$

Then the N-layer Timoshenko composite beams are transformed into a PIDE-ODE system that verifies Assumption 2.1. Following Section II–Section V we can design a boundary control law. See [11] for full details of the Timoshenko plant written in Riemann coordinates and the controller.

VII. Numerical simulation

To illustrate the stabilization result with a numerical example, we consider the case of $N = 2$ with $\beta_1 = 1, \beta_2 = 2, \eta_1 = \eta_2 = 1, \zeta_1 = 1, \zeta_2 = 2, h_1^1 = 1, h_2^1 = 1, k_T^1 = 1, k_N^1 = 1, \alpha_1 = 1, \alpha_2 = 1, \xi_1 = \xi_3 = -1, \xi_2 = \xi_4 = 1$, where the transport speeds verify $\frac{\sqrt{\eta_1}}{\sqrt{\beta_1}} = \frac{\sqrt{\alpha_1}}{\sqrt{\zeta_1}} > \frac{\sqrt{\eta_2}}{\sqrt{\beta_2}} = \frac{\sqrt{\alpha_2}}{\sqrt{\zeta_2}}$. Thus, there are two "blocks" with the same transport speeds. The value of $\Phi(0)$ can be set arbitrarily to specifiy the decay rate and is chosen as $\Phi_{11}(0) = -11, \Phi_{12}(0) = 1, \Phi_{22}(0) = -5, \Phi_{33}(0) = -11.4142, \Phi_{34}(0) = 1, \Phi_{44}(0) = 5$ and other elements in $\Phi(0)$ are set as zero, which leads to the decay rate $C_2 = c + 1 = 6$. For computing the 48 highly coupled kernel equations with triangle domains, a power series method is applied. Specifically, we start by solving $k_{1j}, L_{1j}, \Phi_{1j}, j = 1, 2, 3, 4$ since they are coupled with each other and independent of other kernel functions.The kernel equations are solved taking into account that K_{13}, K_{14} are all discontinuous due to the fact that (38) and (40) needs to be simultaneously verified (but not K_{12} due to it belonging to the same "block" as K_{11}). Thus, they possess a "line of discontinuity" along which they should be split in two analytic parts by dividing the triangular domain $\mathcal{T}$ into two parts which means we equivalently solve 2x4+2x4+4=20 coupled kernel functions. This procedure is followed until all the kernels are found. The open-loop Timoshenko states are divergent over time, but, when we apply the the proposed controller the states converge to zero as shown in Fig. 1.

VIII. Concluding remarks

This work presented an extension of the backstepping hyperbolic design to isotachic systems by working in blocks, motivated by the boundary control problem of a N-layer Timoshenko composite beams with anti-damping and anti-stiffness at the uncontrolled boundary. The plant can be written, in general, as such a isotachic system, and we are able to achieve arbitrarily fast decay by applying the block design. Independent multiple actuations are also required for rapidly stabilizing controllers for multi-layer fluids [15] and

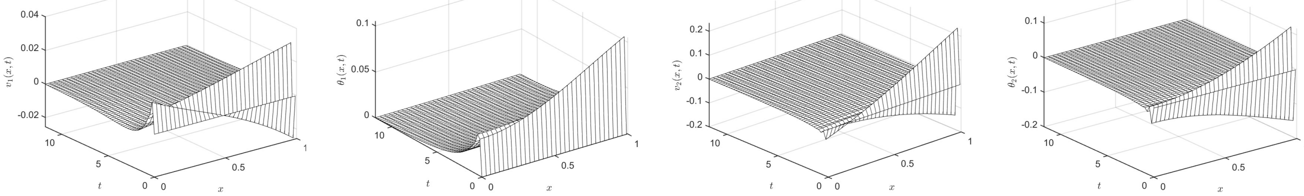

Fig. 1. Evolution of closed-loop Timoshenko states $v_1(x,t), \theta_1(x,t), v_2(x,t), \theta_2(x,t)$ (from left to right).

multi-lane traffic [32]. It is only when the subsystems occupy the same physical space, as in multi-class traffic [7], that a single actuator suffices.

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