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Robust Finite-time Dissipative Output-feedback Control of Linear Time-varying Discrete-time Uncertain Systems

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***Abstract*— This paper presents the design of a dissipativity-based output feedback control for transient performance improvement in linear time-varying discrete-time systems. The uncertain system dynamics is described in a convex polytope, and is subjected to bounded external disturbances. The proposed reduced-order time-varying dynamic controller guarantees that the system trajectories are bounded below a specified threshold for a given finite time interval. Moreover, using the notion of *QSR*-dissipativity, we aim for the system to simultaneously satisfy dissipativity to external disturbances within the considered finite time interval. Sufficient difference linear matrix inequality (DLMI) conditions are derived to design output feedback finite-time dissipative controller gains, considering the augmented form of the closed-loop system-controller dynamics. The paper unifies the procedure of designing finite-time robust controllers, as performance indices such as finite-time passivity and finite $\mathscr{L}_2$-gain are special cases of finite-time dissipativity. Numerical simulations demonstrate the effectiveness of the proposed scheme in bounding the state trajectories within the prescribed limit, for a specified finite time interval.**

***Index Terms*— Finite-time boundedness, Dissipativity, Difference linear matrix inequalities, Output feedback.**

I. Introduction

Robust control techniques are widely used to enhance the transient and asymptotic performance of dynamical systems, which are adversely affected by parameter perturbations and exogenous disturbances. While most of the robust control techniques aim to ensure the stability of the system in the asymptotic sense [1], finite-time robust control techniques improve the behaviour of system trajectories for a specified time interval, including transients [2]. Such finite-time robustness analysis is crucial in applications like active suspensions [3], transient performance improvement in power systems [4], satellites and spacecrafts [5] etc.

To bound the system trajectories during transients in discrete-time linear systems, the concept of finite-time boundedness (FTB) is introduced in [6]. According to this, during undesirable events like faults or sudden exogenous disturbances, the system trajectories are confined within a predefined region for a specified finite interval of time. Following this, the works in [7] carried out finite-time bounded control of discrete-time non-linear systems, with sufficient conditions for state feedback controller design. Further, necessary and sufficient conditions for the design of discrete-time state feedback controllers for FTB are presented in [8]. Recently, FTB analysis and control synthesis have been carried out for different classes of discrete-time systems [9], [10]. However, most of these works are based on a nominal system model, assuming the availability of all states for measurement. In practice, the system model parameters may not be accurate due to unaccounted dynamics or due to model parameter changes [11]. Furthermore, measuring all the system states may not always be feasible.

To guarantee the acceptable robust transient performance of uncertain systems with available measurements, this paper presents the design of reduced-order output feedback finite-time controllers. Finite-time output feedback control of linear discrete-time nominal systems is carried out in [12], leveraging the concept of input-output finite-time stabilization. Using observer-based output feedback controllers, FTB analysis of uncertain discrete-time systems is presented in [13]. However, such formulation adds constraints of finite-time detectability [14]. To eliminate such requirements in output feedback-based finite-time boundedness problems and to simplify the control design and implementation procedure, we propose a reduced-order controller formulation, which can be extended to static output feedback control design.

Besides bounding the system trajectories, to improve the robust performance of the system in the finite interval, we consider the system to satisfy dissipative properties [15] to external disturbances. Incorporating robustness specifications like finite-time passivity and finite $\mathscr{L}_2$-gain, robust controllers for FTB are designed in [16] and [17], respectively. Using the concept of dissipativity, such concepts can be unified, and a generalized concept of *finite-time dissipativity* can be given with energy-based interpretation. Moreover, as dissipative systems do not generate energy of their own, the analysis of finite-time control problems with dissipativity constraints is important in safety-critical applications [18].

In this work, we generalize the procedure of designing output feedback-based robust controllers for improving finite-time behaviour in time-varying discrete-time dynamical systems. For a prescribed finite time interval, the designed robust reduced-order dynamic controllers ensure the confinement of system trajectories within a specified region, when subjected to process and measurement disturbances. Since the polytopic uncertainty description can better represent parameter perturbations and model changes due to faults than the norm-bounded uncertainty [19], we consider the time-varying parameters to lie in a convex polytope. The rate of supply of energy from adversaries is represented by a quadratic supply rate, which provides freedom to incorporate robustness specifications like passivity and finite $\mathscr{L}_2$-gain as

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special cases. The main contributions of the paper are:

1) Unlike state feedback-based methods, conditions for designing output feedback finite-time dissipative control are given for uncertain time-varying discrete-time systems, using difference matrix inequalities.
2) By reformulating obtained matrix inequalities, parameter-independent DLMI conditions are derived for designing output feedback time-varying controllers, ensuring finite-time dissipativity of closed-loop system.

We leverage the advantage of an augmented form of closed-loop system [20], using which, the reduced-order dynamic control design can be performed as static gains for the augmented system-controller dynamics.

Notations: The set $\mathscr{N}_T$ represents the finite discrete-time interval $\mathscr{N}_T := \{k_0, k_0+1, \ldots, k_0+T\}$ and $\mathscr{L}_2^{\mathscr{N}_T}$ denotes the discrete-time Euclidean space of all square-summable functions over $\mathscr{N}_T$. In block matrices, $\star$ signifies the blocks arising from symmetry. Given a matrix X, $\mathtt{He}(X) := X + X^T$. The notation $diag(X,Y)$ refers to the block diagonal matrix with diagonal entries X and Y. Identity and zero matrices of suitable dimensions are represented by I and 0, respectively.

II. System Dynamics, Preliminaries and Problem Formulation

Consider the polytopic uncertain description of a linear time-varying (LTV) discrete-time system, given by,

$$\begin{aligned} x_s(k+1) &= A_s(\xi)x_s(k) + B_s(\xi)u_s(k) + E_s(\xi)\omega(k) \\ y_s(k) &= C_s(\xi)x_s(k) + D_s(\xi)\omega(k) \\ z_s(k) &= F_s(\xi)x_s(k) + G_s(\xi)u_s(k) + H_s(\xi)\omega(k) \end{aligned} \tag{1}$$

where, $x_s(k) \in \mathbb{R}^n$ denote the system states, $u_s(k) \in \mathbb{R}^p$ is the control input, and $y_s(k) \in \mathbb{R}^m$ and $z_s(k) \in \mathbb{R}^q$ respectively represent measured and controlled outputs. The system is subjected to energy bounded external disturbances, $\omega(k) \in \mathscr{W} \subseteq \mathbb{R}^l$ over $k \in \mathscr{N}_T$, i.e., $\omega(k) \in \mathscr{L}_2^{\mathscr{N}_T}$. Moreover, uncertain matrix valued sequences $A_s(\xi) \in \mathbb{R}^{n\times n}$, $B_s(\xi) \in \mathbb{R}^{n\times p}$, $E_s(\xi) \in \mathbb{R}^{n\times l}$, $C_s(\xi) \in \mathbb{R}^{m\times n}$, $D_s(\xi) \in \mathbb{R}^{m\times l}$, $F_s(\xi) \in \mathbb{R}^{q\times n}$, $G_s(\xi) \in \mathbb{R}^{q\times p}$ and $H_s(\xi) \in \mathbb{R}^{q\times l}$ are defined over $k \in \mathscr{N}_T$ and assumed to belong to a convex polytope $\mathscr{S}$ with N-vertices,

$$\begin{aligned} \mathscr{S} = \Bigg\{ &[A_s(\xi)\; B_s(\xi)\; E_s(\xi)\; C_s(\xi)\; D_s(\xi)\; F_s(\xi)\; G_s(\xi)\; H_s(\xi)] \\ = &\sum_{i=1}^{N} \xi_i \,[A_{s_i}(k)\, B_{s_i}(k)\, E_{s_i}(k)\, C_{s_i}(k)\, D_{s_i}(k)\, F_{s_i}(k)\, G_{s_i}(k)\, H_{s_i}(k)] \Bigg\} \end{aligned}$$

with $\xi_i > 0$, $i = 1,2,\ldots,N$ and $\sum_{i=1}^{N} \xi_i = 1$.

Given the system dynamics (1), considering that all the states are not available for measurement, the control input $u_s(k)$ is taken as the output of a reduced-order discrete-time dynamic controller of the form,

$$\begin{aligned} x_c(k+1) &= A_c(k)x_c(k) + B_c(k)y_s(k) \\ u_s(k) &= C_c(k)x_c(k) + D_c(k)y_s(k) \end{aligned} \tag{2}$$

with order $r \leq n-m$. Here, $x_c(k) \in \mathbb{R}^r$ are the controller states and $A_c(k) \in \mathbb{R}^{r\times r}$, $B_c(k) \in \mathbb{R}^{r\times m}$, $C_c(k) \in \mathbb{R}^{p\times r}$ and $D_c(k) \in \mathbb{R}^{p\times m}$ are time-varying discrete-time controller gains.

A. Finite-time Boundedness and Finite-time Dissipativity

Finite-time boundedness deals with the confinement of system trajectories to a predefined set for a specified finite time interval, in the presence of external disturbances. By characterizing the predefined set in terms of ellipsoids, the FTB of discrete-time systems is defined as follows.

Definition 1. *[21] Given a finite time interval $\mathscr{N}_T$, disturbances $\omega(k) \in \mathscr{W}$ and a positive definite matrix valued sequence $\Omega_s(k)$, the discrete-time LTV uncertain system* (1) *with initial conditions satisfying $x_{s_{k_0}}^T \Theta_s x_{s_{k_0}} \leq 1$, with $\Theta_s > \Omega_s(k_0)$ is said to be finite-time bounded with respect to specified parameters $(\Theta_s, \Omega_s(k), \mathscr{S}, \mathscr{W}, \mathscr{N}_T)$, if $\forall\ \omega(k) \in \mathscr{W}$,*

$$x_{s_{k_0}}^T \Theta_s x_{s_{k_0}} \leq 1 \implies x_s^T(k)\Omega_s(k)x_s(k) < 1,\ \forall k \in \mathscr{N}_T. \tag{3}$$

Remark 1. *FTB is a quantitative property dependent on various parameters, which are determined based on specific requirements. These parameters can be chosen to prevent saturation of system states or activation of nonlinear dynamics.*

Besides bounding system trajectories, we aim to enhance system robustness during transients by ensuring dissipativity to external disturbances. This provides an energy-based input-output relationship, such that the total energy of the system does not increase. When the external energy supply is characterized with a supply rate $s(z_s(k), \omega(k))$, the *finite-time dissipativity* (FTD) can be characterized as,

Definition 2. *The discrete-time dynamics* (1) *subjected to $\omega(k) \in \mathscr{W}$ exhibits FTD with respect to supply rate $s(z_s(k), \omega(k))$, over the time interval $\mathscr{N}_T$, if it is finite-time bounded with respect to $(\Theta_s, \Omega_s(k), \mathscr{S}, \mathscr{W}, \mathscr{N}_T)$ and for all trajectories of $x_s(k)$, the controlled output $z_s(k)$ satisfies the inequality,*

$$V(x_s(k_0+T), k_0+T) \leq V(x_s(k_0), k_0) + \sum_{k=k_0}^{k_0+T-1} s(z_s(k), \omega(k)) \tag{4}$$

with a storage function $V(x_s(k), k) \geq 0$.

Further, when the supply rate $s(z_s(k), \omega(k))$ takes the form of a quadratic function,

$$s(z_s(k), \omega(k)) = z_s^T(k)Q_s z_s(k) + 2z_s^T(k)S_s\omega(k) + \omega^T(k)R_s\omega(k) \tag{5}$$

with real matrices $Q_s = Q_s^T \in \mathbb{R}^{q\times q}$, $R_s = R_s^T \in \mathbb{R}^{l\times l}$ and $S_s \in \mathbb{R}^{q\times l}$, the condition in (4) is referred to as *QSR*-dissipativity. This concept of *QSR*-dissipativity offers more generalized conditions for robustness analysis. It encompasses properties like finite-time passivity and finite $\mathscr{L}_2$-gain as special cases. Accordingly, the weighting matrices in the supply rate $s(z_s(k), \omega(k))$ in (5) of Definition 2 can be chosen as,

1) For finite-time passivity: $Q_s = 0$, $S_s = \frac{1}{2}I$ and $R_s = 0$;
2) For finite-time $\mathscr{L}_2$-gain: $Q_s = -I$, $S_s = 0$ and $R_s = \gamma^2 I$, with a disturbance attenuation level $\gamma > 0$.

Without loss of generality, we assume $Q_s \leq 0$, such that discrete-time passivity and finite $\mathscr{L}_2$-gain with parameter γ can be encompassed as special cases of *QSR*-dissipativity.

B. Problem Definition

Our aim is to design a reduced-order time-varying controller of the form (2), such that the closed-loop system is finite-time dissipative.

By associating the controller (2) with system (1), we get the augmented system-controller dynamics as,

$$\begin{aligned} x(k+1) &= A(\xi)x(k) + B(\xi)u(k) + E(\xi)\omega(k) \\ y(k) &= C(\xi)x(k) + D(\xi)\omega(k) \\ z(k) &= F(\xi)x(k) + G(\xi)u(k) + H(\xi)\omega(k) \end{aligned} \tag{6}$$

where, $x(k) := \begin{bmatrix} x_s(k) & x_c(k) \end{bmatrix}^T \in \mathbb{R}^{n+r}$. Further, the augmented system matrices in (6) can be specified as,

$$A(\xi) = \begin{bmatrix} A_s(\xi) & 0 \\ 0 & 0 \end{bmatrix}, B(\xi) = \begin{bmatrix} 0 & B_s(\xi) \\ I & 0 \end{bmatrix}, E(\xi) = \begin{bmatrix} E_s(\xi) \\ 0 \end{bmatrix},$$
$$C(\xi) = \begin{bmatrix} 0 & I \\ C_s(\xi) & 0 \end{bmatrix}, D(\xi) = \begin{bmatrix} 0 \\ D_s(\xi) \end{bmatrix}, F(\xi) = \begin{bmatrix} F_s(\xi) & 0 \\ 0 & 0 \end{bmatrix},$$
$$G(\xi) = \begin{bmatrix} 0 & G_s(\xi) \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad H(\xi) = \begin{bmatrix} H_s(\xi) \\ 0 \end{bmatrix}.$$

The output feedback control function $u(k)$ to the augmented dynamics is given by,

$$u(k) = K(k)y(k) = K(k)\left(C(\xi)x(k) + D(\xi)\omega(k)\right) \tag{7}$$

where, $K(k) = \begin{bmatrix} A_c(k) & B_c(k) \\ C_c(k) & D_c(k) \end{bmatrix}$ is the time-varying discrete-time controller gain matrix.

For specifying the FTD of the closed-loop system according to Definition 2, the parameters for FTB can be chosen as $\Theta := diag(\Theta_s, \Theta_c) > 0$ and $\Omega(k) := diag(\Omega_s(k), \Omega_c(k)) > 0$, consisting of weighting matrices for system and controller states. Further, the *QSR*-dissipativity conditions in (4) with supply rate (5) are defined in terms of matrices $Q \in \mathbb{R}^{(q+r)\times(q+r)}$, $S \in \mathbb{R}^{(q+r)\times l}$ and $R \in \mathbb{R}^{l\times l}$.

Thus, the problem of finite-time dissipativity analysis and synthesis for the discrete-time dynamics in (1) with reduced-order controller (2) can be formally stated as follows.

Problem 1. *Consider the discrete-time uncertain LTV system* (1) *defined in convex polytopic set $\mathscr{S}$, with the initial conditions of the augmented dynamics satisfying $x_{k_0}^T \Theta x_{k_0} \leq 1$.*

1) *Given output feedback control of the form* (2)*, obtain sufficient difference matrix inequality conditions such that the closed loop system* (6) *is finite-time dissipative with respect to $(\Theta_s, \Omega_s(k), \mathscr{S}, \mathscr{W}, \mathscr{N}_T)$ and given matrices Q, S, R, i.e., $\forall\ k \in \mathscr{N}_T$ and $\omega(k) \in \mathscr{W}$, the system trajectories satisfy,*

$$\begin{aligned} & x^T(k)\Omega(k)x(k) < 1 \quad \text{and} \\ & V(x(k_0+T), k_0+T) \leq V(x(k_0), k_0) \\ & \qquad + \sum_{k=k_0}^{k_0+T-1} \begin{bmatrix} z(k) \\ \omega(k) \end{bmatrix}^T \begin{bmatrix} Q & \star \\ S^T & R \end{bmatrix} \begin{bmatrix} z(k) \\ \omega(k) \end{bmatrix} \end{aligned} \tag{8}$$

2) *Using the obtained sufficient conditions, derive parameter-independent difference linear matrix inequality conditions for the design of reduced-order time-varying controller gains $K(k)$, such that the closed-loop system* (6) *is finite-time dissipative.*

III. Finite-time Dissipative Output Feedback Controller Design for Discrete-time Systems

In this section, initially, we derive the sufficient conditions for the discrete-time system to be finite-time dissipative, and further discuss the design of a reduced-order controller.

A. Finite-time Dissipativity with Output Feedback

Using the conditions for FTD in Definition 2, the following lemma describes sufficient conditions for achieving FTD of uncertain time-varying systems using output feedback.

Lemma 1. *Consider the polytopic uncertain system subjected to external disturbances $\omega(k) \in \mathscr{W}$, described by the dynamics in* (1)*. Given matrices Q, S, R, Θ and a matrix sequence $\Omega(k)$, the closed-loop system* (6) *with control function* (7) *is finite-time dissipative, if there exists a symmetric positive definite matrix sequence $P(\xi(k))$, $k \in \mathscr{N}_T$, and time-varying control gains $K(k)$ that satisfy the difference matrix inequality conditions,*

$$P(\xi(k_0)) < \Theta, \ P(\xi(k)) > \Omega(k) \text{ and}$$
$$\begin{bmatrix} -P(\xi(k)) & \star & \star & \star \\ -S^T\Gamma(\xi) & -\mathtt{He}(\Upsilon^T(\xi)S) - R & \star & \star \\ \Phi(\xi) & \Psi(\xi) & -P^{-1}(\xi(k+1)) & \star \\ \hat{Q}^{\frac{1}{2}}\Gamma(\xi) & \hat{Q}^{\frac{1}{2}}\Upsilon(\xi) & 0 & -I \end{bmatrix} \leq 0 \tag{9}$$

where, $\Phi(\xi) := A(\xi) + B(\xi)K(k)C(\xi)$, $\Psi(\xi) := B(\xi)K(k)D(\xi) + E(\xi)$, $\Gamma(\xi) := F(\xi) + G(\xi)K(k)C(\xi)$, $\Upsilon(\xi) := G(\xi)K(k)D(\xi) + H(\xi)$ and $-Q = (\hat{Q}^{\frac{1}{2}})^2$.

Proof. From dissipativity constraints in (4), we have,

$$V(x(k+1), k+1) \leq V(x(k), k) + s(z(k), \omega(k)). \tag{10}$$

For the augmented LTV discrete-time system defined in polytope $\mathscr{S}$, considering the *QSR*-supply rate as $s(z(k), \omega(k)) = z^T(k)Qz(k) + 2z^T(k)S\omega(k) + \omega^T(k)R\omega(k)$, the non-negative storage function can be specified as $V(x(k), k) = x^T(k)P(\xi(k))x(k)$ with $P(\xi(k)) > 0$ [22]. Now, the condition in (10) can be simplified and written as,

$$\begin{aligned} & x^T(k)\left(\Phi^T(\xi)P(\xi(k+1))\Phi(\xi)\right)x(k) + \omega^T(k)\Psi^T(\xi) \\ & P(\xi(k+1))\Phi(\xi)x(k) + x^T(k)\Phi(\xi)P(\xi(k+1))\Psi(\xi)\omega(k) \\ & + \omega^T(k)\Psi^T(\xi)P(\xi(k+1))\Psi(\xi)\omega(k) - x^T(k)P(\xi(k))x(k) \\ & - z^T(k)Qz(k) - 2z^T(k)S\omega(k) - \omega^T(k)R\omega(k) \leq 0. \end{aligned} \tag{11}$$

where, $\Phi(\xi) := A(\xi) + B(\xi)K(k)C(\xi)$ and $\Psi(\xi) := B(\xi)K(k)D(\xi) + E(\xi)$. Upon substituting $z(k)$ in (6) and $u(k)$ in (7), the difference matrix inequality conditions to satisfy (11) can be written as,

$$\begin{bmatrix} \Phi^T(\xi)P(\xi(k+1))\Phi(\xi) - P(\xi(k)) - \Gamma^T(\xi)Q\Gamma(\xi) & \star \\ \Psi^T(\xi)P(\xi(k+1))\Phi(\xi) - \left(\Upsilon^T(\xi)Q + S^T\right)\Gamma(\xi) & \Psi^T(\xi)P(\xi(k+1))\Psi(\xi) - \Upsilon^T(\xi)Q\Upsilon(\xi) - \mathtt{He}(\Upsilon^T(\xi)S) - R \end{bmatrix} \leq 0 \tag{12}$$

with $\Gamma(\xi) := F(\xi) + G(\xi)K(k)C(\xi)$ and $\Upsilon(\xi) := G(\xi)K(k)D(\xi) + H(\xi)$. Taking Schur complements, the matrix inequality (12) can be simplified and written as (9).

For FTB, the state trajectories of the system (6) starting from $x_{k_0}^T \Theta x_{k_0} \leq 1$ are to be confined within the set $x^T(k)\Omega(k)x(k) < 1$, $\forall\ k \in \mathcal{N}_T$. Let the initial stored energy in the system be $V(x_{k_0},k_0) = x_{k_0}^T P(\xi(k_0))x_{k_0}$. Provided that the inequalities in (9) are satisfied, we have,

$$\begin{aligned} x^T(k)\Omega(k)x(k) &< x^T(k)P(\xi(k))x(k) \text{ (since } P(\xi(k)) > \Omega(k)) \\ &< x_{k_0}^T P(\xi(k_0))x_{k_0} \text{ (from (4))} \\ &< x_{k_0}^T \Theta x_{k_0} \leq 1 \text{ (since } P(\xi(k_0)) < \Theta) \end{aligned}$$

indicating that the closed-loop system (6) is finite-time bounded with respect to $(\Theta_s, \Omega_s(k), \mathscr{S}, \mathscr{W}, \mathcal{N}_T)$. $\square$

B. Finite-time Dissipative Control Synthesis

Based on the sufficient conditions for FTD presented in Lemma 1, in the following, we obtain a set of parameter-independent DLMI conditions for controller design.

We utilize the following lemma to convert the matrix inequalities obtained in Lemma 1 into a linear form appropriate for controller design.

Lemma 2. *[19] Let $J < 0$ and $J + LM + M^T L^T < 0$ be given matrix inequalities with J being a symmetric matrix and matrices M and L be of appropriate dimensions. Then, for a scalar β and matrix variable U, the inequality,*

$$\begin{bmatrix} J & \star \\ \beta L^T + UM & -\beta \mathtt{He}(U) \end{bmatrix} < 0$$

is equivalent to the given two matrix inequalities.

By using Lemma 1 and Lemma 2, we state the following theorem, which presents DLMI conditions for designing linear time-varying dynamic output feedback finite-time dissipative controllers.

Theorem 1. *Consider the linear time-varying discrete-time dynamical system* (1) *subjected to disturbances $\omega(k) \in \mathscr{W}$, and defined in polytope $\mathscr{S}$. For given matrices Q, S, R, Θ and a matrix sequence $\Omega(k)$, if there exist a positive definite matrix sequence $X_i(k)$, $i = 1,2,\ldots,N$, a scalar β, and time-varying matrices $V(k)$ and $U(k)$, $k \in \mathcal{N}_T$, that satisfy DLMIs,*

$$\begin{gathered} X_i(k_0) > \Theta^{-1},\ i = 1,2,\ldots,N \\ X_i(k) < \Omega^{-1}(k),\ i = 1,2,\ldots,N \\ \Lambda_{ii} \leq 0,\ i = 1,2,\ldots,N \text{ and} \\ \Lambda_{ij} + \Lambda_{ji} \leq 0,\ i < j,\ i,j,= 1,2,\ldots,N \end{gathered} \tag{13}$$

with Λ_{ij} is specified by (14)*, then the closed-loop system with control gains $K(k) = V(k)U^{-1}(k)$ is finite-time dissipative.*

Proof. By performing a transformation $P(\xi) = X^{-1}(\xi)$ and pre- and post-multiplying the dissipative inequality in Lemma 1 by $diag(X(\xi),I,I,I)$, we obtain,

$$X(\xi(k_0)) > \Theta^{-1},\ X(\xi(k)) < \Omega^{-1}(k) \text{ and}$$

$$\begin{bmatrix} -X(\xi(k)) & \star & \star & \star \\ -S^T\Gamma(\xi)X(\xi(k)) & -\mathtt{He}(\Upsilon^T(\xi)S) - R & \star & \star \\ \Phi(\xi)X(\xi(k)) & \Psi(\xi) & -X(\xi(k+1)) & \star \\ \hat{Q}^{\frac{1}{2}}\Gamma(\xi)X(\xi(k)) & \hat{Q}^{\frac{1}{2}}\Upsilon(\xi) & 0 & -I \end{bmatrix} \leq 0 \tag{15}$$

By choosing the control gain $K(k)$ as $K(k) = V(k)U^{-1}(k)$, we can decompose the inequality (15) as,

$$\begin{aligned} &\begin{bmatrix} -X(\xi(k)) & \star & \star & \star \\ -S^T F(\xi)X(\xi(k)) & -\mathtt{He}(H^T(\xi)S) - R & \star & \star \\ A(\xi)X(\xi(k)) & E(\xi) & -X(\xi(k+1)) & \star \\ \hat{Q}^{\frac{1}{2}}F(\xi)X(\xi(k)) & \hat{Q}^{\frac{1}{2}}H(\xi) & 0 & -I \end{bmatrix} \\ &+ \mathtt{He}\left(\begin{bmatrix} 0 \\ -S^T G(\xi)V \\ B(\xi)V \\ \hat{Q}^{\frac{1}{2}}G(\xi)V \end{bmatrix} U^{-1} \begin{bmatrix} C(\xi)X(\xi(k)) & D(\xi) & 0 & 0 \end{bmatrix} \right) \leq 0. \end{aligned} \tag{16}$$

The inequality (16) is equivalent to,

$$\begin{aligned} &\begin{bmatrix} -X(\xi(k)) & \star & \star & \star \\ -S^T F(\xi)X(\xi(k)) & -\mathtt{He}(H^T(\xi)S) - R & \star & \star \\ A(\xi)X(\xi(k)) & E(\xi) & -X(\xi(k+1)) & \star \\ \hat{Q}^{\frac{1}{2}}F(\xi)X(\xi(k)) & \hat{Q}^{\frac{1}{2}}H(\xi) & 0 & -I \end{bmatrix} \\ &+ \mathtt{He}\left(\begin{bmatrix} 0 \\ -S^T G(\xi)V \\ B(\xi)V \\ \hat{Q}^{\frac{1}{2}}G(\xi)V \end{bmatrix} U^{-1} \begin{bmatrix} C(\xi)X(\xi(k)) - U(k)C(\xi) \\ D(\xi) - U(k)D(\xi) \\ -U(k)C(\xi) \\ -U(k)G(\xi) \end{bmatrix}^T \right) \\ &+ \mathtt{He}\left(\begin{bmatrix} 0 \\ -S^T G(\xi)V \\ B(\xi)V \\ \hat{Q}^{\frac{1}{2}}G(\xi)V \end{bmatrix} U^{-1} \begin{bmatrix} C(\xi)X(\xi(k)) \\ U(k)D(\xi) \\ UC(\xi) \\ UG(\xi) \end{bmatrix}^T \right) \leq 0. \end{aligned} \tag{17}$$

Further, define the matrices in (17) as,

$$J := \begin{bmatrix} -X(\xi(k)) & \star & \star & \star \\ -S^T F(\xi)X(\xi(k)) & -\mathtt{He}(H^T(\xi)S) - R & \star & \star \\ A(\xi)X(\xi(k)) & E(\xi) & -X(\xi(k+1)) & \star \\ \hat{Q}^{\frac{1}{2}}F(\xi)X(\xi(k)) & \hat{Q}^{\frac{1}{2}}H(\xi) & 0 & -I \end{bmatrix},$$

$L := \begin{bmatrix} 0 \\ -S^T G(\xi)V \\ B(\xi)V \\ \hat{Q}^{\frac{1}{2}}G(\xi)V \end{bmatrix}$ and $M := U^{-1} \begin{bmatrix} C(\xi)X(\xi(k)) - UC(\xi) \\ D(\xi) - UD(\xi) \\ -UC(\xi) \\ -UG(\xi) \end{bmatrix}^T$. By utilizing Lemma 2, the conditions presented in (17) can be transformed into a linear matrix inequality, as in (18).

It is worth noting that the matrix variables appearing in (18) are expressed in terms of the parameter ξ. In order to convert the inequalities in (18) into parameter-independent form, we exploit the fact that the system parameters lie in an N-vertex convex polytope $\mathscr{S}$, and select the storage function $P(\xi(k))$ as a linear function in ξ [19], i.e., $P(\xi(k)) = \sum_{i=1}^N \xi_i P_i(k)$. This allows us to reformulate the matrix inequalities in (18) as a convex combination, i.e.,

$$\begin{gathered} \sum_{i=1}^N \xi_i X_i(k_0) > \Theta^{-1}, \quad \sum_{i=1}^N \xi_i X_i(k) < \Omega^{-1}(k) \quad \text{and} \\ \sum_{i=1}^N \sum_{j=1}^N \xi_i \xi_j \Lambda_{ij} \leq 0 \end{gathered} \tag{19}$$

where, Λ_{ij} is as specified in (14). Using the properties of double convex combination [23], the expression $\sum_{i=1}^N \sum_{j=1}^N \xi_i \xi_j \Lambda_{ij}$ in (19) can be expressed as $\sum_{i=1}^N \xi_i^2 \Lambda_{ii} + \sum_{i=1}^N \sum_{i<j}^N \xi_i \xi_j (\Lambda_{ij} + \Lambda_{ji})$. By substituting this into (19), the resulting inequality can be represented as a set of DLMIs given in (13), which provide sufficient conditions for the design of controller gains $K(k) = V(k)U^{-1}(k)$. $\square$

$$\Lambda_{ij} := \begin{bmatrix} -X_j(k) & \star & \star & \star & \star \\ -S^T\left(F_iX_j(k)-G_iVC_j\right) & -\mathtt{He}(H_i^TS)-\mathtt{He}(S^TG_iVD_j)-R & \star & \star & \star \\ A_iX_j(k)+B_iVC_j & E_i-C_i^TV^TG_j^TS+B_iVD_j & -X_j(k+1)-\mathtt{He}(B_iVC_j) & \star & \star \\ \hat{Q}^{\frac{1}{2}}\left(F_iX_j(k)+G_iVC_j\right) & \hat{Q}^{\frac{1}{2}}\left(H_i+G_iVD_j\right)-G_i^TV^TG_i^TS & \hat{Q}^{\frac{1}{2}}G_jVC_i+G_j^TV^TB_i^T & -I+\mathtt{He}(\hat{Q}^{\frac{1}{2}}G_iVG_i) & \star \\ CX_j(k)-UC_i & -\beta V^TG_i^TS+D_i-UD_i & \beta V^TB_i^T-UC_i & \beta V^TG_i^T\hat{Q}^{\frac{1}{2}}-UG_i & -\beta\mathtt{He}(U) \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} -X(\xi(k)) & \star & \star & \star & \star \\ -S^T\left(F(\xi)X(\xi(k))-G(\xi)VC(\xi)\right) & \phi_{22} & \star & \star & \star \\ A(\xi)X(\xi(k))+B(\xi)VC(\xi) & \phi_{23} & -X(\xi(k+1))-\mathtt{He}(B(\xi)VC(\xi)) & \star & \star \\ \hat{Q}^{\frac{1}{2}}\left(F(\xi)X(\xi(k))+G(\xi)VC(\xi)\right) & \phi_{24} & \hat{Q}^{\frac{1}{2}}G(\xi)VC(\xi)+G^T(\xi)V^TB^T(\xi) & -I+\mathtt{He}(\hat{Q}^{\frac{1}{2}}G(\xi)VG(\xi)) & \star \\ C(\xi)X(\xi(k))-UC(\xi) & \phi_{25} & \beta V^TB^T(\xi)-UC(\xi) & \beta V^TG^T(\xi)\hat{Q}^{\frac{1}{2}}-UG(\xi) & -\beta\mathtt{He}(U) \end{bmatrix} < 0 \tag{18}$$

where, $\phi_{22} := -\mathtt{He}(H^T(\xi)S) - \mathtt{He}(S^TG(\xi)VD(\xi)) - R$, $\phi_{23} := E(\xi) - C^T(\xi)V^TG^T(\xi)S + B(\xi)VD(\xi)$, $\phi_{24} := \hat{Q}^{\frac{1}{2}}\left(H(\xi)+G(\xi)VD(\xi)\right) - G^T(\xi)V^TG^T(\xi)S$ and $\phi_{25} := -\beta V^TG^T(\xi)S + D(\xi) - UD(\xi)$.

Remark 2. *When the dynamical controller order r in (2) is set to zero, it becomes a static output feedback control described by the function,*

$$u_s(k) = K_s(k)y_s(k) \tag{20}$$

where, $y_s(k) = C_s(\xi)x_s(k) + D_s(\xi)\omega(k)$ and $K_s(k) = D_c(k) \in \mathbb{R}^{p\times m}$. Therefore, the finite-time static output feedback control design becomes a special case of the proposed method and follows a similar procedure as outlined in Theorem 1.

IV. Numerical Simulations

This section presents simulation results that demonstrate the effectiveness of the proposed controller design.

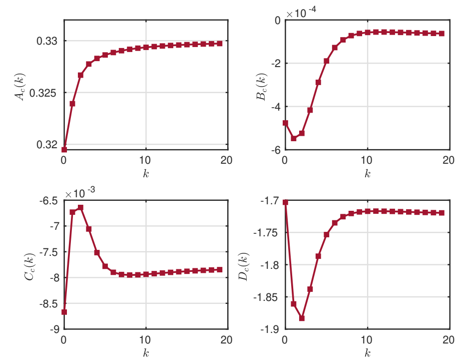
Example 1. *Consider the following discrete-time LTV uncertain system defined by a 2-vertex convex polytope, with the parameters,*

$$A_{s1} = \begin{bmatrix} 0.99+0.05k & 0.1 \\ -0.1\sin 0.5k & 0.97 \end{bmatrix},\ A_{s2} = \begin{bmatrix} 0.99 & 0.1 \\ -0.07+0.07k & 0.96 \end{bmatrix},$$
$$B_{s1} = \begin{bmatrix} 0.4\cos 0.6k \\ 0.15 \end{bmatrix},\ B_{s2} = \begin{bmatrix} 0.2 \\ 0.06k \end{bmatrix},\ E_{s1} = E_{s2} = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix},$$
$$C_{s1} = C_{s2} = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix}, F_{s1} = F_{s2} = \begin{bmatrix} 0.5 & 0 \end{bmatrix},$$
$$D_{s1} = D_{s2} = \begin{bmatrix} 0.1 \end{bmatrix}, G_{s1} = G_{s2} = \begin{bmatrix} 0.3 \end{bmatrix},\ \textit{and}\ H_{s1} = H_{s2} = \begin{bmatrix} 0.1 \end{bmatrix}.$$

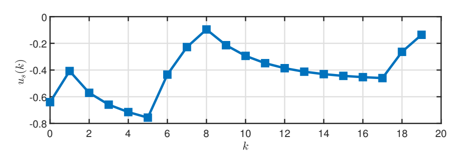
Here, since we have one measured output, the order of the dynamic controller can be either 0 or 1. Hence, it is possible to design both static and reduced-order output feedback controllers, providing flexibility in the controller design.

The corresponding parameters for the design of static and dynamic controllers are selected as, $\Theta := diag(\Theta_s, \Theta_c) = 3I_3$, $\Omega(k) := diag(\Omega_s(k), \Omega_c(k)) = diag(0.5I_2, 0.2)$, $Q = diag(-1.6, -10)$, $S = \begin{bmatrix} 0.3 & 0.4 \end{bmatrix}^T$ and $R = 3$. Using the DLMI conditions derived in Theorem 1 and based on Remark 2, the time-varying control gains are designed over the interval $[1, 20]$, with the scalar variable selected as $\beta = 2$. The simulations are carried out in MATLAB®, on a Core-i7 CPU with 16GB of RAM, using YALMIP optimization toolbox [24] and MOSEK solver [25].

The designed time-varying reduced-order control gains $A_c(k)$, $B_c(k)$, $C_c(k)$ and $D_c(k)$, and the control corresponding control function $u_s(k)$ are plotted in Fig. 1(a) and Fig. 1(b), respectively. With the given CPU specifications, it took 16.42 seconds to design reduced-order control gains.



(a) Control gains, $A_c(k)$, $B_c(k)$, $C_c(k)$ and $D_c(k)$



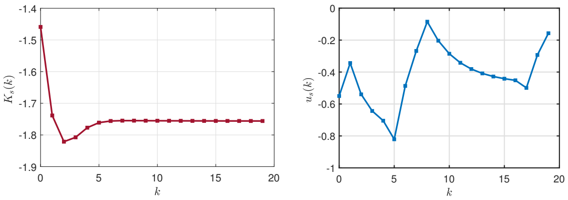
(b) Control function, $u_s(k)$

Fig. 1: Control with reduced-order dynamic controller

Further, the static output feedback time-varying control gains are obtained as shown in Fig. 2(a), with the corresponding control function as in Fig. 2(b). In this case, the time taken for controller design is 14.42 seconds.

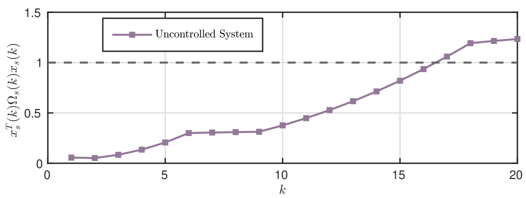
The results of numerical simulations, which show the variation of controlled and uncontrolled weighted system states, are depicted in Fig. 3. Here, the initial conditions of augmented system-controller states are chosen as $x_{k_0} = \begin{bmatrix} 0.45 & -0.28 & -0.22 \end{bmatrix}^T$, which satisfies $x_{k_0}^T\Theta x_{k_0} \le 1$ and the disturbance affecting the system is considered as,

$$\omega(k) = \begin{cases} 0.7, & 2 \le k \le 5 \\ 0.4, & 9 \le k \le 17 \\ 0, & \text{otherwise} \end{cases}.$$

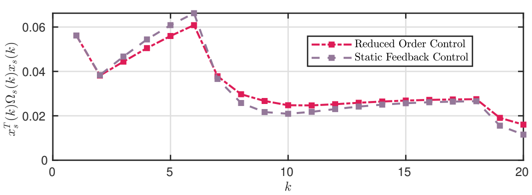


(a) Control gains, $K_s(k)$ (b) Control function, $u_s(k)$

Fig. 2: Control with static output feedback controller



(a) Weighted states of uncontrolled system



(b) Weighted states with reduced-order and static output feedback control

Fig. 3: Weighted system states, $x_s^T(k)\Omega_s(k)x_s(k)$

When subjected to disturbances, the uncontrolled system trajectories cross the specified threshold, as evident from Fig. 3(a). By associating the designed reduced-order and static output feedback control functions, the weighted states are bounded within the prescribed limit, as shown in Fig. 3(b). This indicates the effectiveness of the proposed method.

V. Conclusion

This paper studies the design of finite-time dissipative output feedback controllers for uncertain discrete-time linear time-varying systems. The proposed controller aims to bind the state trajectories below a specified threshold, while ensuring dissipativity properties to external disturbances during transients. We demonstrate the effectiveness of the proposed controller through numerical simulations on linear polytopic uncertain system. The proposed approach can be extended to design decentralized finite-time dissipative controllers for complex and large-scale systems, where using a decentralized control formulation is more convenient and efficient to bound the states of individual subsystems.

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