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# Nash Equilibrium Seeking over Time-Varying Networks with Time-Varying Delays

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***Abstract*—In this paper, we propose a distributed algorithm to seek the Nash equilibrium of the non-cooperative game over time-varying networks with time-varying delays. The slack block parameters, updated adaptively at each iteration, are designed to predict the players' future actions based on their current and past information. Each player then selects the appropriate slack block parameter according to the delay information and updates its local parameters accordingly. This method guarantees the convergence of the proposed algorithm, which can be proved via a non-cooperative game over a generalized delay-free network.**

***Index Terms*—Non-Cooperative Game; Time-Varying Networks; Time-varying Delays.**

## I. Introduction

Non-cooperative games are mathematical models that capture the strategic interactions among rational agents who pursue their interests. A key concept in non-cooperative games is Nash Equilibrium (NE), a state where no player can benefit from unilaterally changing their action. Professor John F. Nash first proposed NE in [1], and it has wide applications in smart grids [2], social networks [3], sensor networks [4], and so on. Thus, an efficient NE-seeking algorithm (NESA) is critical in the non-cooperative game theory. Many important NESAs have been proposed recently [5]–[8], [10]. The geometric convergence rates of NESAs are realized in non-cooperative games over unconstrained actions and constrained actions in [5] and [6], respectively. The distributed generalized NESA is constructed for aggregative games in [7]. Linear convergence of fully distributed NESA is realized over time-varying networks [8]. The algorithm is proposed to seek the time-varying NE of non-cooperative games in [9]. The hybrid NESA with partial-decision information under an adaptive event-triggered scheme is investigated in [10].

The above NESAs are constructed based on the assumption that each player can receive instantaneous information from neighboring players. However, communication delays are unavoidable in information transmission. Moreover, the communication delays may be time-varying and unpredictable rather than constants due to variable encoding/decoding times, varying data rates, and so on [11]. Thus, time-varying delays should be considered in NESAs for more practical applications. If delayed information is directly applied in existing NESAs, the convergence may not be guaranteed. Thus, how to construct NESAs with delayed information for good convergence property is a significant research topic.

This paper constructs a new NESA for non-cooperative games over time-varying networks with time-varying delays. The main idea is to introduce a set of slack block parameters updated adaptively and exchanged among the players at each iteration. These slack block parameters are designed to predict the future actions of players based on their current and past information. Then, some slack block parameters are selected appropriately based on time-varying delays to update the local parameters of each player. By doing so, the effect of time-varying delays can be eliminated, and the convergence of the NESA can be ensured.

To prove the convergence of the proposed NESA, a generalized network is constructed by adding virtual players and redefining edges. The virtual players represent the effect of slack block parameters and their cost functions are defined over one-point sets. The original time-varying network and adaptive updating rules of slack block parameters determine the communication links of the generalized network. The convergence of our proposed NESA with slack block parameters can be verified through the theoretical results of [12] over the proposed generalized network.

*Notations*: The $m$-dimension column vector set and $m \times n$ dimension matrix set are represented by $\mathbb{R}^m$ and $\mathbb{R}^{m\times n}$, respectively. For a positive integer $M$, $[M] := \{1, 2, \cdots, M\}$. For $z_j \in \mathbb{R}^{m_j}$, $\mathrm{col}\{z_1, z_2, \cdots, z_n\} = [z_1^T, z_2^T, \cdots, z_n^T]^T \in \mathbb{R}^{\sum_{j=1}^n m_j}$, where $z_j^T$ is the transpose of $z_j$. For a vector $x_a \in \mathbb{R}^{\sum_{j=1}^N m_j}$, there exist $n$ vectors $x_i \in \mathbb{R}^{m_i}$ ($i \in [N]$) such that $x_a = \mathrm{col}\{x_1, x_2, \cdots, x_N\}$. We define that $x_{a,i} = x_i$ and $x_{a,-i} = \mathrm{col}\{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N\}$. For a matrix $W$, $[W]_a^b$ denotes the element in the $a^{th}$ row, $b^{th}$ column. The norm $\|\cdot\|$ is the Euclidean norm. $\Pi_{\mathcal{X}}\{a\}$ is the projection of $a$ onto $\mathcal{X}$, i.e., $\Pi_{\mathcal{X}}\{a\} = \arg\min_{x\in\mathcal{X}} \|x - a\|$. For a time-varying network $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t), \mathcal{W}(t)\}$, $\mathcal{V}$, $\mathcal{E}(t)$, $\mathcal{W}(t)$, are the player set, the edge set, and the weighted adjacency matrix at time $t$, respectively. At time $t$, if player $a$ sends information to player $b$ if and only if $(b, a) \in \mathcal{E}(t)$, which is equivalent to $[\mathcal{W}(t)]_b^a > 0$. Thus, $(b, a) \notin \mathcal{E}(t)$ is equivalent to $[\mathcal{W}(t)]_b^a = 0$. For agent $a$ at time $t$, the in-neighboring set $N_a^{in}(t) = \{b \in \mathcal{V} | (a, b) \in \mathcal{E}(t)$ and $a \neq b\}$, and out-

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neighboring set $N_a^{out}(t) = \{b \in \mathcal{V} | (b,a) \in \mathcal{E}(t) \text{ and } a \neq b\}$.

## II. Problem Formulation

The objective of this paper is to seek the NE of the non-cooperative game via each players local computation and information exchange, where time-varying networks describe communication links among players and the information transmission has communication delays. The non-cooperative game, time-varying networks, and communication delays shall be introduced in detail in Section II-A, Section II-B, and Section II-C, respectively.

### A. Non-Cooperative Game

We consider the non-cooperative game of $N$ players, where each player $i \in [N]$ has a cost function $J_i(\cdot)$ and an action set $\mathcal{X}_i \subset \mathbb{R}^{n_i}$. Let $n = \sum_{i=1}^N n_i$ be the size of the joint action vector of all players. The cost function $J_i(x_i, x_{-i})$ depends on the action of player $i$, denoted by $x_i \in \mathcal{X}_i$, and the joint action of the other players, denoted by $x_{-i} = \text{col}\{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N\} \in \mathcal{X}_{-i} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_{i-1} \times \mathcal{X}_{i+1} \times \cdots \times \mathcal{X}_N$. The game is represented by $\Gamma = ([N], J_i, \mathcal{X}_i)$. A solution to $\Gamma$ is a Nash equilibrium (NE) $x^* = \text{col}\{x_1^*, x_2^*, \cdots, x_N^*\} \in \mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$ satisfying

$$J_i(x_i^*, x_{-i}^*) \leq J_i(x_i, x_{-i}^*), \qquad \forall x_i \in \mathcal{X}_i. \tag{1}$$

We make the following standard assumptions about the game $\Gamma = ([N], J_i, \mathcal{X}_i)$ (see [12]).

*Assumption 1:* Given $x_{-i} \in \mathbb{R}^{n-n_i}$, $J_i(x_i, x_{-i})$ is continuously differentiable and convex for $x_i \in \mathbb{R}^{n_i}$. In addition, the action set $\mathcal{X}_i$ is non-empty, compact, and convex for each player $i \in \mathcal{V}$.

We define the mapping $F(x) : \mathcal{X} \to \mathbb{R}^n$,

$$F(x) = [\nabla_1 J_1(x_1, x_{-1}), \cdots, \nabla_m J_m(x_m, x_{-m})]^T, \tag{2}$$

and have the following assumption.

*Assumption 2:* The game mapping $F$ in (2) is strictly monotone and $l$-Lipschitz continuous: $\langle a-b, F(a)-F(b)\rangle > 0$ for any $a, b \in \mathbb{R}^n$ and $a \neq b$; there exists a constant $L$ such that $\|F(a) - F(b)\| \leq L\|a-b\|$ for any $a, b \in \mathbb{R}^n$.

By Theorem 2.3.3 of [13], Assumption 2 implies that the game $\Gamma = ([N], J_i, \mathcal{X}_i)$ has a unique NE $x^*$ satisfying (1).

### B. Time-Varying Networks

We assume that the players are connected by a time-varying unbalanced directed network $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t), \mathcal{W}(t)\}$ with the following assumptions.

*Assumption 3:* The directed graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t), \mathcal{W}(t)\}$ is $B$-strongly-connected for $t \geq 0$, i. e., there exists a positive integer $B$ such that $\{\mathcal{V}, \cup_{t=k}^{k+B-1} \mathcal{E}(t)\}$ is strongly connected for any $k \geq 0$.

*Assumption 4:* (Weight Requirements) For $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t), \mathcal{W}(t)\}$, there exist a constant $\beta \in (0,1)$ such that

- $[\mathcal{W}(t)]_i^i \geq \beta$ for $i \in \mathcal{V}$ and $t \geq 0$;
- $[\mathcal{W}(t)]_i^j \geq \beta$ for $(i,j) \in \mathcal{E}(t)$;
- $\mathcal{W}(t)$ is row stochastic, $\sum_{j=1}^N [\mathcal{W}(t)]_i^j = 1$ for $i \in \mathcal{V}$.

### C. Communication Delays

In this paper, time-varying communication delays are considered in information transmission. Suppose player $j$ sends information to player $i$ at time $t_1$ and player $i$ receives it at time $t_2$. Then, the delay from player $j$ to player $i$ at time $t_2$ is $\tau_i^j(t_2) = t_2 - t_1$. We adopt the same assumption about time-varying delays as in [14].

*Assumption 5:* (Communication Delays)

- The delay from player $j$ to player $i$ at time $t$ is a non-negative integer $\tau_i^j(t)$;
- Each player can use its parameters without any delay;
- The delay $\tau_i^j(t)$ for any $i, j \in \mathcal{V}$ is uniformly upper bounded, i.e. $\tau_i^j(t) \leq D_j$ for $t \geq 0$ and $i \in \mathcal{V}$.

*Remark 1:* Assumption 3 is the standard assumption of $\mathcal{G}(t)$ in distributed networked systems (see [14]–[17]). Assumption 4 gives the weight requirements of information transmission between players, which requires the weighted adjacency matrix to be row stochastic but not necessarily doubly stochastic. This requirement guarantees the convergence of the NESA in [12], which will be utilized to prove the convergence of Algorithm 2 (see Theorem 1 and Corollary 1). A doubly stochastic weighted adjacency matrix is a common assumption of NESAs over time-varying networks (see [7]). However, many unbalanced networks do not satisfy this assumption. This paper's assumption on weight requirements is more relaxed than that in [7].

## III. NE-Seeking Algorithm with Slack Block Parameters and Convergence Analysis

### A. Algorithm Setting

In [12], a NESA is proposed for the non-cooperative game (1) over time-varying networks $\mathcal{G}(t)$ without communication delays, which is shown as follows:

$$\begin{cases} z_i(t+1) = \sum_{j=1}^N [\mathcal{W}(t)]_i^j z_j(t), \\ x_i(t+1) = \Pi_{\mathcal{X}_i}\{z_{i,i}(t+1) - \alpha(t)\nabla_i J_i(z_i(t))\}, \\ z_{i,i}(t+1) = x_i(t+1), \end{cases} \tag{3}$$

where $x_i(t) \in \mathbb{R}^{n_i}$ is the action of player $i$ at time $t$, $z_{i,j}(t) \in \mathbb{R}^{n_j}$ is the player $i$'s estimation for the action of player $j$ at time $t$, and $z_i(t) = \text{col}\{z_{i,1}(t), z_{i,2}(t), \cdots, z_{i,n}(t)\} \in \mathbb{R}^n$. Based on the result of the algorithm (3) in [12], we know that

$$\lim_{t\to\infty} \|x_i(t) - x_i^*\| = 0,$$

if Assumptions 1-4 hold.

For seeking NE of the non-cooperative game (1) over time-varying networks $\mathcal{G}(t)$ with communication delays, our strategy is to design slack block parameters (inspired from [17]) for the algorithm (3) to guarantee the convergence. Specifically, slack block parameters $p_i^{j_i}(t) \in \mathbb{R}^n$ with adaptive updating schemes of player $i \in \mathcal{V}$ are designed for predicting its estimation $z_i(t+j_i) \in \mathbb{R}^n$ at time $t$. Due to the condition $\tau_j^i(t) \leq D_i$ in Assumption 5, player $i \in \mathcal{V}$ only need to predict actions of players in the future moments from $t+1$ to $t+D_i$. Thus, $D_i$ slack block parameters, i.e., $\{p_i^1(t), p_i^2(t), \cdots, p_i^{D_i}(t)\}$, will be designed for player $i \in \mathcal{V}$. The convergence of the algorithm (3) over $\mathcal{G}(t)$ with slack block parameters and communication delays can be proved from (3) over a generalized network without delays (see (14)-(16) for more details).

The initial values of $\{p_i^{j_i}(-k_i)\}$ with $k_i = j_i, j_i - 1, \cdots, 1, 0$ should satisfy $p_{i,i}^{j_i}(t) \in \mathcal{X}_i$ and $p_{i,-i}^{j_i}(t) \in \mathbb{R}^{n-n_i}$. In addition, $z_i(0)$ is the player $i$'s initial estimation to actions of other players. Then, Algorithm 1 will be used to set values of $p_i^{j_i}(-k_i)$, with $i \in \mathcal{V}$, $j_i = 1, 2, \cdots, D_i$ and $k_i = j_i, j_i - 1, \cdots, 1, 0$. Algorithm 1 assigns initial values to

**Algorithm 1** Initial Value Set Algorithm of Player $i$

**Input:** $z_{i,i}(0) \in \mathcal{X}_i$, $\zeta_i \in (0,1)$, $p_i^{j_i}(-j_i)$, for $i \in \mathcal{V}$ and $j_i \in [D_i]$.

For Player $i \in \mathcal{V}$, define that

$$p_i^1(0) = (1-\zeta_i)z_i(0) + \zeta_i p_i^1(-1);$$

**for** $j_i = 2$ **to** $D_i$ **do**
  **for** $t = -j_i + 1$ **to** $0$ **do**

$$p_i^{j_i}(t) = (1-\zeta_i)p_i^{j_i-1}(t) + \zeta_i p_i^{j_i}(t-1);$$

  **end for**
**end for**
**Output:** $p_i^{j_i}(t)$ for $j_i = 1, 2, \cdots, D_i$ and $t \leq 0$.

each player's slack block parameters. There is no communication delay in Algorithm 1 since slack block parameters are updated locally without any transmission process. Then, the NESA with time-varying delays is shown as Algorithm 2.

Algorithm 2 consists of 4 steps. In Step 1, at time $t$, player $i$ receives the delayed information $\{z_j(t-\tau_i^j(t)), p_j^1(t-\tau_i^j(t)), p_j^2(t-\tau_i^j(t)), \cdots, p_j^{D_j}(t-\tau_i^j(t))\}$ from its neighbors $j \in N_i^{in}(t)$. In Step 2, each player selects the appropriate information based on time-varying delays. In Step 3, each player updates its parameters based on (4)-(8). The last step is to transmit $\{z_j(t+1), p_j^1(t+1), p_j^2(t+1), \cdots, p_j^{D_j}(t+1)\}$ to its neighbors $j \in N_i^{out}(t+1)$. Next, we shall explain time-varying delays in Remarks 2 and 3, then provide the convergence analysis of Algorithm 2 in Section III-B.

*Remark 2:* The time-varying delay $\tau_i^j(t)$ from player $j$ to player $i$ can be calculated by player $i$ at each iteration. In Step 2, player $i$ selects the appropriate information based on time-varying delay $\tau_i^j(t)$ to define $a_i^j(t)$ for updating parameters in Step 3. If player $i$ receives the information $\{z_j(s), p_j^1(s), \cdots, p_j^{D_j}(s)\}$ from player $j$ at time $t$, the time-varying delay $\tau_i^j(t)$ is calculated as $t-s$.

*Remark 3:* Due to time-varying delays, two information transmission cases should be considered in Algorithm 2.

- **Case 1**: Player $i$ may receive more than one set of information from player $j$ at time $t$. For example, player $i$ receives the information $\{z_j(s_1), p_j^1(s_1), \cdots, p_j^{D_j}(s_1)\}$, $\{z_j(s_2), p_j^1(s_2), \cdots, p_j^{D_j}(s_2)\}, \cdots, \{z_j(s_k), p_j^1(s_k), \cdots, p_j^{D_j}(s_k)\}$ from player $j$ at time $t$. Then, the time-varying delay $\tau_i^j(t)$ can be defined as any one of $t-s_h$ with $h = 1, 2, \cdots, k$. If $\tau_i^j(t)$ is defined as $t - s_{h_0}$ $(1 \leq h_0 \leq k)$, then $a_i^j(t)$ should be chosen from the set $\{z_j(s_{h_0}), p_j^1(s_{h_0}), \cdots, p_j^{D_j}(s_{h_0})\}$.
- **Case 2**: Player $i$ may receive no information from player $j$ at time $t$. In this case, player $i$ can select the previous information to define $\tau_i^j(t)$ and $a_i^j(t)$. For example, player $i$ receives the information $\{z_j(s), p_j^1(s), \cdots, p_j^{D_j}(s)\}$ from player $j$ at time $t_0$ $(t_0 < t$ and $t - s \leq D_i)$ but does not receive any information at time $t$. Then, $\tau_i^j(t)$ can be defined as $t-s$ and $a_i^j(t)$ can be chosen from $\{z_j(s), p_j^1(s), \cdots, p_j^{D_j}(s)\}$.

**Algorithm 2** Nash Equilibrium Seeking Method with Delays

**Input:** $x_i(0)$, $p_i^{j_i}(-k_i)$, $i \in \mathcal{V}$, $j_i = 1, \cdots, D_i$, $k_i = j_i, \cdots, 0$.
**for** $t = 1$ to $T$ **do**
  **Step 1**: Player $i$ received $z_j(t-\tau_i^j(t))$ and $p_j^{h_j}(t-\tau_i^j(t))$ with $h_j = 1, 2, \cdots, D_j$ from player $j \in N_i^{in}(t)$.
  **Step 2**: Select appropriate received information
    **for** $j \in N_i^{in}(t)$ **do**

$$a_i^j(t) = \begin{cases} p_j^{\tau_i^j(t)}(t-\tau_i^j(t)), & \tau_i^j(t) \neq 0, \\ z_j(t), & \tau_i^j(t) = 0, \end{cases}$$

    **end for**
  **Step 3**: Update local parameters of player $i$

$$z_i(t+1) = \zeta_i z_i(t) + \sum_{j \in N_i^{in}(t)} [\mathcal{W}(t)]_i^j a_i^j(t) + ([\mathcal{W}(t)]_i^i - \zeta_i)p_i^{D_i}(t-D_i), \tag{4}$$

$$x_i(t+1) = \Pi_{\mathcal{X}_i}\{z_{i,i}(t+1) - \alpha(t)\nabla_i J_i(z_i(t))\}, \tag{5}$$

$$z_{i,i}(t+1) = x_i(t+1), \tag{6}$$

$$p_i^1(t+1) = (1-\zeta_i)z_i(t+1) + \zeta_i p_i^1(t), \tag{7}$$

    **for** $h_i = 2$ **to** $D_i$ **do**

$$p_i^{h_i}(t+1) = (1-\zeta_i)p_i^{h_i-1}(t+1) + \zeta_i p_i^{h_i}(t). \tag{8}$$

    **end for**
  **Step 4**: Player $i$ sends the information $z_i(t+1)$ and $p_i^{j_i}(t+1)$ with $j_i = 1, 2, \cdots, D_i$ to player $j \in N_i^{out}(t+1)$.
**end for**
**Output:** A sequence of $x_i(t+1)$ for $i \in \mathcal{V}$ and $t = 1, 2, 3 \cdots, T$.

### B. Generalized Game and Convergence Analysis

In this subsection, a generalized graph $\widehat{\mathcal{G}}(t) = \{\widehat{\mathcal{V}}, \widehat{\mathcal{E}}(t), \widehat{\mathcal{W}}(t)\}$ with virtual players can be constructed from $\mathcal{G}(t)$ and $\tau_j^i(t)$ to prove the convergence of Algorithm 2. Firstly, the player set is defined as $\widehat{\mathcal{V}} = \{1, 2, \cdots, N, \cdots, N + \sum_{h=1}^N D_h\}$. Then, we assign the weight requirements of $\widehat{\mathcal{W}}(t)$. To simplify the expression, the symbol $A[a;b]$ is defined as follows

$$A[a;b] = \begin{cases} N + \sum_{h=1}^{a-1} D_h + b, & b \neq 0, \\ a, & b = 0, \end{cases} \quad (9)$$

with $1 \leq a \leq N$ and $0 \leq b \leq D_a$. Then, $\widehat{\mathcal{W}}(t)$ can be constructed from a constants $\zeta_i \in (0, \beta)$, $\mathcal{W}(t)$ and $\tau_i^j(t)$ with $i, j \in \mathcal{V}$ as follows:

- For $i = 1, 2, \cdots, A[N; D_N]$, $[\widehat{\mathcal{W}}(t)]_i^i = \zeta_i$;
- For $i \in [N]$ and $j_i \in [D_i]$, $[\widehat{\mathcal{W}}(t)]_{A[i;j_i]}^{A[i;j_i-1]} = 1 - \zeta_i$;
- For $i \in [N]$, $[\widehat{\mathcal{W}}(t)]_i^{A[i;D_i]} = [\mathcal{W}(t)]_i^i - \zeta_i$;
- For $i, j \in [N]$, $i \neq j$, $[\widehat{\mathcal{W}}(t)]_j^{A[i;\tau_j^i(t)]} = [\mathcal{W}(t)]_j^i$;
- Otherwise, $[\widehat{\mathcal{W}}(t)]_j^i = 0$.

The edge set $\widehat{\mathcal{E}}(t)$ can be derived from the weighted adjacency matrix $\widehat{\mathcal{W}}(t)$. The following Lemmas 1 and 2 hold for $\widehat{\mathcal{G}}(t) = \{\widehat{\mathcal{V}}, \widehat{\mathcal{E}}(t), \widehat{\mathcal{W}}(t)\}$.

*Lemma 1:* As for $\widehat{\mathcal{G}}(t) = \{\widehat{\mathcal{V}}, \widehat{\mathcal{E}}(t), \widehat{\mathcal{W}}(t)\}$, there is $\widehat{\beta} = \min\{\zeta_1, \cdots, \zeta_N, \beta - \zeta_1, \cdots, \beta - \zeta_N\} \in (0, 1)$ such that

- $[\widehat{\mathcal{W}}(t)]_i^i \geq \widehat{\beta}$ for any $i \in \widehat{\mathcal{V}}$;
- $[\widehat{\mathcal{W}}(t)]_i^j \geq \widehat{\beta}$ for any $(i, j) \in \widehat{\mathcal{E}}(t)$;
- $\sum_{j=1}^{A[N;D_N]} [\widehat{\mathcal{W}}(t)]_i^j = 1$ for any $i \in \widehat{\mathcal{V}}$.

*Lemma 2:* The direct graph $\widehat{\mathcal{G}}(t)$ is $B$-strongly-connected for any $t \geq 0$.

The proof of Lemmas 1 and 2 can be extended from Lemma 1 in [17] for non-cooperative games. We also define the generalized non-cooperative game $\widehat{\Gamma} = ([A[N, D_N]], \{\widehat{J}_i\}, \{\widehat{\mathcal{X}}_i\})$ based on the generalized network $\widehat{\mathcal{G}}(t)$. Cost functions can be defined as follows:

$$\widehat{J}_i(\widehat{x}_i, \widehat{x}_{-i}) = \begin{cases} J_i(\widehat{x}_i, x_{-i}), & i \in \mathcal{V}, \\ \widehat{x}_i^2, & i \in \widehat{\mathcal{V}}/\mathcal{V}, \end{cases} \quad (10)$$

where $\widehat{J}_i(\widehat{x}_i, \widehat{x}_{-i}) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_{A[N,D_N]}}$ and $n_i = 1$ for $i \in \mathcal{V}$. Cost functions in (10) will be further explained:

- For the player $i \in \mathcal{V}$, $\widehat{J}_i$ is defined based on cost functions in $\Gamma = ([N], J_i, \mathcal{X}_i)$. Specifically, $\widehat{J}_i$ only depends on $i$'s action $x_i \in \mathcal{X}_i$ and the joint actions of other players of $\mathcal{V}$: $x_{-i} = \mathrm{col}\{\widehat{x}_1, \cdots, \widehat{x}_{i-1}, \widehat{x}_{i+1}, \cdots, \widehat{x}_N\} \in \mathcal{X}_{-i} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_{i-1} \times \mathcal{X}_{i+1} \times \cdots \times \mathcal{X}_N$.
- Moreover, for player $i \in \widehat{\mathcal{V}}/\mathcal{V}$, $\widehat{J}_i$ only depends on player $i$'s action and the dimension of $x_i$ is 1 ($n_i = 1$ for $i \in \widehat{\mathcal{V}}/\mathcal{V}$). The reason to design it as $\widehat{x}_i^2$ is to ensure that $\widehat{\Gamma} = ([A[N, D_N]], \{\widehat{J}_i\}, \{\widehat{\mathcal{X}}_i\})$ satisfies Assumption 1 and Assumption 2. Alternatively, $\widehat{J}_i(\widehat{x}_i, \widehat{x}_{-i})$ of virtual players $i \in \widehat{\mathcal{V}}/\mathcal{V}$ can also be designed as $\widehat{x}_i^3$, $\widehat{x}_i^4$, $\widehat{x}_i^5$, etc.

Action sets of cost functions can be designed as follows:

$$\widehat{\mathcal{X}}_i = \begin{cases} \mathcal{X}_i, & i \in \mathcal{V}, \\ \{x_0\}, & i \in \widehat{\mathcal{V}}/\mathcal{V}, \end{cases} \quad (11)$$

where $x_0 \in \mathbb{R}$. For $i \in \mathcal{V}$, $\widehat{\mathcal{X}}_i$ is the same as the action set $\mathcal{X}_i$ in $\Gamma = ([N], J_i, \mathcal{X}_i)$. The action set of virtual player $i \in \widehat{\mathcal{V}}/\mathcal{V}$ is the same one point set $\{x_0\}$ with dimension $n_i = 1$. Thus, the action set of the virtual player is convex, nonempty, and closed. The generalized NE point $\widehat{x}^* \in \widehat{\mathcal{X}}_1 \times \widehat{\mathcal{X}}_2 \times \cdots \times \widehat{\mathcal{X}}_{A[N,D_N]}$ of the generalized non-cooperative game $\widehat{\Gamma} = ([A[N, D_N]], \{\widehat{J}_i\}, \{\widehat{\mathcal{X}}_i\})$ satisfies

$$\widehat{J}_i(\widehat{x}_i^*, \widehat{x}_{-i}^*) \leq \widehat{J}_i(\widehat{x}_i, \widehat{x}_{-i}^*), \quad \forall \widehat{x}_i \in \widehat{X}_i, \quad (12)$$

Since cost functions $\widehat{J}_i(\widehat{x}_i, \widehat{x}_{-i})$ and action sets $\widehat{\mathcal{X}}_i$ of players in $\widehat{\mathcal{G}}(t)$ are designed as (10) and (11), we have Lemma 3.

*Lemma 3:* The non-cooperative game $\widehat{\Gamma} = ([A[N, D_N]], \{\widehat{J}_i\}, \{\widehat{\mathcal{X}}_i\})$ satisfies Assumption 1 and Assumption 2.

The proof of Lemma 3 is omitted here. From Lemma 3 and Theorem 2.3.3 of [13], a unique generalized NE point $\widehat{x}^* \in \widehat{\mathcal{X}}_1 \times \widehat{\mathcal{X}}_2 \times \cdots \times \widehat{\mathcal{X}}_{A[N,D_N]}$ satisfying (12) exists. In addition, the relationship between $x^* \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$ and $\widehat{x}^* \in \widehat{\mathcal{X}}_1 \times \widehat{\mathcal{X}}_2 \times \cdots \times \widehat{\mathcal{X}}_{A[N,D_N]}$ is shown as follows:

$$\widehat{x}^* = \mathrm{col}\{x^*, x_0, \cdots, x_0\}. \quad (13)$$

We will use the following algorithm from [12] to seek NE for $\widehat{\Gamma} = ([A[N, D_N]], \{\widehat{J}_i\}, \{\widehat{\mathcal{X}}_i\})$:

$$\widehat{z}_i(k+1) = \sum_{j=1}^{A[N;D_N]} [\widehat{\mathcal{W}}(k)]_i^j \widehat{z}_j(k), \quad (14)$$

$$\widehat{x}_i(k+1) = \Pi_{\widehat{\mathcal{X}}_i}\{\widehat{z}_{i,i}(k+1) - \alpha(t)\nabla_i \widehat{J}_i(\widehat{z}_i(k))\}, \quad (15)$$

$$\widehat{z}_{i,i}(k+1) = \widehat{x}_i(k+1), \quad (16)$$

where $i = 1, 2, \cdots, A[N; D_N]$. For $\widehat{\Gamma}$, the initial values with $i \in \mathcal{V}$ and $j_i = 1, 2, \cdots, D_i$ are defined as follows:

$$\begin{cases} \widehat{z}_{N+\sum_{h=1}^{i-1} D_h + j_i}(0) = \mathrm{col}\{p_i^{j_i}(-j_i), x_0, \cdots, x_0\}, \\ \widehat{z}_i(0) = \mathrm{col}\{z_i(0), x_0, \cdots, x_0\}, \\ \widehat{x}_i(0) = x_i(0). \end{cases} \quad (17)$$

*Assumption 6:* The non-increasing stepsize $\{\alpha(t)\}$ of (15) satisfies $\alpha(t) > 0$, $\sum_{t=1}^{\infty} \alpha(t) = \infty$, and $\sum_{t=1}^{\infty} \alpha^2(t) < \infty$.

From [12], $\widehat{x}_i(t)$ converges to $\widehat{x}_i^*$ as $t \to \infty$. However, this is a theoretical result based on a simplified model that assumes the existence of virtual players and generalized NE, which are not realistic in the real world. To show the practical relevance of Algorithm 2, we establish the relationship between Algorithm 2 and (14)-(16) in Theorem 1 (proved in Appendix VI-A).

*Theorem 1:* If generalized game $\widehat{x}^* \in \widehat{\mathcal{X}}_1 \times \widehat{\mathcal{X}}_2 \times \cdots \times \widehat{\mathcal{X}}_{A[N,D_N]}$ is designed as (10) and (11), then we have

$$\widehat{z}_i(t) = \mathrm{col}\{z_i(t), x_0, \cdots, x_0\}, \quad (18)$$

$$\widehat{x}_i(t) = x_i(t), \quad (19)$$

where $\{\widehat{z}_i(t), \widehat{x}_i(t)\}$ are the output of (14)-(16) and $\{z_i(t), x_i(t)\}$ are the output of Algorithm 2, the initial values are designed as (17) and Algorithm 1.

*Corollary 1:* If the step size $\alpha(t)$ in Algorithm 2 satisfies Assumption 6, then Algorithm 2 converges to the NE $x^*$ of the original game $\Gamma = ([N], J_i, \mathcal{X}_i)$, i.e.,

$$\lim_{k \to \infty} \|x_i(k) - x_i^*\| = 0,$$

where $x_i(k)$ is the output of Algorithm 2 and $x_i^*$ is defined in (1) for $i \in \mathcal{V}$.

From Lemmas 1, 2, 3, and [12], we know that $\lim_{k\to\infty} \|\widehat{x}_i(k) - \widehat{x}_i^*\| = 0$, where $\widehat{x}_i(k)$ is calculated from (14)-(16) and $\widehat{x}_i^*$ is defined in (12). From Theorem 1, Corollary 1 is proved.

*Remark 4:* The method of slack parameters for distributed optimization with communication delays was first proposed in our previous work [17], where delays from player $i$ to its neighbors were assumed to be a positive integer $\tau_i$. In this paper, we extend this method to the NESA with time-varying delays. Compared with [17], we consider a more general case of time-varying delays (see Assumption 5). Moreover, the proposed method is expected to be applied to realize distributed optimization problems with time-varying delays.

*Remark 5:* The key to realizing the effect of slack block parameters through virtual players is to eliminate the influence of the gradient descent processes of virtual players (see (15)). The methods in [17] and this paper are different. In [17], the objective functions of virtual agents are set as 0. However, in this paper, to satisfy Assumption 1 and Assumption 2, cost functions cannot be designed as 0. Thus, we adopt another approach to design the action sets of virtual players as one-point sets (see (11)). By (15) in Algorithm 2, $\widehat{x}_i(k+1) = x_0$ for any $i \in \widehat{\mathcal{V}}/\mathcal{V}$ and $k \geq 0$. Nevertheless, adaptive updating methods of slack parameters in [17] are similar to the method in this paper. The design of virtual players' action sets is the key point in this paper.

## IV. Simulation

In this section, a Nash-Cournot game in [18] is considered over time-varying networks with time-varying communication delays. In this game, $N$ firms produce and compete for $m$ markets. The cost function of the firm $i$ is

$$J_i(x_i, x_{-i}) = C_i(x_i) - (P(Ax))^{\mathrm{T}} A_i x_i, \tag{20}$$

where $C_i(x_i) = x_i^{\mathrm{T}} Q_i x_i + q_i^{\mathrm{T}} x_i$ is the production cost of firm $i$ and $P \in \mathbb{R}^{m\times n}$ is the price function that maps the total supply of each market to the marker price vector.

The parameters can be selected to satisfy Assumptions 1, 2 and $\mathcal{G}(t)$ can be selected to satisfy Assumptions 3, 4. The communication delay $\tau_i^j(t)$ is randomly selected from $[0, D_i]$ for $i = 1, 2, \cdots, N$. We apply Algorithm 1 and Algorithm 2 to seek the NE of (20). We use the error $e_i(t) = \|x_i(t) - x_i^*\|$ with $i = 1, 2, \cdots, N$ to show the convergence. As shown in Fig. 1, $e_i(t) = \|x_i(t) - x_i^*\|$ converges to 0 with $t \to \infty$. The simulation result demonstrates the convergence result in Corollary 1.

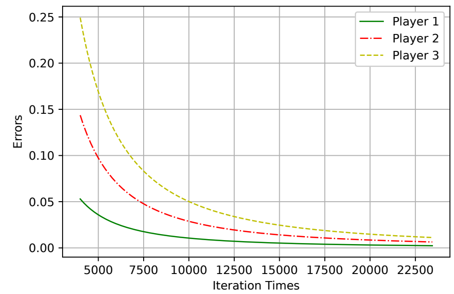


Fig. 1. Convergence Performance

## V. Conclusion

This paper investigates a NESA for non-cooperative games over time-varying networks with time-varying delays. Slack block parameters are proposed to predict players' future actions and are transmitted among different players with delays. The convergence of the proposed NESA can be proved by transforming the original game into a generalized game over a delay-free network. Numerical examples are given to demonstrate the effectiveness of the proposed NESA.

## VI. Appendix

### A. Proof of Theorem 1

To simplify the symbol, $\mathrm{col}\{z, x_0, \cdots, x_0\}$ denotes $\mathrm{col}\{z, \underbrace{x_0, \cdots, x_0}_{A[i,D_i]-N}\}$ for any column vector $z$. Mathematical induction shall be used to show

$$\begin{cases} \widehat{z}_i(t) = \mathrm{col}\{z_i(t), x_0, \cdots, x_0\}, \\ \widehat{x}_i(t) = x_i(t), \\ \widehat{z}_{A[i;j_i]}(t) = \mathrm{col}\{p_i^{j_i}(t - j_i), x_0, \cdots, x_0\}, \end{cases} \tag{21}$$

for $t \geq 1$, $i \in [N]$ and $j_i \in [D_i]$. For $t = 1$, for player $i \in [N]$, we know that $\widehat{z}_i(1) = \sum_{j=1}^{A[N;D_N]} [\widehat{\mathcal{W}}(0)]_i^j \widehat{z}_j(0) = \zeta_i \widehat{z}_i(0) + ([\mathcal{W}(0)]_i^i - \zeta_i)\widehat{z}_{A[i;D_i]}(0) + \sum_{j=1, j\neq i}^{N} [\mathcal{W}(0)]_i^{A[j;\tau_i^j(0)]} \widehat{z}_{A[j;\tau_i^j(0)]}(0)$. From (4), it holds that $z_i(1) = \zeta_i z_i(0) + \sum_{j=1,j\neq i}^{N} [\mathcal{W}(0)]_i^j a_i^j(0) + ([\mathcal{W}(0)]_i^i - \zeta_i) p_i^{D_i}(-D_i)$. From (17), we know that $\widehat{z}_{A[i,j_i]}(0) = \mathrm{col}\{p_i^{j_i}(-j_i), x_0, \cdots, x_0\}$ and $\widehat{z}_i(0) = \mathrm{col}\{z_i(0), x_0, \cdots, x_0\}$. Thus, with $j \in [N]$ $\widehat{z}_{A[j;\tau_i^j(0)]}(0) = \mathrm{col}\{p_j^{\tau_i^j(0)}(-\tau_i^j(0)), x_0, \cdots, x_0\}$ for $\tau_i^j(0) \neq 0$ and $\widehat{z}_{A[j;\tau_i^j(0)]}(0) = \mathrm{col}\{z_j(0), x_0, \cdots, x_0\}$ for $\tau_i^j(0) = 0$. In addition, from Algorithm 2, we know that $a_i^j(0) = p_j^{\tau_i^j(0)}(-\tau_i^j(0))$ for $\tau_i^j(0) \neq 0$ and $a_i^j(0) = z_j(0)$ for $\tau_i^j(0) = 0$. Thus, it holds that

$$\widehat{z}_i(1) = \mathrm{col}\{z_i(1), x_0, \cdots, x_0\}, \tag{22}$$

for $i \in [N]$. From (5), (10), (17), (15), and (22), we know that

$$\widehat{x}_i(1) = x_i(1), \tag{23}$$

for $i = 1, 2, \cdots, N$. In addition, for player $i \in \mathcal{V}$, $p_i^1(0) = (1 - \zeta_i) z_i(0) + \zeta_i p_i^1(-1)$ and $\widehat{z}_{A[i;1]}(1) = \sum_{j=1}^{A[N;D_N]} [\widehat{\mathcal{W}}(0)]_{A[i;1]}^j \widehat{z}_j(0) = (1 - \zeta_i)\widehat{z}_i(0) + \zeta_i \widehat{z}_{A[i;1]}(0)$. Since the action set $\widehat{\mathcal{X}}_h = \{x_0\}$ with $N + 1 \leq h \leq A[i, D_i]$, from (15) and (16), it holds that

$$\widehat{z}_{h,h}(t) = x_0, \tag{24}$$

for any $t \geq 0$ and $N+1 \leq h \leq A[i, D_i]$. In addition, from (14) and initial values setting (17), it holds that

$$\widehat{z}_{i,h}(t) = x_0, \tag{25}$$

for any $t \geq 0$, $i \in \mathcal{V}$, and $N+1 \leq h \leq A[i, D_i]$. Due to $\widehat{z}_i(0) = \mathrm{col}\{z_i(0), x_0, \cdots, x_0\}$ and $\widehat{z}_{A[i;1]}(0) = \mathrm{col}\{p_i^1(-1), x_0, \cdots, x_0\}$, combining with (25), it holds that

$$\widehat{z}_{A[i;1]}(1) = \mathrm{col}\{p_i^1(0), x_0, \cdots, x_0\}. \tag{26}$$

For $j_i = 2, 3, \cdots, N$, $p_i^{j_i}(1-j_i) = \zeta_i p_i^{j_i}(-j_i) + (1-\zeta_i) p_i^{j_i-1}(1-j_i)$ and $\widehat{z}_{A[i;j_i]}(1) = \sum_{j=1}^{A[N;D_N]} [\widehat{\mathcal{W}}(0)]^j_{A[i;j_i]} \widehat{z}_j(0) = (1-\zeta_i)\widehat{z}_{A[i;j_i-1]}(0) + \zeta_i \widehat{z}_{A[i;j_i]}(0)$. Due to $\widehat{z}_{A[i;j_i]}(0) = \mathrm{col}\{p_i^{j_i}(-j_i), x_0, \cdots, x_0\}$ and $\widehat{z}_{A[i;j_i-1]}(0) = \mathrm{col}\{p_i^{j_i-1}(1-j_i), x_0, \cdots, x_0\}$, it implies that for $j_i = 2, 3, \cdots, D_i$,

$$\widehat{z}_{A[i;j_i]}(1) = \mathrm{col}\{p_i^{j_i}(1-j_i), x_0, \cdots, x_0\}. \tag{27}$$

for $j_i = 1, 2, \cdots, D_i$. Thus, from (22), (23), and (27), we know that (21) is true for $t = 1$. Then, we assume that (21) is true for $t = k$, i.e.,

$$\begin{cases} \widehat{z}_i(k) = \mathrm{col}\{z_i(k), x_0, \cdots, x_0\}, \\ \widehat{x}_i(k) = x_i(k), \\ \widehat{z}_{A[i;j_i]}(k) = \mathrm{col}\{p_i^{j_i}(k-j_i), x_0, \cdots, x_0\}. \end{cases} \tag{28}$$

Then, for $t = k+1$, (14) implies that $\widehat{z}_i(k+1) = \sum_{j=1}^{A[N;D_N]} [\widehat{\mathcal{W}}(k)]_i^j \widehat{z}_j(k) = ([\mathcal{W}(k)]_i^i - \zeta_i)\widehat{z}_{A[i;D_i]}(k) + \zeta_i \widehat{z}_i(k) + \sum_{j=1, j\neq i}^{N} [\mathcal{W}(k)]_i^{A[j;\tau_i^j(k)]} \widehat{z}_{A[j;\tau_i^j(k)]}(k)$. From (4), $z_i(k+1) = \zeta_i z_i(k) + \sum_{j=1, j\neq i}^{N} [\mathcal{W}(k)]_i^j a_i^j(k) + ([\mathcal{W}(k)]_i^i - \zeta_i) p_i^{D_i}(k - D_i)$. From Algorithm 2, we know that $a_i^j(k) = p_j^{\tau_i^j(k)}(k - \tau_i^j(k))$ for $\tau_i^j(k) \neq 0$ and $a_i^j(k) = z_j(k)$ for $\tau_i^j(k) = 0$. Thus, for any $\tau_j^i(k)$, from (28), we have

$$\widehat{z}_{A[j;\tau_i^j(k)]}(k) = \mathrm{col}\{a_i^j(k), x_0, \cdots, x_0\}. \tag{29}$$

Thus, we know that

$$\widehat{z}_i(k+1) = \mathrm{col}\{z_i(k+1), x_0, \cdots, x_0\}, \tag{30}$$

for $i \in [N]$. From (5), (10), (15), (22), (28), and (29), we have

$$\widehat{x}_i(k+1) = x_i(k+1), \tag{31}$$

for $i \in [N]$. Moreover, for player $i \in \mathcal{V}$, $p_i^1(k) = (1-\zeta_i) z_i(k) + \zeta_i p_i^1(k-1)$ and $\widehat{z}_{A[i;1]}(k+1) = \sum_{j=1}^{A[N;D_N]} [\widehat{\mathcal{W}}(k)]^j_{A[i;1]} \widehat{z}_j(k) = (1-\zeta_i)\widehat{z}_i(k) + \zeta_i \widehat{z}_{A[i;1]}(k)$. From (28), we know that $\widehat{z}_i(k) = \mathrm{col}\{z_i(k), x_0, \cdots, x_0\}$ and $\widehat{z}_{A[i;1]}(k) = \mathrm{col}\{p_i^1(k-1), x_0, \cdots, x_0\}$. Combining with (25), it holds that

$$\widehat{z}_{A[i;1]}(k+1) = \mathrm{col}\{p_i^1(k), x_0, \cdots, x_0\}. \tag{32}$$

For $j_i = 2, 3, \cdots, N$, $p_i^{j_i}(k+1-j_i) = (1-\zeta_i) p_i^{j_i-1}(k+1-j_i) + \zeta_i p_i^{j_i}(k-j_i)$ and $\widehat{z}_{A[i;j_i]}(k+1) = \sum_{j=1}^{A[N;D_N]} [\widehat{\mathcal{W}}(k)]^j_{A[i;j_i]} \widehat{z}_j(k) = (1-\zeta_i)\widehat{z}_{A[i;j_i-1]}(k) + \zeta_i \widehat{z}_{A[i;j_i]}(k)$. From (25) and (28), we know that

$$\widehat{z}_{A[i;j_i-1]}(k) = \mathrm{col}\{p_i^{j_i-1}(k+1-j_i), x_0, \cdots, x_0\}, \tag{33}$$

$$\widehat{z}_{A[i;j_i]}(k) = \mathrm{col}\{p_i^{j_i}(k-j_i), x_0, \cdots, x_0\}. \tag{34}$$

Combining with (25), (32), (33), and (34), it holds that

$$\widehat{z}_{A[i;j_i]}(k+1) = \mathrm{col}\{p_i^{j_i}(k+1-j_i), x_0, \cdots, x_0\}, \tag{35}$$

for $j_i \in [D_i]$. From (30), (31), and (35), we know that (21) is true for $t = k+1$. Based on the mathematical induction, (21) is true for all $t \geq 0$.

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