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Switching $\mathcal{L}_2$ gain for evaluating the fluctuations around an output of a specified transfer property*

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Abstract— **In this paper, we introduce the difference structure to pre- and post-switch systems, to propose a new switching $\mathcal{L}_2$ gain for evaluating the fluctuations around an output of a specified transfer property after a system switch. Moreover, we apply it to the initial state design of a newly-activated controller designed by model matching to establish its potential practicality as a design index.**

I. Introduction

A control system switch is not a rare event. A typical example is a controller switch from an existing one to a more desirable one has been widely performed for letting a control system adjust to a failure in a control device, such as an actuator. It is so-called "active fault-tolerance." Here, it is needless to say that we should reduce the undesirable effects of a system switch. Concretely speaking, we should suppress the fluctuations in transient responses caused by a switch.

Many studies have addressed the issue of evaluating the magnitude of the fluctuations in transient responses caused by a system switch by using bumpless transfer [1] and other concepts. In [2], a theoretical procedure for evaluating the effects of a system switch by using the $\mathcal{L}_2$ gain of a Hankel-like operator was proposed. In [3], the $\mathcal{L}_2$ performance was first discussed as an important property over a long time interval including successive system switches. In [4], the performance was formulated as a switching $\mathcal{L}_2$ gain with doubly infinite time support for a single and unpredictable switch. In [5], a switching $\mathcal{L}_2$ gain based only on transient responses after a switch was proposed, whose value is not affected by the output before a switch. By using the switching $\mathcal{L}_2$ gain, we can evaluate the magnitude of the fluctuations in transient responses in the output. In [6], it was extended to a switching $\mathcal{L}_2$ gain for evaluating the smoothness of transient responses after a system switch by including the evaluation of their differential.

There are many cases where the shape of transient responses, as well as the magnitude of their fluctuations, is important. For example, in [7], as a support technology for safely performing preventive maintenance of LTI control systems, a safe shutdown process for a maintenance-object subsystem was proposed. The proposed process is based on the "fail-soft" concept that the subsystem gradually makes its way toward a complete stoppage and the overall control system is lead to a safe situation by using the remaining function in the overall system including the subsystem. The safety of such operating-state transitions can be guaranteed by suppressing the magnitude of the fluctuations in transient responses. In addition, by suppressing the difference with a desirable trajectory, we can improve the safety of the transitions. However, no studies have addressed the evaluation of the fluctuations in transient responses around a specified trajectory.

In this paper, we introduce the difference structure to pre- and post-switch systems to propose a new switching $\mathcal{L}_2$ gain for evaluating the fluctuations around an output of a specified transfer property after a system switch. Moreover, we applied it to the initial state design of a newly-activated controller designed by model matching to establish its potential practicality as a design index.

As the structure of pre- and post-switch systems, we consider the difference between a specified and fixed transfer property and a switched transfer property. Here, an output of the specified and fixed transfer property has a desirable shape. That is, we indirectly consider it as a desirable trajectory. Then, by considering the switching $\mathcal{L}_2$ gain presented in [5], we propose a new switching $\mathcal{L}_2$ gain for evaluating the fluctuations around an output of the specified transfer property after a system switch.

For example, a practical technique for reducing the undesirable effect of a controller switch is to appropriately initialize a newly-activated controller. In [1], [8], [9], [10], and [11], the initial state was obtained by making the output after a controller switch close to the virtual output in the case where the switch does not occur. In [12] and [13], the initial state was obtained by minimizing the $\mathcal{L}_2$ norm of the error between the output and reference signal. In [5], a switching $\mathcal{L}_2$ gain was minimized for obtaining the optimal switching matrix that determines the initial state of a newly-activated controller. In these preceding studies, the initial state of a newly-activated controller was designed for suppressing the magnitude of the fluctuations in transient responses after a controller switch.

Conversely, in this paper, by minimizing the proposed switching $\mathcal{L}_2$ gain, we can find the optimal switching matrix for suppressing the fluctuations of transient responses around an output of the specified transfer property as a desirable trajectory after the controller switch. That is, the proposed switching $\mathcal{L}_2$ gain provides a new viewpoint, the desirable shape of transient responses after a system switch, to the research area of the initial state design.

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Notations. I : an identity matrix of appropriate dimensions, O: a zero matrix of appropriate dimensions, $A \succ B$: a square Hermitian matrix $A - B$ is positive definite, $\bar{\sigma}(M)$: the maximum singular value of a matrix M, $\|G\|_\infty$: the $\mathcal{H}_\infty$ norm of a transfer function matrix G, $\mathcal{L}_2(a, b)$: the Lebesgue space of all square-integrable and vector-valued functions defined on an interval (a, b), i.e., $\mathcal{L}_2(a, b) = \left\{ x(t) \,\middle|\, \|x(t)\|_{2\,(a,\,b)} < \infty \right\}$, where $\|x(t)\|_{2\,(a,\,b)}$ denotes the $\mathcal{L}_2$ norm.

II. Switching $\mathcal{L}_2$ gain around a specified transfer property

A. Switch to be analyzed

Suppose that a linear time-invariant (LTI) system switches to another LTI system with a state transition at a switching time $t = t_0$.

1) Pre-switch system: The pre-switch syatem H_p is shown as Fig. 1(a). Here, $G_{\rm ref}$ is a specified and fixed transfer property described by

$$G_{\rm ref}: \begin{cases} \dot{x}_{\rm ref}(t) = A_{\rm ref} x_{\rm ref}(t) + B_{\rm ref} w(t) \\ y_{\rm ref}(t) = C_{\rm ref} x_{\rm ref}(t) + D_{\rm ref} w(t). \end{cases} \tag{1}$$

On the other hand, G_p described by

$$G_p: \begin{cases} \dot{x}_{g_p}(t) = A_{g_p} x_{g_p}(t) + B_{g_p} w(t) \\ y(t) = C_{g_p} x_{g_p}(t) + D_{g_p} w(t), \end{cases} \qquad t \le t_0 \tag{2}$$

is switched to G_f at the switching time $t = t_0$. Then, defining

$$x_p(t) = \begin{bmatrix} x_{\rm ref}(t) \\ x_{g_p}(t) \end{bmatrix}, \qquad t \le t_0, \tag{3}$$

we can describe the overall pre-switch system H_p as follows:

$$H_p: \begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p w(t) \\ z(t) = C_p x_p(t) + D_p w(t), \end{cases} \qquad t \le t_0, \tag{4}$$

where

$$\begin{aligned} A_p &= \begin{bmatrix} A_{\rm ref} & O \\ O & A_{g_p} \end{bmatrix}, \qquad B_p = \begin{bmatrix} B_{\rm ref} \\ B_{g_p} \end{bmatrix}, \\ C_p &= \begin{bmatrix} -C_{\rm ref} & C_{g_p} \end{bmatrix}, \quad D_p = -D_{\rm ref} + D_{g_p}. \end{aligned} \tag{5}$$

Strictly speaking, the output equation in (4) is not necessary, because we focus only on transient responses after a switch in this paper. We assume that A_p is stable, (A_p, B_p) is controllable, and (C_p, A_p) is observable.

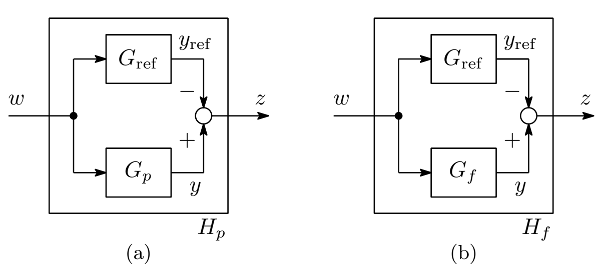

Fig. 1. (a) Pre- and (b) post-switch systems.

2) Post-switch system: The pre-switch syatem H_f is shown as Fig. 1(b). Here, G_f is described by

$$G_f: \begin{cases} \dot{x}_{g_f}(t) = A_{g_f} x_{g_f}(t) + B_{g_f} w(t) \\ y(t) = C_{g_f} x_{g_f}(t) + D_{g_f} w(t), \end{cases} \qquad t > t_0. \tag{6}$$

Then, defining

$$x_f(t) = \begin{bmatrix} x_{\rm ref}(t) \\ x_{g_f}(t) \end{bmatrix}, \qquad t > t_0, \tag{7}$$

we can describe the overall pre-switch system H_p as follows:

$$H_f: \begin{cases} \dot{x}_f(t) = A_f x_f(t) + B_f w(t) \\ z(t) = C_f x_f(t) + D_f w(t), \end{cases} \qquad t > t_0, \tag{8}$$

where

$$\begin{aligned} A_f &= \begin{bmatrix} A_{\rm ref} & O \\ O & A_{g_f} \end{bmatrix}, \qquad B_f = \begin{bmatrix} B_{\rm ref} \\ B_{g_f} \end{bmatrix}, \\ C_f &= \begin{bmatrix} -C_{\rm ref} & C_{g_f} \end{bmatrix}, \quad D_f = -D_{\rm ref} + D_{g_f}. \end{aligned} \tag{9}$$

We assume that A_f is stable, (A_f, B_f) is controllable, and (C_f, A_f) is observable.

3) State transition at the switching time: Suppose that the following state transition occurs at the switching time $t = t_0$:

$$x_{g_f}(t_{0+}) = S_0 x_{g_p}(t_0), \tag{10}$$

where $t = t_{0+}$ denotes an infinitesimal time increment of $t = t_0$, and S_0 is a real constant switching matrix. Then,

$$x_f(t_{0+}) = S x_p(t_0), \tag{11}$$

where

$$S = \begin{bmatrix} I & O \\ O & S_0 \end{bmatrix}. \tag{12}$$

B. Definition

In [5], the following switching $\mathcal{L}_2$ gain was proposed for analyzing the fluctuations in transient responses after an unpredictable system switch:

$$\hat{\gamma}_{\rm tr} = \sup_{w(t) \in \mathcal{L}_2(-\infty,\,\infty) \setminus \{0\}} \frac{\|z(t)\|_{2\,(t_0,\,\infty)}}{\|w(t)\|_{2\,(-\infty,\,\infty)}}. \tag{13}$$

Although this definition includes the switching time t_0, it does not affect the value of $\hat{\gamma}_{\rm tr}$, which depends on the fixed transfer property $G_{\rm ref}$, pre-switch part G_p, post-switch part G_f, and switching matrix S_0, as shown in [5]. That is,

$$\hat{\gamma}_{\rm tr} = \hat{\gamma}_{\rm tr}(G_{\rm ref}, G_p, G_f, S_0) = \hat{\gamma}_{\rm tr}(H_p, H_f, S). \tag{14}$$

By using this switching $\mathcal{L}_2$ gain, we can evaluate the fluctuations in transient responses after the system switch around an output $y_{\rm ref}$ of the specified transfer property $G_{\rm ref}$.

C. $\mathcal{L}_2$ gain condition

1) Equation-based $\mathcal{L}_2$ gain condition: The following theorem that presents an equation-based $\mathcal{L}_2$ gain condition. It implies that the switching time does not affect the value of the augmented switching $\mathcal{L}_2$ gain.

Theorem 2.1: For a given $\gamma > 0$, the switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\mathrm{tr}}$ in (14) satisfies $\hat{\gamma}_{\mathrm{tr}} < \gamma$ if and only if the following conditions are satisfied.
(a) It holds that $\bar{\sigma}(D_f) < \gamma$.
(b) There exists the stabilizing solution $X_f \succ O$ to the Riccati equation

$$\begin{aligned} &X_f A_f + A_f^{\mathrm{T}} X_f + C_f^{\mathrm{T}} C_f \\ &\quad + (X_f B_f + C_f^{\mathrm{T}} D_f)(\gamma^2 I - D_f^{\mathrm{T}} D_f)^{-1} \\ &\qquad \times (X_f B_f + C_f^{\mathrm{T}} D_f)^{\mathrm{T}} = O. \end{aligned} \tag{15}$$

(c) It holds that

$$\gamma^2 X_p^{-1} - S^{\mathrm{T}} X_f S \succ 0, \tag{16}$$

where $X_p \succ O$ is the unique solution to the Lyapunov equation:

$$A_p X_p + X_p A_p^{\mathrm{T}} + B_p B_p^{\mathrm{T}} = O. \tag{17}$$

In (b), using the stabilizing solution X_f, we have that $A_f + B_f K_f$ is stable, where

$$K_f = (\gamma^2 I - D_f^{\mathrm{T}} D_f)^{-1} (X_f B_f + C_f^{\mathrm{T}} D_f)^{\mathrm{T}}. \tag{18}$$

In (c), there exists the unique solution $X_p \succ O$ to the Lyapunov equation (17), because A_p is stable and (A_p, B_p) is controllable. Thus, there also exists $X_p^{-1} \succ O$. *Proof:* Using the relationship (14) and Theorem 3.1 in [5], we can prove this theorem. □
An input providing the switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\mathrm{tr}}$ can be described explicitly in the following corollary to Theorem 2.1.

Corollary 2.2: Suppose that for a given $\gamma > 0$, Conditions (a) and (b) in Theorem 2.1 are satisfied, and the matrix $\gamma^2 X_p^{-1} - S^{\mathrm{T}} X_f S$ is positive semidefinite with a zero eigenvalue. Then, the switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\mathrm{tr}}$ in (14) is given by $\hat{\gamma}_{\mathrm{tr}} = \gamma$. Furthermore, the input providing the value of $\hat{\gamma}_{\mathrm{tr}}$ is given by

$$\hat{w}(t) = \begin{cases} B_p^{\mathrm{T}} e^{-A_p^{\mathrm{T}}(t-t_0)} X_p^{-1} v, & t \le t_0 \\ K_f e^{(A_f + B_f K_f)(t-t_0)} S v, & t > t_0, \end{cases} \tag{19}$$

where v is an eigenvector v corresponding to the zero eigenvalue.

Proof: See the proof of Corollary 3.1 in [5]. □

2) LMI-based $\mathcal{L}_2$ gain condition: The gain condition using LMIs in the following theorem has the advantage that we can compute more efficiently the value of the switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\mathrm{tr}}$ in (14). It also implies that the switching time does not affect the value of the switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\mathrm{tr}}$. We will use it for designing the switching matrix for obtaining the initial state of a newly-activated controller in Section III.

Theorem 2.3: For a given $\gamma > 0$, the switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\mathrm{tr}}$ in (14) satisfies $\hat{\gamma}_{\mathrm{tr}} < \gamma$ if and only if there exist $\tilde{X}_p \succ O$ and $\tilde{X}_f \succ O$ satisfying the following conditions:

$$\begin{bmatrix} \tilde{X}_p A_p + A_p^{\mathrm{T}} \tilde{X}_p & \tilde{X}_p B_p \\ B_p^{\mathrm{T}} \tilde{X}_p & -\gamma I \end{bmatrix} \prec O \tag{20}$$

$$\begin{bmatrix} \tilde{X}_f A_f^{\mathrm{T}} + A_f \tilde{X}_f & B_f & \tilde{X}_f C_f^{\mathrm{T}} \\ B_f^{\mathrm{T}} & -\gamma I & D_f^{\mathrm{T}} \\ C_f \tilde{X}_f & D_f & -\gamma I \end{bmatrix} \prec O \tag{21}$$

$$\begin{bmatrix} \tilde{X}_p & S^{\mathrm{T}} \\ S & \tilde{X}_f \end{bmatrix} \succ O. \tag{22}$$

Proof: Using the relationship (14) and Proposition 5.2 in [5], we can prove this theorem. □

Consider the $\mathcal{H}_\infty$ norm of the post-switch system:

$$\gamma_f = \|H_f\|_\infty. \tag{23}$$

Then, we have the following corollary from Theorem 2.3.

Corollary 2.4: The switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\mathrm{tr}}$ in (14) satisfies

$$\gamma_f \le \hat{\gamma}_{\mathrm{tr}}. \tag{24}$$

Proof: For a given $\gamma > 0$, $\gamma_f < \gamma$ if and only if there exists $\tilde{X}_f \succ O$ satisfying (21). □

III. Initial state design of a newly-activated controller designed by model matching

A. Problem statement

1) Situation: Consider the pre- and post-switch systems as shown in Fig. 2. In the both systems, P is the plant, and G_{ref} is the model-matching target. Then, the model matching performance is evaluated by $\|H_p\|_\infty$ and $\|H_f\|_\infty$.

Consider the situation where due to the failure in a control device, change in the surrounding circumstances, and so on, the plant P varied from a previous property to the current property. We suppose that the pre-switch controller K_p is designed for the previous property of P, thus the model matching performance deteriorated. Therefore, we should the controller to K_f designed for the current property, which can achieve a more desirable performance. For example, we can obtain K_f via the optimization

$$\underset{K_f}{\text{minimize}} \ \|H_f\|_\infty. \tag{25}$$

Due to (24), the optimization (25) is important for the initial state design by using γ_{tr}.

2) Pre-switch system: In Fig. 2(a), let $x_{\mathrm{ref}}(t)$, $x_{\mathrm{plant}}(t)$, $x_{\mathrm{int}}(t)$, and $x_{k_p}(t)$ denote the state-variable vectors of the model-matching target G_{ref}, integrators, plant P, and pre-switch controller K_p, respectively. Then, by using

$$x_p(t) = \begin{bmatrix} x_{\mathrm{ref}}(t) \\ x_{\mathrm{plant}}(t) \\ x_{\mathrm{int}}(t) \\ x_{k_p}(t) \end{bmatrix}, \qquad t \le t_0, \tag{26}$$

for the given G_{ref}, P, and K_p, we can obtain the following description of the pre-switch system:

$$H_p : \begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p w(t) \\ z(t) = C_p x_p(t) + D_p w(t), \end{cases} \qquad t \le t_0. \tag{27}$$

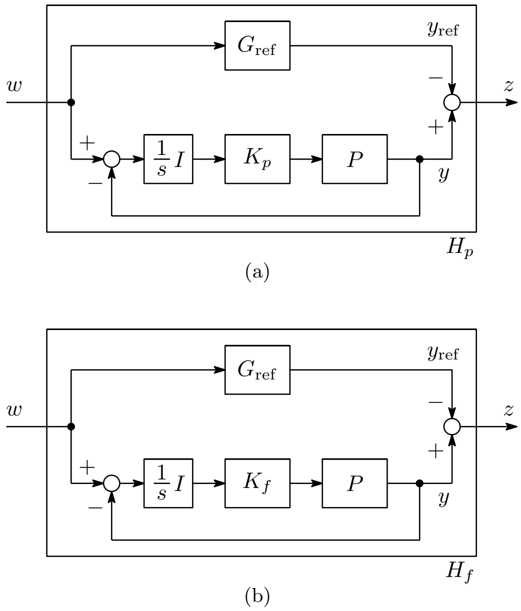

Fig. 2. (a) Pre- and (b) post-switch systems for model matching.

3) Post-switch system: In Fig. 2(b), let $x_{\mathrm{ref}}(t)$, $x_{\mathrm{plant}}(t)$, $x_{\mathrm{int}}(t)$, and $x_{k_f}(t)$ denote the state-variable vectors of the model-matching target G_{ref}, integrators, plant P, and post-switch controller K_f, respectively. Then, by using

$$x_f(t) = \begin{bmatrix} x_{\mathrm{ref}}(t) \\ x_{\mathrm{plant}}(t) \\ x_{\mathrm{int}}(t) \\ x_{k_f}(t) \end{bmatrix}, \qquad t > t_0, \tag{28}$$

for the given G_{ref}, P, and K_f, we can obtain the following description of the post-switch system:

$$H_f : \begin{cases} \dot{x}_f(t) = A_f x_f(t) + B_f w(t) \\ z(t) = C_f x_f(t) + D_f w(t), \end{cases} \qquad t > t_0. \tag{29}$$

4) State transition at the switching time: Suppose that the following state transition occurs at the switching time $t = t_0$:

$$x_{k_f}(t_{0+}) = S_k \begin{bmatrix} x_{\mathrm{plant}}(t_0) \\ x_{\mathrm{int}}(t_0) \\ x_{k_p}(t_0) \end{bmatrix}, \tag{30}$$

where S_k is a real constant switching matrix. This is the initial state of the newly-activated controller K_f. Then, as the transition of the overall state-variable vectors of the pre- and post-switch systems,

$$x_f(t_{0+}) = S x_p(t_0), \tag{31}$$

where

$$S = \begin{bmatrix} I & O & O & O \\ O & I & O & O \\ O & O & I & O \\ O & [& S_k &] \end{bmatrix}. \tag{32}$$

5) Initial state design via the switching matrix: At the controller switch from K_p to K_f, we should assign the initial state of K_f, $x_f(t_{0+})$. In this paper, we consider the following problem via the switching matrix.

Problem 3.1: For the given G_{ref}, P, K_p, and K_f, find the optimal S_k in (31) that minimizes the switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\mathrm{tr}}$ in (14).

By assigning the initial state of the newly-activated controller K_f by using the optimal switching matrix S_k, the fluctuations in transient responses around the desirable output y_{ref} of the model-matching target G_{ref} after the switch can be suppressed.

B. Design procedure

By considering

$$S_1 = \begin{bmatrix} O & O & O & O \\ O & O & O & O \\ O & O & O & O \\ O & [& S_k &] \end{bmatrix} \tag{33}$$

$$S_2 = \begin{bmatrix} I & O & O & O \\ O & I & O & O \\ O & O & I & O \\ O & O & O & O \end{bmatrix}, \tag{34}$$

we can decompose S in (32) as

$$S = S_1 + S_2. \tag{35}$$

Thus, the LMI (22) is equivalent to

$$\begin{bmatrix} \tilde{X}_p & S_1^{\mathrm{T}} + S_2^{\mathrm{T}} \\ S_1 + S_2 & \tilde{X}_f \end{bmatrix} \succ O. \tag{36}$$

Therefore, by solving the LMI problem under the constraint conditions (20), (21), and (36) to minimize γ by using the variables $\tilde{X}_p$, $\tilde{X}_f$, and S_1, we can obtain the optimal S_k in S_1.

Consequently, the procedure for designing S_k is summarized as follows.

Algorithm 3.2:

Step 1: From the given G_{ref}, P, K_p, and K_f, obtain H_p in (27) and H_f in (29), i.e., (A_p, B_p, C_p, D_p) and (A_f, B_f, C_f, D_f).

Step 2: Solve the LMI problem consisting of (20), (21), and (36) with the variable the variables $\tilde{X}_p$, $\tilde{X}_f$, and S_1 to obtain the optimal S_k in S_1.

C. Example

The plant is given by

$$P : \left[\begin{array}{rr|r} -1 & 1 & 0 \\ 0 & -2 & 1 \\ \hline 1 & 0 & 0 \end{array}\right]. \tag{37}$$

The model-matching target is

$$G_{\mathrm{ref}}(s) = \frac{1}{0.03s^4 + 0.15s^3 + 0.5s^2 + s + 1}. \tag{38}$$

This transfer function is known for its desirable response against a unit step input (see $y_{\rm ref}$ in Fig. 3). The pre- and post-switch controllers are as follows:

$$K_p : \left[\begin{array}{ccccccc|c} -2.5 & -0.5 & -6.3 & 6.6 & -11.5 & -19.0 & -40.1 & 1.8 \\ 0.5 & -3.0 & 4.2 & -1.7 & 4.2 & 6.0 & 16.3 & 2.6 \\ 2.1 & 2.8 & -5.4 & -1.0 & 0.1 & 1.5 & -9.4 & -5.8 \\ 0.9 & 1.0 & -1.4 & -3.6 & -4.5 & -18.6 & 19.5 & -2.5 \\ -2.2 & -0.8 & 2.2 & 5.0 & -19.5 & -79.4 & 25.0 & 3.9 \\ -4.9 & -4.1 & 7.8 & 4.8 & -22.3 & -97.1 & 33.7 & 10.8 \\ 1.1 & 2.3 & -3.8 & -1.0 & 8.5 & 32.4 & -39.3 & -3.8 \\ \hline -0.3 & 0.1 & -7.6 & 6.2 & -11.7 & -18.7 & -44.2 & 0 \end{array}\right] \tag{39}$$

$$K_f : \left[\begin{array}{ccccccc|c} -58.9 & 1589.9 & 192.9 & 133.6 & 655.6 & 22.7 & 33.0 & 1736.9 \\ 90.9 & -1563.2 & -189.0 & -129.8 & -631.7 & 18.3 & -366.9 & -1703.6 \\ -179.1 & -295.4 & -37.9 & -27.3 & -103.6 & -274.0 & 4365.5 & -311.1 \\ 10.1 & -178.6 & -20.6 & -15.9 & -72.7 & 1.9 & -50.3 & -194.8 \\ 85.6 & -1839.9 & -222.8 & -153.8 & -753.1 & -2.2 & -339.8 & -2008.2 \\ 41.3 & -1209.8 & -146.8 & -101.6 & -497.6 & -23.1 & 72.9 & -1321.0 \\ -7.8 & -13.8 & -1.8 & -1.1 & -4.8 & -11.0 & -180.4 & -14.6 \\ \hline -192.6 & -10.3 & -1.4 & -3.5 & 12.2 & -273.5 & 4395.0 & 0 \end{array}\right]. \tag{40}$$

TABLE I shows the model matching performance values of the $\mathcal{L}_2$ gain. Consider the situation where due to the failure in a control device, change in the surrounding circumstances, and so on, the plant P varied to (37) from a slightly different property. We suppose that the pre-switch controller K_p is designed for the previous property of P, thus the performance index value with the current (37) is not good. On the other hand, the post-switch controller K_f is designed for the current (37), thus it can achieve the desirable performance index value as shown in TABLE I. That is, the controller switch from K_p to K_f is for improving the model matching performance.

TABLE I
PERFORMANCE INDEX.

	$\mathcal{H}_\infty$ norm
H_p	0.0763
H_f	0.0027

In order to obtain the initial state $x_{k_f}(t_{0+})$ of the post-switch controller K_f, we consider the following two switching matrix designs (i) and (ii).

(i) Switching matrix design by using $\hat{\gamma}_{\rm tr}$. By using Algorithm 3.2, we obtain

$$S_k = \begin{bmatrix} -29.5 & -124.5 & 0.3 & 118.5 & -16.1 & 33.5 & -17.6 & -4.7 & -46.3 & -161.9 \\ 24.5 & 103.8 & -0.4 & -99.1 & 13.0 & -26.8 & 13.8 & 4.4 & 39.9 & 139.8 \\ 0.2 & 1.9 & -0.2 & -0.9 & 0.1 & -0.3 & 0.3 & -0.1 & 0.5 & 1.4 \\ 7.4 & 27.2 & 0.0 & -26.0 & 3.5 & -7.0 & 3.6 & 1.2 & 10.3 & 36.1 \\ 31.4 & 133.2 & -0.4 & -126.9 & 17.2 & -35.1 & 18.3 & 5.5 & 50.2 & 176.4 \\ 22.4 & 94.7 & -0.3 & -90.1 & 12.3 & -25.4 & 13.4 & 3.6 & 35.4 & 123.9 \\ 2.9 & 12.1 & 0.0 & -11.5 & 1.5 & -3.2 & 1.6 & 0.5 & 4.3 & 15.3 \end{bmatrix}. \tag{41}$$

(ii) Zero initial state. The following switching matrix sets the zero initial state of the post-switch controller K_f, i.e., $x_{k_f}(t_{0+}) = 0$:

$$S_k = O. \tag{42}$$

TABLE II
DESIGN RESULTS.

	$\hat{\gamma}_{\rm tr}$
(i) Proposed design	0.0117
(ii) Zero initial state	0.8776

TABLE II shows the design results. The switching matrix design (i) minimizes $\hat{\gamma}_{\rm tf}$ to obtain the value 0.0117. On the other hand, in the zero initial state case (ii), the value of $\hat{\gamma}_{\rm tf}$ is extremely worse.

We now present the simulation results. The controller switch occurs at $t = 3$. The model matching performance of the pre-switch system deteriorated, thus the difference between y and $y_{\rm ref}$ grows large before the switch, i.e., $0 \le t \le 3$.

After the switch with the switching matrix (41), the proposed design, the fluctuations in transient responses around the desirable output $y_{\rm ref}$ of the model-matching target $G_{\rm ref}$ can be well suppressed.

On the other hand, in the case of the zero initial state, there is violent fluctuation around $y_{\rm ref}$. This implies that the importance of the initial state design of newly-activated controller in active fault torelance.

IV. CONCLUSIONS

By introducing the difference structure to the pre- and post-switch systems, we proposed a new switching $\mathcal{L}_2$ gain for evaluating the fluctuations around an output of a specified transfer property after a system switch. Moreover, we applied it to the initial state design of a newly-activated controller designed by model matching to establish its potential practicality as a design index.

There are many practical situations where a desirable shape of transient responses in physical values is required due to safety aspects and equipment protection, such as the operating-state transitions considered in [7]. In such a

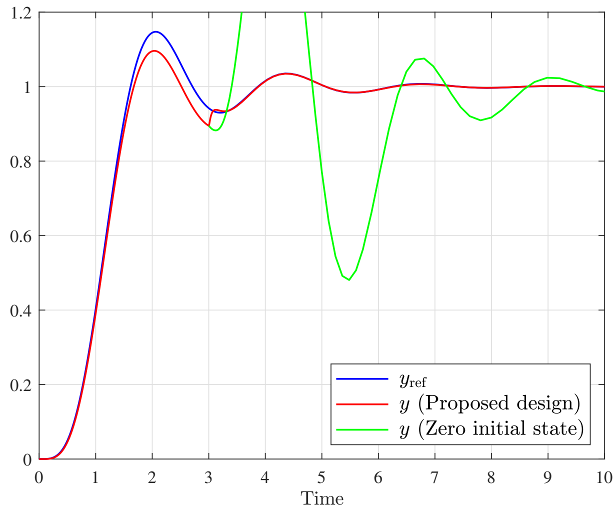

Fig. 3. Responses against a unit step input.

situation, we can use the proposed switching $\mathcal{L}_2$ gain by specifying a transfer property with its desirable output. The result obtained in this paper is the first and significant step of theoretical challenges to a desirable shape of transient responses in control theory.

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