

# On $\ell_2$ -performance of weakly-hard real-time control systems

Marc Seidel, Simon Lang, Frank Allgöwer

**Abstract**—This paper considers control systems with failures in the feedback channel, that occasionally lead to loss of the control input signal. A useful approach for modeling such failures is to consider window-based constraints on possible loss sequences, for example that at least  $r$  control attempts in every window of  $s$  are successful. A powerful framework to model such constraints are weakly-hard real-time constraints. Various approaches for stability analysis and the synthesis of stabilizing controllers for such systems have been presented in the past. However, existing results are mostly limited to asymptotic stability and do not consider performance measures such as the resulting  $\ell_2$ -gain. To address this problem, we adapt a switched system description where the switching sequence is constrained by a graph that captures the loss information. We present an approach for  $\ell_2$ -performance analysis involving linear matrix inequalities (LMI). Further, leveraging a system lifting method, we propose an LMI-based approach for synthesizing state-feedback controllers with guaranteed  $\ell_2$ -performance. The results are illustrated by a numerical example.

**Index Terms**—control over communication, switched systems

## I. INTRODUCTION

In control systems with an unreliable feedback channel, failures can lead to the occasional loss of the control input signal. Such losses appear for example in Networked Control Systems (NCS), where the feedback loop is closed via a communication channel that is subject to packet loss [1]. Similarly, in real-time control applications deadline misses can occur, where the control signal computation does not finish in time and thus no new actuator command is issued [2], i.e., the control signal can be considered as lost. In both scenarios, the system runs in open-loop from time to time whenever the attempt to control the system fails, e.g., when a control input sent over the network is lost or its computation is not completed before the deadline. A number of approaches model these losses as random variables, see [3] and references therein. For such models, only stochastic guarantees can be given, e.g., in the mean-square sense. Further approaches deal with deterministic bounded loss [4]. In this case, classical guarantees in the form of stability in the sense of Lyapunov can be given. However, these approaches neglect information on the past losses, which is rather conservative.

To avoid this conservatism, a new window-like deterministic packet loss model was suggested in the literature [5]. As a generalization of bounded loss and scheduling constraints

like  $(m, k)$ -firm deadlines [6], [7], where at least  $m$  out of  $k$  consecutive tasks are guaranteed to meet their deadline, the notion of weakly-hard real-time (WHRT) constraints has been proposed [5]. Originally presented as scheduling constraints for real-time applications, these type of constraints have been used recently to model the packet loss in NCS [8], [9], [10]. WHRT constraints pose a window-like constraint, where in a moving time window of certain length at least a specified minimum number of control attempts are successful. This allows to describe losses more accurately compared to a stochastic description, that allows the losses to be distributed unevenly. For example, a 10% loss chance allows that 100 unsuccessful control attempts are followed by 900 successful ones, or 10 unsuccessful ones followed by 90 successful control attempts, which is vastly different from a control perspective [5], [8]. The notion of WHRT control systems includes NCS with WHRT packet loss description as well as real-time control systems in which the deadline misses are modeled by a WHRT constraint.

For linear NCS, [8], [9] present linear matrix inequality (LMI) conditions for asymptotic stability analysis and asymptotic state-feedback stabilization. The work [10] extends this to an event-triggered setup, switching controllers and output-feedback, while [11], [12] deal with asymptotic stability for a nonlinear WHRT control system based on the concept of non-monotonic Lyapunov functions. Another approach in the area of real-time control systems focuses on asymptotic stability analysis using the constrained joint spectral radius [2], [13], which characterizes the maximum asymptotic growth rate over all possible switching sequences. However, the aforementioned works focus solely on asymptotic stability and lack performance considerations, despite the fact that performance guarantees are crucial from an application point of view. First progress towards performance analysis using a quadratic state cost function has been made in [14], [15], however no theoretical guarantees are given. In [7], performance guarantees in an LQR setting are studied. To the best of our knowledge, classical control performance in the sense of, e.g.,  $\ell_2$ -gains has not been considered so far.

This paper aims at addressing the analysis of the  $\ell_2$ -performance and the synthesis of state-feedback controllers with guaranteed  $\ell_2$ -performance for WHRT control systems. To formally describe them, we adapt the switched system description from [7], [8], [9], [10], where the switching is described by a graph, that captures the WHRT constraint, called the WHRT graph [10]. Leveraging the switched system performance results from [16] and prior work [17], we derive sufficient LMI conditions for  $\ell_2$ -performance analysis of WHRT control systems. These novel result can also be used

F. Allgöwer thanks the German Research Foundation (DFG) for support of this work within grant AL 316/13-2 and within the German Excellence Strategy under grant EXC-2075 - 285825138; 390740016.

The authors are with the University of Stuttgart, Institute for Systems Theory and Automatic Control, Stuttgart, Germany, {seidel, lang, allgower}@ist.uni-stuttgart.de

for  $\ell_2$ -performance analysis of general *graph-constrained* switched systems. In contrast to previous works, this has not been done yet for an  $\ell_2$ -performance measure, but only for stability. Moreover, we utilize a classical system lifting method [18] to derive an LMI-based approach for synthesis of switched and non-switched state-feedback controllers with guaranteed  $\ell_2$ -performance, which additionally allows for a more efficient performance analysis. The lifting method is based on combining multiple past inputs and outputs and “discretizing” the system at time instants of successful control attempts, which directly enables controller synthesis.

The remainder of this paper is organized as follows. In Section II, we define our setup and state preliminaries. We then present some first  $\ell_2$ -performance analysis results in Section III. Section IV defines the lifting method and gives an improved analysis results for the lifted system. In Section V, controller synthesis is presented and in Section VI we emphasize our findings by a numerical example. Section VII concludes the paper.

### A. Notation

The set of real and natural numbers are denoted  $\mathbb{R}$  and  $\mathbb{N}$ , respectively, and the notation  $A \succ 0$  means that  $A = A^\top$  is positive definite. Matrix blocks which can be inferred from symmetry are abbreviated with  $*$ . A (block) diagonal matrix with the blocks  $\lambda_1, \dots, \lambda_n$  is denoted  $\text{diag}(\lambda_1, \dots, \lambda_n)$ .

## II. SETUP

In this paper, we consider the linear discrete-time plant

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k^a + B^w w_k \\ z_k &= Cx_k + Du_k^a + D^w w_k \end{aligned} \quad (1)$$

with initial state  $x_0 \in \mathbb{R}^n$ , actuator input  $u_k^a \in \mathbb{R}^m$ , performance input  $w_k \in \mathbb{R}^q$ , performance output  $z_k \in \mathbb{R}^p$ , and real system matrices. At every time instant  $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ , the control input  $u_k^c \in \mathbb{R}^m$  is computed by means of a linear state-feedback controller as

$$u_k^c = Kx_k, \quad (2)$$

which may be lost, e.g., due to transmission failure or a deadline miss. These losses are described by the binary loss sequence  $\mu := (\mu_k)_{k \in \mathbb{N}_0}$ , where  $\mu_k = 1$  indicates a successful control attempt and  $\mu_k = 0$  a loss. Lost control inputs are discarded and not sent or computed again. Without loss of generality, we assume for simplicity that the first control attempt is always successful, i.e.,  $\mu_0 = 1$ . The actuator input  $u_k^a$  depends on the control input  $u_k^c$  and the loss sequence, but also on the chosen strategy in case of a loss. In literature, typically the zero strategy (input is set to zero if control attempt is not successful) and the hold strategy (last input is held until a new control input is received) are considered, and none of them is seen to be in general superior to the other [19]. In particular, the two strategies result in the actuator inputs

$$\begin{aligned} u_k^a &= \mu_k u_k^c && \text{for zero strategy} \\ u_k^a &= \mu_k u_k^c + (1 - \mu_k) u_{k-1}^a && \text{for hold strategy,} \end{aligned} \quad (3)$$

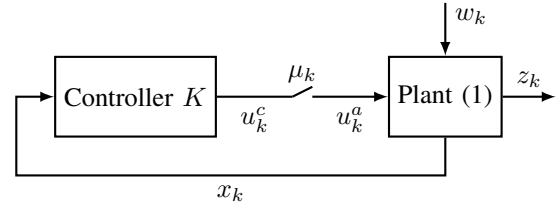


Fig. 1. The WHRT control system with losses in the controller-actuator connection.

with some initial condition  $u_{-1}^a$  for the hold strategy. An overview of the system setup is given in Fig. 1.

### A. WHRT constraints

We assume that the loss sequence  $\mu$  satisfies a WHRT constraint  $\lambda$ . These type of constraints can be interpreted as a window specification: Within a moving time window, a minimum number of control attempts has to be successful. More precisely, the standard type of WHRT constraints is defined as follows.

*Definition 1 ([5]):* A loss sequence  $\mu$  “meets any  $r$  in  $s$  control attempts” ( $r, s \in \mathbb{N}, r \leq s$ ), denoted  $\mu \vdash \binom{r}{s} = \lambda$ , if in any window of  $s$  consecutive control attempts there are at least  $r$  of them successful (in any order).

Other types of WHRT constraints focus on the number of successful consecutive control attempts, or on the number of unsuccessful control attempts.

*Definition 2 ([5]):* A loss sequence  $\mu$  “meets row  $r$  in  $s$  control attempts” ( $r, s \in \mathbb{N}, r \leq s$ ), denoted  $\mu \vdash \langle \overline{r} \rangle_s = \lambda$ , if in any window of  $s$  consecutive control attempts, at least  $r$  consecutive of them are successful.

*Definition 3 ([5]):* A loss sequence  $\mu$  “misses any  $r$  in  $s$  control attempts” ( $r, s \in \mathbb{N}, r \leq s$ ), denoted  $\mu \vdash \overline{\binom{r}{s}} = \lambda$ , if in any window of  $s$  consecutive control attempts there are not more than  $r$  of them unsuccessful (in any order).

*Definition 4 ([5]):* A loss sequence  $\mu$  “misses row  $r$  in  $s$  control attempts” ( $r, s \in \mathbb{N}, r \leq s$ ), denoted  $\mu \vdash \overline{\langle \overline{r} \rangle_s} = \lambda$ , if in any window of  $s$  consecutive control attempts, there are not more than  $r$  consecutive of them unsuccessful.

Moreover, some WHRT constraints can be related to each other as weaker or harder [5], [20].

*Definition 5 ([5]):* Given two WHRT constraints  $\lambda_1$  and  $\lambda_2$ , we say that  $\lambda_1$  is easier than  $\lambda_2$  ( $\lambda_2$  is harder than  $\lambda_1$ ), denoted  $\lambda_2 \preceq \lambda_1$  if  $\mu \vdash \lambda_2 \Rightarrow \mu \vdash \lambda_1$ .

The combination of (1), (2), (3), and the WHRT loss description forms the overall system model. More formally, we define:

*Definition 6 (WHRT control system):* The WHRT control system is defined as the system resulting from (1), (2), and (3), whose control input losses are described by the sequence  $\mu$  that satisfies a given WHRT constraint, i.e.,  $\mu \vdash \lambda$ .

### B. Switched system

To cope with the WHRT control system we will rewrite it as a constrained switched (linear) system (CSS), where the loss sequence is constrained by a WHRT constraint. This

type of systems has been used in the past to deal with similar setups, e.g., in [7], [8], [9], [10], and proves to be useful in ours as well.

The CSS takes the form

$$\begin{aligned} x_{k+1} &= \mathcal{A}_{\mu_k}^{\text{cl}} x_k + \mathcal{B}_{\mu_k}^w w_k \\ z_k &= \mathcal{C}_{\mu_k}^{\text{cl}} x_k + \mathcal{D}_{\mu_k}^w w_k, \end{aligned} \quad (4)$$

where the real system matrices depend on the loss sequence  $\mu$ . Thus, the CSS has two modes:  $\mu_k = 1$  for a successful control attempt and  $\mu_k = 0$  representing a loss at time instant  $k$ , e.g.,  $\mathcal{A}_{\mu_k}^{\text{cl}} \in \{\mathcal{A}_0^{\text{cl}}, \mathcal{A}_1^{\text{cl}}\}$ . Note that the system matrices can be chosen such that (4) is an interchangeable representation of the WHRT control system. For example,

$$\begin{aligned} \mathcal{A}_0^{\text{cl}} &= A & \mathcal{A}_1^{\text{cl}} &= A + BK & \mathcal{B}^w &= B^w \\ \mathcal{C}_0^{\text{cl}} &= C & \mathcal{C}_1^{\text{cl}} &= C + DK & \mathcal{D}^w &= D^w \end{aligned} \quad (5)$$

is obtained for the zero strategy, cf. [8], [9]. For the respective hold strategy matrices we refer to [8], [9], which require an augmented state vector with the additional auxiliary state  $u_{k-1}$  for storing the last applied input. We will later state the full system matrices for both strategies in the lifted system formulation.

### C. $\ell_2$ -performance

The  $\ell_2$ -gain is a classical performance measure for control systems, that can be interpreted as the worst-case energy amplification from a performance input (e.g., a disturbance)  $w_k$  to a performance output  $z_k$  [21]. More formally, we define  $\ell_2$ -performance and the  $\ell_2$ -gain as follows.

*Definition 7 ( $\ell_2$ -performance):* The CSS (4) has  $\ell_2$ -performance with gain  $\gamma$  if it is asymptotically stable and for  $x_0 = 0$

$$\sum_{k=0}^{\infty} z_k^\top z_k < \gamma^2 \sum_{k=0}^{\infty} w_k^\top w_k \quad \forall w \in \ell_2, w \neq 0, \quad \forall \mu \vdash \lambda, \quad (6)$$

where  $\ell_2$  denotes the set of all square-summable signals, i.e., all  $w$  for which  $\sum_{k=0}^{\infty} w_k^\top w_k < \infty$ . The smallest  $\gamma$  such that (6) is still satisfied is called the  $\ell_2$ -gain of the system.

In this work, we are interested in the  $\ell_2$ -performance of the WHRT control system. For that, we will first study the  $\ell_2$ -performance of the CSS (4) and then transfer the results to the WHRT control system.

## III. $\ell_2$ -PERFORMANCE ANALYSIS FOR CSSs

In this section, we derive a sufficient condition to verify that the CSS (4) has  $\ell_2$ -performance with a certain gain  $\gamma$ . There already exist  $\ell_2$ -performance results for switched systems in the literature [16], [22]. However, these results are limited to arbitrary switching between the modes of the switched system. Since in our setup the switching sequence is not arbitrary but constrained by the WHRT constraint, using arbitrary switching results induces conservatism. We leverage these results by explicitly taking the constrained switching into consideration.

### A. Graph representation

To formalize the notion of constrained switching, a particular concept has been proven useful in several previous works [7], [9], [10], [23]: representing the WHRT constraint  $\lambda$  as an automaton described by a graph, that generates the allowed switching sequences.

*Definition 8 (WHRT graph):* For a WHRT constraint  $\lambda$ , the corresponding WHRT graph  $\mathcal{G}$  satisfies the following properties:

- It is a labeled directed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  with the node set  $\mathcal{V} = \{v_1, \dots, v_{n_\mathcal{V}}\}$  and the edge set  $\mathcal{E} = \{e_1, \dots, e_{n_\mathcal{E}}\}$  with  $e_p = (i_p, j_p, l_p)$ , where  $e_p \in \mathcal{E}$  if there exists an edge with label  $l_p$  from node  $v_{i_p}$  to node  $v_{j_p}$ .
- The labels  $l_p$  take values in the set  $\{0, 1\}$ , representing a loss of the control input signal or a successful control attempt.
- All sequences  $\mu \vdash \lambda$  can be generated by the graph.
- Every node of  $\mathcal{G}$  has at least one incoming and one outgoing edge.

As a consequence, we can represent the loss sequence  $\mu$  by moving along the edges of the corresponding graph. The labels thereby characterize the current mode of the CSS, whereas the graph specifies how the modes can be concatenated such that  $\mu \vdash \lambda$ . Based on Definition 8, define further the indicator function  $\eta_i: \mathbb{N}_0 \rightarrow \{0, 1\}$ ,  $i = 1, \dots, n_\mathcal{V}$  with

$$\eta_i(k) = \begin{cases} 1, & \text{for } i = i_p \text{ with } \mu_k = l_p \\ 0, & \text{otherwise,} \end{cases} \quad (7)$$

i.e.,  $\eta$  indicates the initial node of the current edge, that is the current mode  $\mu_k$  at time instant  $k$ . Note that it is always possible to generate such a graph for any given WHRT constraint. We will discuss the corresponding details in Section IV. The size of the graph is discussed in [20].

### B. $\ell_2$ -performance analysis result

Given the previous graph definition, we can now state our first result. We use [16, Theorem 2] as a basis and modify it to account for the constrained switching.

*Theorem 1 ( $\ell_2$ -performance analysis):* The CSS (4), whose switching is captured by a graph  $\mathcal{G}$  according to Definition 8, has  $\ell_2$ -performance with gain  $\gamma$  if there exist symmetric matrices  $S_i \in \mathbb{R}^{n \times n}$  and matrices  $G_i \in \mathbb{R}^{n \times n}$ ,  $i = 1, \dots, n_\mathcal{V}$ , such that

$$\begin{bmatrix} G_i + G_i^\top - S_i & * & * & * \\ 0 & \gamma I & * & * \\ \mathcal{A}_i^{\text{cl}} G_i & \mathcal{B}_i^w & S_j & * \\ \mathcal{C}_i^{\text{cl}} G_i & \mathcal{D}_i^w & 0 & \gamma I \end{bmatrix} \succ 0 \quad \forall (i, j, l) \in \mathcal{E}. \quad (8)$$

A Lyapunov function is then given by

$$V(x_k) = x_k^\top \left( \sum_{i=1}^{n_\mathcal{V}} \eta_i(k) S_i^{-1} \right) x_k. \quad (9)$$

*Proof:* Since every node has at least one incoming and one outgoing edge, (8) guarantees  $S_i^{-1} \succ 0 \forall i$ . Note that (8)

implies

$$\begin{bmatrix} G_i + G_i^\top - S_i & * \\ \mathcal{A}_l^{\text{cl}} G_i & S_j \end{bmatrix} \succ 0 \quad \forall (i, j, l) \in \mathcal{E}, \quad (10)$$

which implies asymptotic stability by [9, Corollary 9]. The remaining proof is adapted from [16, Theorem 2], but modified to cope with the graph induced constrained switching. Applying the congruence transformation  $\text{diag}(0, \gamma^{-1}I, 0, \gamma^{-1}I)$  on the LMI (8), it is equivalent to

$$\begin{bmatrix} \bar{G}_i + \bar{G}_i^\top - \bar{S}_i & * \\ \bar{\mathcal{A}}_l^{\text{cl}} \bar{G}_i & \bar{S}_j \end{bmatrix} \succ 0 \quad \forall (i, j, l) \in \mathcal{E},$$

where

$$\begin{aligned} \bar{G}_i &:= \begin{bmatrix} G_i & 0 \\ 0 & \gamma^{-1}I \end{bmatrix} & \bar{S}_i &:= \begin{bmatrix} S_i & 0 \\ 0 & \gamma^{-1}I \end{bmatrix} \\ \bar{\mathcal{A}}_l^{\text{cl}} &:= \begin{bmatrix} \mathcal{A}_l^{\text{cl}} & \mathcal{B}_l^w \\ \gamma^{-1}\mathcal{C}_l^{\text{cl}} & \gamma^{-1}\mathcal{D}_l^w \end{bmatrix} & \bar{S}_j &:= \begin{bmatrix} S_j & 0 \\ 0 & \gamma^{-1}I \end{bmatrix}. \end{aligned}$$

Following the same steps as in the proof of [17, Theorem 2] (step iii)  $\Rightarrow$  ii) yields

$$\begin{bmatrix} \bar{S}_i^{-1} & * \\ \bar{S}_j^{-1} \bar{\mathcal{A}}_l^{\text{cl}} & \bar{S}_j^{-1} \end{bmatrix} \succ 0 \quad \forall (i, j, l) \in \mathcal{E}. \quad (11)$$

Using the Schur complement and resubstituting the  $(\bar{\cdot})$  variables, we obtain for all  $(i, j, l) \in \mathcal{E}$

$$[*]^\top \begin{bmatrix} S_j^{-1} & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} \mathcal{A}_l^{\text{cl}} & \mathcal{B}_l^w \\ \gamma^{-1}\mathcal{C}_l^{\text{cl}} & \gamma^{-1}\mathcal{D}_l^w \end{bmatrix} - \begin{bmatrix} S_i^{-1} & 0 \\ 0 & \gamma I \end{bmatrix} \prec 0.$$

Since the graph represents the constrained switching with the labels taking the values in  $\{0, 1\}$ , we can set  $l_p = \mu_k$ , and utilize (9), (7) for the initial node  $i_p$  of the edge at time instant  $k$  and for the receiving node  $j_p$  of the edge at time instant  $k + 1$ . This results in

$$[*]^\top \begin{bmatrix} P_{i,k+1} & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} \mathcal{A}_{\mu_k}^{\text{cl}} & \mathcal{B}_{\mu_k}^w \\ \gamma^{-1}\mathcal{C}_{\mu_k}^{\text{cl}} & \gamma^{-1}\mathcal{D}_{\mu_k}^w \end{bmatrix} - \begin{bmatrix} P_{i,k} & 0 \\ 0 & \gamma I \end{bmatrix} \prec 0.$$

We multiply this equation with  $[x_k^\top \ w_k^\top]$  from the left and its transpose from the right and use (9) and (4) to end up with  $V(x_{k+1}) - V(x_k) < \gamma w_k^\top w_k - \gamma^{-1} z_k^\top z_k \ \forall w_k \in \ell_2, \forall x_k$ , which is a sufficient condition for (4) having  $\ell_2$ -performance with gain  $\gamma$ , see, e.g., [16], [22]. ■

Note that Theorem 1 can be used to analyze the  $\ell_2$ -performance of an arbitrary CSS, whose switching sequence can be captured by a graph. Previous results only captured the  $\ell_2$ -performance of *unconstrained* switched systems. In our setup, the WHRT control system can be formulated as a CSS (4), see (5), and thus Theorem 1 can directly be used to analyze a given WHRT control system not only regarding its stability as in [9], but additionally regarding its  $\ell_2$ -performance. Moreover, we can minimize  $\gamma$  under the LMI constraints (8) to obtain an upper bound on the  $\ell_2$ -gain of the CSS. However, the underlying graph and thus the number of decision variables of the LMIs (8) may be relatively large, since we need to solve one LMI per edge, and have  $n^2$  decision variables in  $G_i$  per node plus  $\frac{1}{2}(n^2 + n)$  decision variables in  $S_i = S_i^\top$  per node. Therefore, the conditions of Theorem 1 might be computationally expensive

to solve. Moreover, controller synthesis becomes difficult, because it is not possible to isolate  $K$  in (8). The reason is that the matrices  $\mathcal{A}_{\mu_k}^{\text{cl}}$  and  $\mathcal{C}_{\mu_k}^{\text{cl}}$  might not be linear in  $K$ , dependent on the mode. For example in (5),  $\mathcal{A}_0^{\text{cl}}$  is independent of  $K$  since the computed control signal does not arrive at the actuator whenever a loss occurs and thus the system runs open-loop. In contrast,  $K$  appears linearly in  $\mathcal{A}_1^{\text{cl}}$ , i.e., when the control attempt is successful. In the next section, we show how we can address these issues by lifting the system in an appropriate way.

#### IV. LIFTED SYSTEM

In this section, we apply a lifting method to the CSS (4) to enable controller synthesis and a more efficient analysis.

##### A. Main idea and lifting

The fundamental concept is to “discretize” the system at time instants of successful control attempts, similarly as in [9]. This approach was called *alternative discretization* and constitutes a special case of the lifting presented in this section, which is adapted from [18]. To formalize, define  $\tau := (\tau_k)_{k \in \mathbb{N}_0}$  as in [9] as the sequence of time indices of successful control attempt instants  $k$ , and  $\alpha := (\alpha_k)_{k \in \mathbb{N}_0}$  as the sequence counting the number of losses between two successful control attempts. The sequence  $\alpha$  plays an important role in the lifting and its elements take values in the finite set  $\{0, \dots, s-1\}$ , since we assumed  $s$  to be bounded and hence there can only be a finite amount of consecutive losses. As an example, for  $\mu = (1011001\dots)$  the corresponding sequences for the lifting are  $\tau = (0236\dots)$  and  $\alpha = (102\dots)$ .

The goal is to find an alternative representation of (4), but based on the successful control attempt time instants. It is therefore defined at the time instants  $\tau_k$  only. All variables which refer to the lifted system are marked with  $(\tilde{\cdot})$ , e.g.,  $\tilde{x}$  represents the lifted state of  $x$ . In [9], this alternative representation is relatively straightforward, because autonomous systems with no performance in- and output are considered therein. In contrast, our setup features such an in- and output, which cannot be discarded at time instants between successful control attempts without changing the  $\ell_2$ -performance of the system. For this reason, we lift the input and output as follows

$$\begin{aligned} \tilde{w}_k &= \begin{bmatrix} w_{\tau_{k-1}}^\top & \cdots & w_{\tau_k}^\top \end{bmatrix}^\top \\ \tilde{z}_k &= \begin{bmatrix} z_{\tau_{k-1}}^\top & \cdots & z_{\tau_k}^\top \end{bmatrix}^\top. \end{aligned} \quad (12)$$

Note that at each time instant  $\tau_k$ , the lifted in- and output  $\tilde{w}_k$  and  $\tilde{z}_k$  consist of past in- and outputs. Since we are interested in an equivalent formulation of (4), the lifted and the non-lifted system states are required to coincide at these time instants, i.e.,  $x_k = \tilde{x}_{\tau_k}$ . The lifted CSS is then defined as

$$\begin{aligned} \tilde{x}_{k+1} &= \tilde{\mathcal{A}}_{\alpha_k}^{\text{cl}} \tilde{x}_k + \tilde{\mathcal{B}}_{\alpha_k}^w \tilde{w}_k \\ \tilde{z}_k &= \tilde{\mathcal{C}}_{\alpha_k}^{\text{cl}} \tilde{x}_k + \tilde{\mathcal{D}}_{\alpha_k}^w \tilde{w}_k \end{aligned} \quad (13)$$

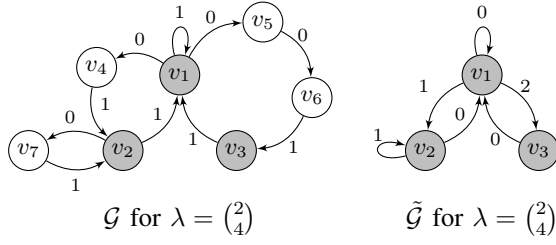


Fig. 2. Left: non-lifted WHRT graph with labels representing the loss sequence  $\mu$ ; right: its lifted counterpart with labels representing  $\alpha$ .

with the closed-loop matrices

$$\begin{aligned} \tilde{\mathcal{A}}_{\alpha_{\bar{k}}}^{\text{cl}} &:= \tilde{\mathcal{A}}_{\alpha_{\bar{k}}} + \tilde{\mathcal{B}}_{\alpha_{\bar{k}}} K \\ \tilde{\mathcal{C}}_{\alpha_{\bar{k}}}^{\text{cl}} &:= \tilde{\mathcal{C}}_{\alpha_{\bar{k}}} + \tilde{\mathcal{D}}_{\alpha_{\bar{k}}} K. \end{aligned} \quad (14)$$

Since the number of time instants between two successful control attempts is not constant, the lifted CSS has varying in- and output dimensions, depending on the current mode  $\alpha_{\bar{k}}$ . For WHRT control systems, we state the lifted system matrices explicitly in the next subsection.

### B. Application to WHRT control systems

The modes of the lifted system (13) are not dependent on the loss sequence  $\mu$  anymore, because it is defined at time instants of successful control attempts only. Instead, the switching is now based on the sequence  $\alpha$ , whose elements take values in  $\{0, \dots, s-1\}$ . Therefore, we need to redefine the WHRT graph, similar as in [10].

*Definition 9 (Lifted WHRT graph):* For a WHRT constraint  $\lambda$ , the lifted WHRT graph  $\tilde{\mathcal{G}}$  is defined as in Definition 8, but redefining

- (b) The labels  $l_p$  take values in the set  $\{0, \dots, s-1\}$ , representing the number of losses between successful control attempts.

The lifted WHRT graph has  $\tilde{n}_V$  nodes and  $\tilde{n}_E$  edges. Further, the indicator function (7) is redefined to  $\tilde{\eta}_i(\bar{k})$  by replacing  $\mu_k$  by  $\alpha_{\bar{k}}$  and  $n_V$  by  $\tilde{n}_V$ . Similar to the non-lifted graph, the  $\alpha$ -sequence can be generated by moving along the edges of  $\tilde{\mathcal{G}}$ . It is always possible to generate a lifted WHRT graph for any given WHRT constraint. An algorithm for its automatic generation has been presented in [10] and an implementation is referenced therein, while an algorithm for generating a non-lifted WHRT graph is given in [20]. Therein, the authors furthermore present a method for combining different WHRT constraints into a joint graph. Further, a (non-lifted) WHRT graph can be generated from the lifted version as follows. Any edge with  $l_p \geq 1$  represents  $l_p$  consecutive losses followed by one successful control attempt. Replacing these edges by a series of  $l_p$  edges with label 0 followed by one with label 1, including the necessary nodes in-between, leads to a (non-lifted) WHRT graph for a given WHRT constraint. An example WHRT graph and its corresponding lifted version can be found in Fig. 2, where the relation between both graphs can be observed.

We are now able to state the WHRT control system as a lifted CSS of the form (13) with (14). Performing the

calculations for the WHRT control system (equations (1) – (3)), “discretized” at time instants of successful control attempts, one obtains using the sequence  $\alpha$

$$\begin{aligned} \tilde{\mathcal{B}}_{\alpha_{\bar{k}}} &= A^{\alpha_{\bar{k}}} B \\ \tilde{\mathcal{D}}_{\alpha_{\bar{k}}} &= \begin{cases} \text{for } \alpha_{\bar{k}} = 0: D \\ \text{for } \alpha_{\bar{k}} \geq 1: \\ \left[ D^\top \quad (CA^0 B)^\top \quad \dots \quad (CA^{\alpha_{\bar{k}}-1} B)^\top \right]^\top \end{cases} \end{aligned} \quad (15)$$

for the zero strategy and

$$\begin{aligned} \tilde{\mathcal{B}}_{\alpha_{\bar{k}}} &= \sum_{i=0}^{\alpha_{\bar{k}}} A^i B \\ \tilde{\mathcal{D}}_{\alpha_{\bar{k}}} &= \begin{cases} \text{for } \alpha_{\bar{k}} = 0: D \\ \text{for } \alpha_{\bar{k}} \geq 1: \\ \left[ D^\top \quad (CB + D)^\top \quad \dots \quad (C \sum_{i=0}^{\alpha_{\bar{k}}-1} A^i B + D)^\top \right]^\top \end{cases} \end{aligned} \quad (16)$$

for the hold strategy, while

$$\begin{aligned} \tilde{\mathcal{A}}_{\alpha_{\bar{k}}} &= A^{\alpha_{\bar{k}}+1} \\ \tilde{\mathcal{C}}_{\alpha_{\bar{k}}} &= [(CA^0)^\top \quad (CA)^\top \quad \dots \quad (CA^{\alpha_{\bar{k}}})^\top]^\top \\ \tilde{\mathcal{B}}_{\alpha_{\bar{k}}}^w &= [A^{\alpha_{\bar{k}}} B^w \quad \dots \quad AB^w \quad A^0 B^w] \\ \tilde{\mathcal{D}}_{\alpha_{\bar{k}}}^w &= \begin{cases} \text{for } \alpha_{\bar{k}} = 0: D^w \\ \text{for } \alpha_{\bar{k}} \geq 1: \begin{bmatrix} D^w & 0 & \dots & 0 \\ CB^w & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{\alpha_{\bar{k}}-1} B^w & \dots & CB^w & D^w \end{bmatrix} \end{cases} \end{aligned} \quad (17)$$

are identical for both strategies. These matrices also appear for lifting general (non-switched) discrete-time systems, cf. [18, Chapter 8.2]. Note that compared to [10] and previous works, due to our lifting no auxiliary state  $u_{k-1}^a$  is required for the hold strategy, thus reducing the state-space dimension.

Note that we can handle one- or multi-step delays in the feedback channel [2] with our framework. For that, augment the system state  $x_k$  with the delayed inputs  $[u_{k-1}^a, u_{k-2}^a, \dots]$  and adapt the system matrices in (1) accordingly. The later synthesized controller will additionally to  $x_k$  depend on the delayed inputs, which have to be known to the controller, e.g., by acknowledgment mechanisms of the communication channel.

### C. $\ell_2$ -performance for the lifted system

Observe that by definition of  $\tilde{w}$  and  $\tilde{z}$

$$\sum_{k=0}^{\infty} z_k^\top z_k = \sum_{\bar{k}=0}^{\infty} \tilde{z}_{\bar{k}}^\top \tilde{z}_{\bar{k}} \quad \text{and} \quad \sum_{k=0}^{\infty} w_k^\top w_k = \sum_{\bar{k}=0}^{\infty} \tilde{w}_{\bar{k}}^\top \tilde{w}_{\bar{k}}.$$

Therefore, lifting the system loses no input/output information, the  $\ell_2$ -gain stays the same, also because asymptotic stability of the lifted system (13) carries over to the non-lifted one (4) [9]. Further,  $\ell_2$ -performance guarantees given for (13) carry over to (4). The following Lemma states the equivalence.

*Lemma 1:* The CSS (4) and its lifted counterpart (13) have the same  $\ell_2$ -performance gain  $\gamma$ .

We can now state a similar  $\ell_2$ -performance analysis result for the lifted system (13) as for the non-lifted system (4).

*Theorem 2 (Lifted system  $\ell_2$ -performance analysis):* The lifted CSS (13), whose switching is captured by a graph  $\tilde{\mathcal{G}}$  according to Definition 9, has  $\ell_2$ -performance with gain  $\gamma$  if there exist symmetric matrices  $S_i \in \mathbb{R}^{n \times n}$  and matrices  $G_i \in \mathbb{R}^{n \times n}$ ,  $i = 1, \dots, \tilde{n}_\nu$ , such that

$$\begin{bmatrix} G_i + G_i^\top - S_i & * & * & * \\ 0 & \gamma I & * & * \\ \tilde{A}_l^{\text{cl}} G_i & \tilde{B}_l^w & S_j & * \\ \tilde{C}_l^{\text{cl}} G_i & \tilde{D}_l^w & 0 & \gamma I \end{bmatrix} \succ 0 \quad \forall (i, j, l) \in \mathcal{E}. \quad (18)$$

A Lyapunov function is then given by

$$V(\tilde{x}_{\tilde{k}}) = \tilde{x}_{\tilde{k}}^\top \left( \sum_{i=1}^{\tilde{n}_\nu} \tilde{\eta}_i(\tilde{k}) S_i^{-1} \right) \tilde{x}_{\tilde{k}}. \quad (19)$$

*Proof:* The proof follows the same steps as in Theorem 1. The dimensions of  $\tilde{B}_{\mu_k}^w$ ,  $\tilde{C}_{\mu_k}^{\text{cl}}$ , and  $\tilde{D}_{\mu_k}^w$  in the proof of Theorem 1 have to be adapted accordingly. ■

*Corollary 1:* Under the assumptions of Theorem 2, the WHRT control system has  $\ell_2$ -performance with gain  $\gamma$ .

*Proof:* From Theorem 2 and Lemma 1 we obtain that (13) and (4) have  $\ell_2$ -performance with gain  $\gamma$ . The fact that the WHRT control system can be represented by (13) completes the proof. ■

Due to the lifting, the system matrices  $\tilde{B}_{\alpha_{\tilde{k}}}^w$ ,  $\tilde{C}_{\alpha_{\tilde{k}}}^{\text{cl}}$ , and  $\tilde{D}_{\alpha_{\tilde{k}}}^w$  vary in dimension depending on the current mode  $\alpha_{\tilde{k}}$ . Therefore, the size of the LMI (18) increases, but the number of decision variables does not, because the dimension of  $\tilde{A}_{\alpha_{\tilde{k}}}^{\text{cl}}$  is not changing. The state dimensions of  $\tilde{x}$  and  $x$  are equal as well. As mentioned in [9], the lifted graph is typically much smaller, cf. Fig. 2. Hence, the overall computational demand for the analysis of the lifted CSS with Theorem 2 is typically less than for the non-lifted CSS with Theorem 1, although the dimension of the LMIs increase. A comparison by means of an example is given in Section VI.

## V. CONTROLLER SYNTHESIS

Based on the lifted CSS, we are able to state sufficient LMI conditions which allow to synthesize a state-feedback controller such that the WHRT control system is not only asymptotically stable [9], but additionally has  $\ell_2$ -performance with gain  $\gamma$ . The reason is that in (14) the controller enters linearly, because at each time instant  $\tilde{k}$  the control attempt is successful and thus  $K$  influences the dynamics at these time instants only. For all inter-attempt time instants the controller does not appear, because the control signal is lost. Hence, the controller can be isolated in the LMI (18), which enables controller synthesis. The following result aims at synthesizing a non-switched state-feedback controller.

*Theorem 3 ( $\ell_2$ -performance controller synthesis):* There exists a state-feedback controller such that the lifted CSS (13), whose switching is captured by a graph  $\tilde{\mathcal{G}}$  according to Definition 9, has  $\ell_2$ -performance with gain  $\gamma$  if there exist

symmetric matrices  $S_i \in \mathbb{R}^{n \times n}$ ,  $i = 1, \dots, \tilde{n}_\nu$  and matrices  $G \in \mathbb{R}^{n \times n}$ ,  $R \in \mathbb{R}^{m \times n}$  such that

$$\begin{bmatrix} G + G^\top - S_i & * & * & * \\ 0 & \gamma I & * & * \\ \tilde{A}_l G + \tilde{B}_l R & \tilde{B}_l^w & S_j & * \\ \tilde{C}_l G + \tilde{D}_l R & \tilde{D}_l^w & 0 & \gamma I \end{bmatrix} \succ 0 \quad \forall (i, j, l) \in \mathcal{E}. \quad (20)$$

The corresponding controller is then given by  $K = RG^{-1}$  and a Lyapunov function is (19).

*Proof:* Along the lines of [16]:  $G$  is invertible by standard arguments and insert  $R = KG$  in (20) and together with (14) the conditions of Theorem 2 are obtained. ■

Theorem 3 can be used in combination with Corollary 1 to synthesize a state-feedback controller for the WHRT control system with guaranteed  $\ell_2$ -performance, while in previous works it was only possible to synthesize a stabilizing controller [9].

As already annotated, Theorem 3 synthesizes a non-switched controller, which is independent of the past loss sequence. This induces some conservatism, but is a crucial property for NCS in which the network features no acknowledgment mechanism. A switched controller requires the knowledge of the past packet losses, which is challenging to achieve without the presence of acknowledgments. However, our results can directly be extended to switched controllers to be applied to networks that feature such a mechanism by revising the proof by substituting  $G$  with  $G_i$  and  $R$  with  $R_i$  in (20) with the controller  $K_i = R_i G_i^{-1}$ . Switched controllers can improve the feasibility of the control design. In this case, the control input (2) and the controller in (14) have to be adapted accordingly to account for the switching. The resulting controller is dependent on the starting node of the corresponding edge in the lifted WHRT graph, i.e., implicitly depends on the past loss sequence and may switch whenever a control attempt is successful. In [10], a possibility for designing switched controller policies in the absence of acknowledgment mechanisms is presented, with the tradeoff of requiring computational capabilities at the actuator and having to compute and send larger control input packets. This technique can be applied here similarly.

Note finally that if a certain performance gain  $\gamma$  is guaranteed for a specific  $\lambda$ , it is also guaranteed for all WHRT control systems with a harder WHRT constraint.

## VI. NUMERICAL EXAMPLE

We consider the numerical example from [8], [9], but extended by an performance in- and output, namely

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k^a + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_k \\ z_k &= \begin{bmatrix} 1 & 1 \end{bmatrix} x_k + u_k^a + w_k. \end{aligned} \quad (21)$$

The setup is as described in Section II, i.e., the control input losses are described by a WHRT constraint and we use the state-feedback controller (2). We use YALMIP (Version R20210331) [24] and MOSEK [25] to minimize  $\gamma$  with the respective LMIs as constraints. Note that the obtained  $\gamma$  is in general only an upper bound on the  $\ell_2$ -gain of the

WHRT control system, since the presented conditions are only sufficient. In [8], the controller  $K = [-0.35 \quad -0.85]$  for the WHRT constraint  $\lambda = \binom{2}{3}$  and the zero strategy was proposed. For this controller, Theorems 1 and 2 yield  $\gamma = 3.52$ . Simulating this system with  $w_k = 1$  for  $k \geq T$  and  $w_k = 0$  otherwise with sufficiently large, finite  $T$  and the worst-case loss sequence  $\mu = (101101\dots)$ , the simulated  $\gamma$  obtained by computing (6) is  $\gamma_{\text{sim}} = 3.38$ . Note that it is in general difficult to find the worst-case input  $w$  and loss sequence  $\mu$  that lead to the highest  $\gamma$ , i.e., the  $\ell_2$ -gain. By synthesizing a non-switched controller with Theorem 3 we obtain the controller  $K = [-0.61 \quad -1.00]$ , which results in  $\gamma = 2.505$ . The respective simulated gain is  $\gamma_{\text{sim}} = 2.10$ . Therefore, we are able to significantly improve the  $\ell_2$ -performance of (21). If we synthesize a switched controller, the  $\ell_2$ -performance can be improved even a bit further to  $\gamma = 2.488$ . Note that the improvement of a switched controller over a non-switched one strongly depends on the chosen system and the WHRT constraint.

To compare the computational complexity, we consider the WHRT constraint  $\binom{4}{10}$ . The corresponding non-lifted WHRT graph has 462 edges and 336 nodes and its lifted counterpart has 210 edges and 84 nodes. This leads to 462 LMIs with 2352 decision variables, or 210 LMIs with 588 decision variables. This significant reduction can also be seen in the computation time. For above example, the analysis with Theorem 1 took around 1.1 s to compute on a standard computer, while for the lifted system it finished after roughly 0.3 s. Thus, our numerical findings confirm the computational advantages when considering the lifted system instead of the non-lifted one.

## VII. CONCLUSIONS

In this paper, we considered the  $\ell_2$ -performance of linear plants with unreliable feedback, i.e., the control signal can be lost, which is modeled by a WHRT constraint. We presented sufficient LMI conditions under which such systems have  $\ell_2$ -performance with a certain gain  $\gamma$ . Moreover, we proposed an approach for the synthesis of state-feedback controller that ensures that the WHRT control system has  $\ell_2$ -performance with gain  $\gamma$ . This was made possible by lifting the system, which in addition typically reduces the computational complexity of the respective LMI conditions for the  $\ell_2$ -performance analysis. The resulting controller can be a non-switched, loss sequence independent one for NCS without acknowledgment mechanisms, or designed as a switched controller, that switches depending on the past loss sequence. As a side result, we also obtained an  $\ell_2$ -performance analysis result for general switched systems with constrained switching that can be captured by a graph. To make the results more applicable to NCS and real-time control systems, open problems for the future are the extensions to nonlinear plants and output feedback.

## REFERENCES

- [1] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proc. of the IEEE*, vol. 95, no. 1, pp. 138–162, 2007.
- [2] M. Maggio, A. Hamann, E. Mayer-John, and D. Ziegenbein, "Control-system stability under consecutive deadline misses constraints," in *Proc. 32nd Euromicro Conf. Real-Time Systems (ECRTS)*, vol. 165, pp. 21:1–21:24, 2020.
- [3] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. S. Sastry, "Foundations of control and estimation over lossy networks," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 163–187, 2007.
- [4] J. Xiong and J. Lam, "Stabilization of linear systems over networks with bounded packet loss," *Automatica*, vol. 43, no. 1, pp. 80–87, 2007.
- [5] G. Bernat, A. Burns, and A. Liamosi, "Weakly hard real-time systems," *IEEE Trans. on Computers*, vol. 50, pp. 308–321, 2001.
- [6] M. Hamdaoui and P. Ramanathan, "A dynamic priority assignment technique for streams with (m, k)-firm deadlines," *IEEE Trans. on Computers*, vol. 44, no. 12, pp. 1443–1451, 1995.
- [7] E. P. van Horsen, A. R. B. Behrouzian, D. Goswami, D. Antunes, T. Basten, and W. P. M. H. Heemels, "Performance analysis and controller improvement for linear systems with (m,k)-firm data losses," in *Proc. IEEE European Control Conf. (ECC)*, pp. 2571–2577, 2016.
- [8] R. Blind and F. Allgöwer, "Towards networked control systems with guaranteed stability: Using weakly hard real-time constraints to model the loss process," in *Proc. 54th IEEE Conf. Decision and Control (CDC)*, pp. 7510–7515, 2015.
- [9] S. Linsenmayer and F. Allgöwer, "Stabilization of networked control systems with weakly hard real-time dropout description," in *Proc. 56th IEEE Conf. on Decision and Control (CDC)*, pp. 4765–4770, 2017.
- [10] S. Linsenmayer, M. Hertneck, and F. Allgöwer, "Linear weakly hard real-time control systems: Time- and event-triggered stabilization," *IEEE Trans. Automat. Control*, vol. 66, no. 4, pp. 1932–1939, 2021.
- [11] M. Hertneck, S. Linsenmayer, and F. Allgöwer, "Stability analysis for nonlinear weakly hard real-time control systems," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 2594–2599, 2020.
- [12] M. Hertneck, S. Linsenmayer, and F. Allgöwer, "Efficient stability analysis approaches for nonlinear weakly-hard real-time control systems," *Automatica*, vol. 133, p. 109868, 2021.
- [13] N. Vreman, P. Pazzaglia, V. Magron, J. Wang, and M. Maggio, "Stability of linear systems under extended weakly-hard constraints," *IEEE Control Systems Lett.*, vol. 6, pp. 2900–2905, 2022.
- [14] P. Pazzaglia, L. Pannocchi, A. Biondi, and M. Di Natale, "Beyond the weakly hard model: Measuring the performance cost of deadline misses," in *Proc. 30th Euromicro Conf. Real-Time Systems (ECRTS)*, pp. 10:1–10:22, 2018.
- [15] N. Vreman, A. Cervin, and M. Maggio, "Stability and performance analysis of control systems subject to bursts of deadline misses," in *Proc. 33rd Euromicro Conf. Real-Time Systems (ECRTS)*, pp. 15:1–15:23, 2021.
- [16] J. Daafouz and J. Bernussou, "Robust dynamic output feedback control for switched systems," in *Proc. 41st IEEE Conf. Decision and Control (CDC)*, vol. 4, pp. 4389–4394, 2002.
- [17] J. Daafouz, P. Riedinger, and C. Iung, "Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach," *IEEE Trans. Automat. Control*, vol. 47, no. 11, pp. 1883–1887, 2002.
- [18] T. Chen and B. Francis, *Optimal Sampled-Data Control Systems*. Springer London, 1995.
- [19] L. Schenato, "To zero or to hold control inputs with lossy links?," *IEEE Trans. on Automat. Control*, vol. 54, no. 5, pp. 1093–1099, 2009.
- [20] N. Vreman, R. Pates, and M. Maggio, "WeaklyHard.jl: Scalable analysis of weakly-hard constraints," in *28th IEEE Real-Time and Embedded Technology and Applications Symp.*, pp. 228–240, 2022.
- [21] H. Khalil, *Nonlinear systems*. Upper Saddle River, New Jersey: Prentice Hall, 3rd ed., 2002.
- [22] L. Fang, H. Lin, and P. Antsaklis, "Stabilization and performance analysis for a class of switched systems," in *Proc. 43th IEEE Conf. Decision and Control (CDC)*, vol. 3, pp. 3265–3270, 2004.
- [23] M. Philippe, R. Essick, G. E. Dullerud, and R. M. Jungers, "Stability of discrete-time switching systems with constrained switching sequences," *Automatica*, vol. 72, pp. 242–250, 2016.
- [24] J. Löfberg, "YALMIP: a toolbox for modeling and optimization in MATLAB," in *Proc. 2004 IEEE Intern. Conf. Robotics and Automation (ICRA)*, pp. 284–289, 2004.
- [25] MOSEK ApS, *The MOSEK optimization toolbox for MATLAB manual. Version 10.0.37*, 2023.