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Finite-dimensional boundary control for stochastic semilinear 2D parabolic PDEs

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Abstract— **In this paper, we consider state-feedback global stabilization of stochastic semilinear 2D parabolic PDEs with nonlinear multiplicative noise, where the nonlinearities satisfy globally Lipschitz condition. We consider the Dirichlet actuation and design the controller with the shape functions in the form of eigenfunctions corresponding to the first comparatively unstable N eigenvalues. We extend the trigonometric change of variables to the 2D case and further improve it, leading to homogeneous boundary conditions. Employing N-dimensional dynamic extension with the corresponding proportional-integral controller and using modal decomposition, we derive stochastic nonlinear ODEs for the modes of the state with the first N-dimensional part being controllable. By using a direct Lyapunov method and Itô's formula for stochastic ODEs and PDEs, we provide mean-square L^2 exponential stability analysis of the full-order closed-loop system. We provide linear matrix inequality (LMI) conditions for finding N and the controller gain. We prove that the LMIs are always feasible provided the Lipschitz constants are small enough and N is large enough. Numerical examples demonstrate the efficiency of our method and show that the employment of the suggested dynamic extension allows for larger Lipschitz constants than the previously used dynamic extensions.**

I. Introduction

Finite-dimensional controllers for PDEs are attractive in applications. Such controllers were designed by the modal decomposition approach and have been extensively studied since the 1980s [1], [2]. In recent years, estimation and control problems for stochastic PDEs become popular due to their wide applications in many areas of science, engineering, and finance. In [3], [4], finite-dimensional control of linear stochastic PDEs was studied, where constructive conditions for finding the controller dimension were not provided. Recently, inspired by [5] for deterministic 1D parabolic PDEs, in [6] we suggested the first constructive finite-dimensional control for stochastic 1D parabolic PDEs. Considering that semilinear parabolic PDEs arise in many physical models [7] and motivated by recent results [8], [9] for deterministic 1D PDEs, in [10], we studied the finite-dimensional output feedback control of stochastic semilinear 1D parabolic PDEs. However, the constructive results in [5], [6], [8], [9], [10] are confined to 1D parabolic PDEs.

In recent years, control of high-dimensional PDEs has become an active research area. Such systems have promising applications in engineering, water heating [11], material preparation processes [3], as well as in multi-agents deployment [12]. The finite-dimensional boundary state-feedback stabilization of high-dimensional parabolic PDEs was studied in [13], [14] for the linear case and in [15], [16] for the nonlinear case. The finite-dimensional observer-based boundary control for 2D and 3D linear parabolic PDEs was first studied in [17]. Note that in [15], [16], only local stabilization was considered, and constructive conditions for finding the controller dimension were not provided. In our recent paper [18], we design finite-dimensional observer-based control for 2D linear deterministic heat equation under Neumann actuation and provided effective LMI conditions for finding controller and observer dimensions, where as in 1D case (see [19]) dynamic extension is not needed. Boundary control for high-dimensional semilinear parabolic deterministic and stochastic PDEs remains a challenging open problem. The main challenges lie in the following (i) The multiple eigenvalues and slow convergence of the eigenvalues to infinity (compared to 1D case, see [20, Proposition 3.6.9]) complicate the analysis; (ii) In the stochastic case, stability analysis and efficient controller design are more challenging.

In this paper, we consider state-feedback global stabilization of stochastic semilinear 2D parabolic PDEs with nonlinear multiplicative noise, where the nonlinearities satisfy globally Lipschitz condition. We consider the Dirichlet actuation. Following [13], [16], [21], we design the controller with the shape functions in the form of eigenfunctions corresponding to the first comparatively unstable N eigenvalues (N is the controller dimension). We extend the trigonometric change of variables studied in [9], [22] for 1D parabolic PDEs to the 2D case and further improve it, leading to homogeneous boundary conditions. By employing N-dimensional dynamic extension with the corresponding proportional-integral controller and applying modal decomposition, we derive stochastic nonlinear ODEs for the modes of the state, where the first N-dimensional part is controllable. We compensated the deterministic nonlinear term by Parseval's inequality and the stochastic nonlinear term by S-procedure. By suggesting a direct Lyapunov method and employing Itô's formula for stochastic ODEs and PDEs, respectively, we provide mean-square L^2 exponential stability analysis of the full-order closed-loop system. We derive LMIs for obtaining N and the controller gain and prove that the LMIs are always feasible provided the Lipschitz constants are small enough and N is large enough. Numerical examples demonstrate the efficiency of our method and show that the employment of

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the suggested dynamic extension allows for larger Lipschitz constants than the previously used ones.

Notations: Let $(\Omega,\mathscr{F},\mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ of increasing sub σ-fields of $\mathscr{F}$ and let $\mathbb{E}\{\cdot\}$ be the expectation operator. Denote by $\mathscr{W}(t)$ the 1D standard Brownian motion defined on $(\Omega,\mathscr{F},\mathbb{P})$. The Euclidean norm is denoted by $|\cdot|$. For $P\in\mathbb{R}^{n\times n}$, $P>0$ means that P is symmetric and positive definite. The symmetric elements of a symmetric matrix will be denoted by $*$. For $0<P\in\mathbb{R}^{n\times n}$ and $x\in\mathbb{R}^n$, we write $|x|_P^2=x^{\mathrm{T}}Px$. Denote $\mathbb{N}$ by the set of positive integers. For any bounded domain $\mathscr{O}_0\subset\mathbb{R}^n$ $(n=1,2)$, denote by $L^2(\mathscr{O}_0)$ the space of square integrable functions with inner product $\langle f,g\rangle_{\mathscr{O}_0}=\int_{\mathscr{O}_0}f(x)g(x)\mathrm{d}x$ and induced norm $\|f\|^2_{L^2(\mathscr{O}_0)}=\langle f,f\rangle_{\mathscr{O}_0}$. $H^1(\mathscr{O}_0)$ is the Sobolev space of functions $f:\mathscr{O}_0\longrightarrow\mathbb{R}$ with a square integrable weak derivative. The norm defined in $H^1(\mathscr{O}_0)$ is $\|f\|^2_{H^1(\mathscr{O}_0)}=\|f\|^2_{L^2(\mathscr{O}_0)}+\|\nabla f\|^2_{L^2(\mathscr{O}_0)}$, where ∇f represents the gradient of f and $\|\nabla f\|^2_{L^2(\mathscr{O}_0)}=\int_{\mathscr{O}_0}|\nabla f(x)|^2\mathrm{d}x$. Let $\frac{\partial}{\partial\mathbf{n}}$ be the normal derivative.

II. Main results

A. System under consideration

Let $\mathscr{O}\subset\mathbb{R}^2$ be a bounded open connected set. We assume that either the boundary $\partial\mathscr{O}=\Gamma_1\cup\Gamma_2$ is of class C^2 or $\mathscr{O}$ is a rectangular domain. Consider the following stochastic semilinear 2D heat equation under Dirichlet actuation:

$$\begin{aligned}&\mathrm{d}z(x,t)=[\Delta z(x,t)+qz(x,t)+f(z(x,t))]\mathrm{d}t\\&\qquad+g(z(x,t))\mathrm{d}\mathscr{W}(t),\\&z(x,t)=0,\ x\in\Gamma_1,\quad z(x,t)=u(x,t),x\in\Gamma_2,\\&z(x,0)=z_0(x),\end{aligned}\tag{1}$$

where $x\in\mathscr{O}$, Δ is the usual Laplacian, $q\in\mathbb{R}$ is the reaction coefficient, $u(x,t)$ is the control input to be designed, $g(z(x,t))\mathrm{d}\mathscr{W}(t)$ is a nonlinear multiplicative noise that appears due to the random parameter variation of $f(z(x,t))\mathrm{d}t$. Throughout the paper, we assume that functions $f,g:\mathbb{R}\to\mathbb{R}$ satisfy

$$\begin{aligned}&f(0)=0,\ |f(z_1)-f(z_2)|\leq\sigma_f(z_1-z_2),\\&g(0)=0,\ |g(z_1)-g(z_2)|\leq\sigma_g(z_1-z_2),\ \forall z_1,z_2\in\mathbb{R},\end{aligned}\tag{2}$$

for some $\sigma_f,\sigma_g>0$.

Let

$$\begin{aligned}&\mathscr{A}\phi=-\Delta\phi,\ \mathscr{D}(\mathscr{A})=\{\phi|\phi\in H^2(\mathscr{O})\cap H_0^1\},\\&H_0^1=\{\phi\in H^1(\mathscr{O})|\phi(x)=0\text{ for }x\in\partial\mathscr{O}\}.\end{aligned}\tag{3}$$

It follows from [20, Proposition 3.2.12] that the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of $\mathscr{A}$ are real and we can repeat each eigenvalue according to its finite multiplicity to get

$$\lambda_1<\lambda_2\leq\cdots\leq\lambda_n\leq\ldots,\quad \lim_{n\to\infty}\lambda_n=\infty.\tag{4}$$

We denote the corresponding eigenfunctions as $\{\phi_n\}_{n=1}^\infty$. For λ_N, we have the following estimate:

Lemma 1: ([23, Sec. 11.6]) For eigenvalues (4), the following holds: $\lim_{N\to\infty}\frac{\lambda_N}{N}=\frac{4\pi}{|\mathscr{O}|}$, where $|\mathscr{O}|$ is the area of $\mathscr{O}$.

Let $\delta>0$ be a desired decay rate and $N\in\mathbb{N}$ such that

$$-\lambda_n+q+\delta+\sqrt{2}\sigma_f+\sigma_g^2<0,\quad n>N,\tag{5}$$

where N denotes the number of unstable modes. Our controller will be designed by using N modes. For given $\lambda\in\{\lambda_n\}_{n=1}^N$, let m_λ be the geometric multiplicity of λ and $\phi_\lambda^{(1)},\ldots,\phi_\lambda^{(m_\lambda)}$ be the eigenfunctions corresponding to λ. We impose the following assumption that is crucial for the controllability of the finite-dimensional part of the closed-loop system (see above (22)):

Assumption 1: Given $\lambda\in\{\lambda_n\}_{n=1}^N$, let $\{\frac{\partial\phi_\lambda^{(i)}}{\partial\mathbf{n}}\}_{i=1}^{m_\lambda}$ be linearly independent in $L^2(\Gamma_2)$.

Remark 1: Note that Assumption 1 always holds for 1D case (due to simple eigenvalues) and for rectangular domain $\mathscr{O}=(0,a_1)\times(0,a_2)$, $a_1,a_2>0$. Consider the boundary:

$$\partial\mathscr{O}=\Gamma_1\cup\Gamma_2,\quad\Gamma_2=\{(x_1,0),\ x_1\in(0,a_1)\}.\tag{6}$$

Here the eigenvalues of $\mathscr{A}$ are given by

$$\lambda_{m,k}=\pi^2[\tfrac{m^2}{a_1^2}+\tfrac{k^2}{a_2^2}],\quad m,k\in\mathbb{N},\tag{7}$$

whereas the corresponding eigenfunctions have the form

$$\phi_{m,k}(x)=\tfrac{2}{\sqrt{a_1a_2}}\sin(\tfrac{m\pi x_1}{a_1})\sin(\tfrac{k\pi x_2}{a_2}),\quad x=(x_1,x_2).\tag{8}$$

For any pair of multiple eigenvalues $\lambda_{m_1,k_1}=\lambda_{m_2,k_2}$, the relation $m_1\neq m_2$ always implies $k_1\neq k_2$ (and vice versa). Therefore, $\frac{\partial\phi_{m_1,k_1}}{\partial\mathbf{n}}$ and $\frac{\partial\phi_{m_2,k_2}}{\partial\mathbf{n}}$ are always linearly independent in $L^2(\Gamma_2)$. Note that Assumption 1 is much weaker than the assumption (linear independence of $\{\frac{\partial\phi_n(x)}{\partial\mathbf{n}},x\in L^2(\Gamma_2)\}_{n=1}^N$) in [16], which does not hold true for rectangular domain when $N\geq 3$. The assumption in [16] was removed in [15], [17] by slightly perturbing the linear operator $\mathscr{A}$, whereas constructive conditions for finding the controller dimension were not provided in [15], [17].

Remark 2: In [18], the observer-based design for 2D linear heat equation was explored under Neumann actuation, which is not applicable for Dirichlet actuation (similar to the 1D case explained in [10, Remark 2]). In this paper, we manage with the Dirichlet actuation via dynamic extension and the results can be directly extended to the Neumann actuation. In this scenario, we do not need Assumption 1 since for general domain $\mathscr{O}$ and given $\lambda\in\{\lambda_n\}_{n=1}^N$, eigenvectors $\{\phi_\lambda^{(i)}\}_{i=1}^{m_\lambda}$ are always linearly independent in $L^2(\Gamma_2)$ (see [13, Lemma 7.1]).

For given positive constants $\{\mu_i\}_{i=1}^N$ satisfying

$$\mu_i\neq\lambda_n,\quad i\in\{1,\ldots,N\},n\in\mathbb{Z},\quad\mu_i=O(\lambda_i),\tag{9}$$

consider a sequence of functions $\psi_i\in L^2(\mathscr{O})$, $i=1,\ldots,N$, that satisfy

$$\begin{aligned}&\Delta\psi_i(x)=-\mu_i\psi_i(x),\quad x\in\mathscr{O},\\&\psi_i(x)=0,\ x\in\Gamma_1,\quad\psi_i(x)=b_i\tfrac{\partial\phi_i(x)}{\partial\mathbf{n}},\ x\in\Gamma_2,\end{aligned}\tag{10}$$

where $b_i\in\mathbb{R}$ are chosen such that $\|\psi_i\|_{L^2(\mathscr{O})}=\rho$. Here $\rho>0$ is a tuning parameter. Such functions always exist [13]. Since $\mu_i\neq\lambda_n$, by applying Green's first identity, we find that

$$\langle\psi_i,\phi_n\rangle_{\mathscr{O}}=\tfrac{-b_i}{\lambda_n-\mu_i}\langle\tfrac{\partial\phi_i}{\partial\mathbf{n}},\tfrac{\partial\phi_n}{\partial\mathbf{n}}\rangle_{\Gamma_2}.\tag{11}$$

Remark 3: Functions ψ_i in (10) are the extension of the 1D functions considered in [9], [22] where ρ is fixed to

$\frac{1}{\sqrt{2}}$. In our example in Sec. III, we show that an appropriate choice of $\rho \in (0, \frac{1}{\sqrt{2}})$ can lead to larger upper bounds on the Lipschitz constants. Moreover, the tuning parameter ρ allows also us to improve results in the 1D case (see Sec. III).

***Assumption** 2:* Let ψ_i be linearly independent or, equivalently (see Theorem 7.2.10 of [24]), let $\Psi^N = (\langle \psi_i, \psi_j \rangle_{\mathscr{O}})_{i,j=1}^N$ be invertible.

***Remark** 4:* Assumption 2 is crucial for our controller design (see Λ_2^{-1} in (41) with $\Lambda_2 = \mathrm{diag}\{\Psi^N, I_N\}$) and always holds in rectangular domains introduced in Remark 1. Indeed, let us take

$$\begin{aligned} &\psi_{m,k}(x) = \tfrac{2\rho}{\sqrt{a_1 a_2}} \sin(\tfrac{m\pi x_1}{a_2}) \cos(\tfrac{(k-0.5)\pi x_2}{a_2}), \\ &\mu_{m,k} = (\tfrac{m\pi}{a_1})^2 + (\tfrac{(k-0.5)\pi}{a_2})^2, \ b_{m,k} = -\tfrac{\rho a_2}{k\pi}, \end{aligned} \tag{12}$$

where $m,k \in \mathbb{N}$. We reorder the eigenvalues (7) to form a non-decreasing sequence $\{\lambda_n\}_{n=1}^\infty$ satisfying (4) and denote the corresponding eigenfunctions as $\{\phi_n\}_{n=1}^\infty$. Following the corresponding relationship between (7) and (4), we reorder $\{\psi_{m,k}\}_{m,k=0}^\infty$, $\{\mu_{m,k}\}_{m,k=0}^\infty$, and $\{b_{m,k}\}_{m,k=0}^\infty$ as $\{\psi_i\}_{i=1}^\infty$, $\{\mu_i\}_{i=1}^\infty$, and $\{b_i\}_{i=1}^\infty$. We see that $\{\mu_i\}_{i=1}^N$ satisfy (9), $\|\psi_i\|_{L^2(\mathscr{O})} = \rho$, $i \in \mathbb{N}$, and $\{\psi_i\}_{i=1}^\infty$ are linearly independent and satisfy (10).

Following [13], [16], [21], we design the control input with the shape functions in the form of eigenfunctions $\{\phi_i\}_{i=1}^N$:

$$u(x,t) = \textstyle\sum_{i=1}^N b_i \frac{\partial \phi_i(x)}{\partial \mathbf{n}} u_i(t), \ x \in \Gamma_2, \tag{13}$$

where $u_i(t)$, $i = 1, \ldots, N$ are to be designed later. Consider the change of variables:

$$\begin{aligned} &w(x,t) = z(x,t) - \psi^{\mathrm{T}}(x)\mathbf{u}(t), \\ &\psi(x) = \mathrm{col}\{\psi_i(x)\}_{i=1}^N, \quad \mathbf{u}(t) = \mathrm{col}\{u_i(t)\}_{i=1}^N. \end{aligned} \tag{14}$$

Substituting (14) into (1) we obtain

$$\begin{aligned} &\mathrm{d}w(x,t) = [\Delta w(x,t) + qw(x,t) + \psi^{\mathrm{T}}(x)\Xi_0 \mathbf{u}(t) \\ &\quad + f(w(x,t) + \psi^{\mathrm{T}}(x)\mathbf{u}(t))]\mathrm{d}t - \psi^{\mathrm{T}}(x)\mathrm{d}\mathbf{u}(t) \\ &\quad + g(w(x,t) + \psi^{\mathrm{T}}(x)\mathbf{u}(t))\mathrm{d}\mathscr{W}(t), \ x \in \mathscr{O}, \\ &w(x,t) = 0, \ x \in \partial\mathscr{O}, \quad w(x,0) = z(x,0), \end{aligned} \tag{15}$$

where $\Xi_0 = \mathrm{diag}\{-\mu_1 + q, \ldots, -\mu_N + q\}$. We treat $\mathbf{u}(t)$ as an additional state variable, subject to the dynamics:

$$\mathrm{d}\mathbf{u}(t) = [\Xi_0 \mathbf{u}(t) + \mathbf{v}(t)]\mathrm{d}t, \quad \mathbf{u}(0) = 0, \tag{16}$$

where $\mathbf{v}(t) \in \mathbb{R}^N$ is the new control input. From (15) and (16), we have the following equivalent system:

$$\begin{aligned} &\mathrm{d}w(x,t) = [\Delta w(x,t) + qw(x,t) - \psi^{\mathrm{T}}(x)\mathbf{v}(t) \\ &\quad + f(w(x,t) + \psi^{\mathrm{T}}(x)\mathbf{u}(t))]\mathrm{d}t \\ &\quad + g(w(x,t) + \psi^{\mathrm{T}}(x)\mathbf{u}(t))\mathrm{d}\mathscr{W}(t), \ x \in \mathscr{O}, \\ &w(x,t)|_{x\in\partial\mathscr{O}} = 0, \ w(x,0) = z(x,0). \end{aligned} \tag{17}$$

Present the solution to (17) as

$$w(x,t) = \textstyle\sum_{n=1}^\infty w_n(t)\phi_n(x), \quad w_n(t) = \langle w(\cdot,t), \phi_n \rangle_{\mathscr{O}}. \tag{18}$$

By differentiating $w_n(t)$ defined in (18) and using (17) and Green's first identity, we have

$$\begin{aligned} &\mathrm{d}w_n(t) = [(-\lambda_n + q)w_n(t) - \mathbf{b}_n^{\mathrm{T}}\mathbf{v}(t) + f_n(t)]\mathrm{d}t \\ &\quad + g_n(t)\mathrm{d}\mathscr{W}(t), \ t > 0, \\ &w_n(0) = \langle w(\cdot,0), \phi_n \rangle_{\mathscr{O}}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} &\mathbf{b}_n = [\langle \psi_1, \phi_n \rangle_{\mathscr{O}}, \cdots, \langle \psi_N, \phi_n \rangle_{\mathscr{O}}]^{\mathrm{T}} \\ &\quad \overset{(11)}{=} [\tfrac{-b_1}{\lambda_n - \mu_1} \langle \tfrac{\partial \phi_1}{\partial \mathbf{n}}, \tfrac{\partial \phi_n}{\partial \mathbf{n}} \rangle_{\Gamma_2}, \cdots, \tfrac{-b_N}{\lambda_n - \mu_N} \langle \tfrac{\partial \phi_N}{\partial \mathbf{n}}, \tfrac{\partial \phi_n}{\partial \mathbf{n}} \rangle_{\Gamma_2}]^{\mathrm{T}}, \\ &f_n(t) = \langle f(w(\cdot,t) + \psi^{\mathrm{T}}(\cdot)\mathbf{u}(t)), \phi_n \rangle_{\mathscr{O}}, \\ &g_n(t) = \langle g(w(\cdot,t) + \psi^{\mathrm{T}}(\cdot)\mathbf{u}(t)), \phi_n \rangle_{\mathscr{O}}. \end{aligned} \tag{20}$$

Define the notations:

$$\begin{aligned} &A_0 = \mathrm{diag}\{-\lambda_n + q\}_{n=1}^N, \quad \tilde{A} = \mathrm{diag}\{\Xi_0, A_0\}, \\ &\mathbf{B}_0 = [\mathbf{b}_1, \ldots, \mathbf{b}_N]^{\mathrm{T}}, \qquad \tilde{\mathbf{B}} = \begin{bmatrix} I_N \\ -\mathbf{B}_0 \end{bmatrix}, \\ &X(t) = \mathrm{col}\{\mathbf{u}(t), w_1(t), \ldots, w_N(t)\}, \\ &F^N(t) = \mathrm{col}\{0_{N\times 1}, f_1(t), \ldots, f_N(t)\}, \\ &G^N(t) = \mathrm{col}\{0_{N\times 1}, g_1(t), \ldots, g_N(t)\}. \end{aligned} \tag{21}$$

By Lemmas 7.1 and 7.2 in [13], the pair (A_0, B_0) is stabilizable, which implies that the pair $(\tilde{A}, \tilde{\mathbf{B}})$ is stabilizable. Let $K \in \mathbb{R}^{N\times 2N}$ be the controller gain (it will be found from LMIs (41), (42), and (43) below). We propose a controller of the form

$$\mathbf{v}(t) = -KX(t) \tag{22}$$

leading to the following closed-loop system for $t \geq 0$:

$$\mathrm{d}X(t) = [(\tilde{A} - \tilde{\mathbf{B}}K)X(t) + F^N(t)]\mathrm{d}t + G^N(t)\mathrm{d}\mathscr{W}(t), \tag{23a}$$

$$\begin{aligned} \mathrm{d}w_n(t) = {} & [(-\lambda_n + q)w_n(t) + \mathbf{b}_n^{\mathrm{T}}KX(t) + f_n(t)]\mathrm{d}t \\ & + g_n(t)\mathrm{d}\mathscr{W}(t), \quad n > N. \end{aligned} \tag{23b}$$

B. Well-posedness

For the well-posedness, we consider the state $\xi(t) = \mathrm{col}\{\mathbf{u}(t), w(\cdot,t)\}$ to obtain the following stochastic evolution equation

$$\mathrm{d}\xi(t) = [\mathscr{A}_1\xi(t) + \hat{f}_1(\xi(t)) + \hat{f}_2(\xi(t))]\mathrm{d}t + \hat{g}(\xi(t))\mathrm{d}\mathscr{W}(t), \tag{24}$$

where $\hat{\mathscr{A}} = \mathrm{diag}\{\Xi_0, -\mathscr{A}\}$, $K = [K_1, K_2]$,

$$\begin{aligned} &\hat{f}_1(\xi(t)) = \begin{bmatrix} K_1\mathbf{u}(t) + K_2 w^N(t) \\ qw(\cdot,t) + \psi^{\mathrm{T}}(\cdot)[K_1\mathbf{u}(t) + K_2 w^N(t)] \end{bmatrix}, \\ &\hat{f}_2(\xi(t)) = \begin{bmatrix} 0_{N\times 1} \\ f(w(t) + \psi^{\mathrm{T}}(\cdot)\mathbf{u}(t)) \end{bmatrix}, \hat{g}(\xi(t)) = \begin{bmatrix} 0_{N\times 1} \\ g(w(t) + \psi^{\mathrm{T}}(\cdot)\mathbf{u}(t)) \end{bmatrix}. \end{aligned}$$

Let $\mathscr{H} = \mathbb{R}^N \times L^2(\mathscr{O})$ be a Hilbert space with norm $\|\cdot\|_{\mathscr{H}}^2 = |\cdot|^2 + \|\cdot\|_{L^2(\mathscr{O})}^2$. Take $\mathscr{V} = \mathbb{R}^N \times H^1(\mathscr{O})$ with norm $\|\cdot\|_{\mathscr{V}}^2 = |\cdot|^2 + \|\cdot\|_{H^1(\mathscr{O})}^2$, and $\mathscr{V}' = \mathbb{R}^{N+1} \times H_{\partial\mathscr{O}}^{-1}(0,1)$. We see that $\hat{\mathscr{A}}$ satisfies conditions B.1-B.3 on page 198 in [25] and $\hat{f}_1(\xi)$ is a linear function of ξ. From (2), it can be verified for any $\xi_1, \xi_2 \in \mathscr{H}$,

$$\begin{aligned} &\|\hat{f}_2(\xi)\|_{\mathscr{H}}^2 + \|\hat{g}(\xi)\|_{\mathscr{H}}^2 \leq 2(\rho^2 + 1)\max\{\sigma_f^2, \sigma_g^2\}\|\xi\|_{\mathscr{H}}^2, \\ &\|\hat{f}_2(\xi_1) - \hat{f}_2(\xi_2)\|_{\mathscr{H}}^2 + \|\hat{g}(\xi_1) - \hat{g}(\xi_2)\|_{\mathscr{H}}^2 \\ &\leq 2(\rho^2 + 1)\max\{\sigma_f^2, \sigma_g^2\}\|\xi_1 - \xi_2\|_{\mathscr{H}}^2. \end{aligned}$$

Then by [25, Theorem 6.7.4], for any initial value $\xi_0 \in L^2(\Omega; \mathscr{H})$ and $\xi_0 \in \mathscr{D}(\hat{\mathscr{A}})$ almost surely, the closed loop system (24) has a unique strong solution satisfying

$$\xi \in L^2(\Omega; C([0,T]; \mathscr{H})) \cap L^2([0,T] \times \Omega; \mathscr{V})$$

for any $T > 0$, and $\xi(t) \in \mathscr{D}(\hat{\mathscr{A}})$ almost surely.

C. Mean-square L^2 exponential stability

For mean-square L^2 exponential stability of the closed-loop system (23), we consider the Lyapunov function:

$$\begin{aligned} &V(t) = V_P(t) + V_{\text{tail}}(t), \\ &V_P(t) = |X(t)|_P^2, \quad V_{\text{tail}}(t) = \textstyle\sum_{n=N+1}^{\infty} w_n^2(t), \end{aligned} \tag{25}$$

where $0 < P \in \mathbb{R}^{2N\times 2N}$. By Parseval's equality, we present $V_{\text{tail}}(t)$ in (25) as

$$\begin{aligned} &V_{\text{tail}}(t) = -V_1(t) + V_2(w(t)), \\ &V_1(t) = X^{\mathrm{T}}(t)\Lambda_1 X(t), \quad \Lambda_1 = \operatorname{diag}\{0_{N\times N}, I_N\}, \\ &V_2(w(t)) = \|w(t)\|_{L^2(\mathscr{O})}^2. \end{aligned} \tag{26}$$

For functions V_P, V_1, calculating the generator $\mathscr{L}$ along stochastic ODE (23a) (see [26, P. 149]), we have

$$\begin{aligned} &\mathscr{L}V_P(t) + 2\delta V_P(t) = X^{\mathrm{T}}(t)[P(\tilde{A} - \tilde{\mathbf{B}}K) + (\tilde{A} - \tilde{\mathbf{B}}K)^{\mathrm{T}}P \\ &+ 2\delta P]X(t) + 2X^{\mathrm{T}}(t)PF^N(t) + |G^N(t)|_P^2, \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\mathscr{L}V_1(t) + 2\delta V_1(t) = 2X^{\mathrm{T}}(t)\Lambda_1 F^N(t) + (G^N(t))^{\mathrm{T}}\Lambda_1 G^N(t) \\ &\quad + X^{\mathrm{T}}(t)[\Lambda_1(\tilde{A} - \tilde{\mathbf{B}}K) + (\tilde{A} - \tilde{\mathbf{B}}K)^{\mathrm{T}}\Lambda_1 + 2\delta\Lambda_1]X(t) \\ &= \textstyle\sum_{n=1}^{N} 2(-\lambda_n + q + \delta)w_n^2(t) + \sum_{n=1}^{N} 2w_n(t)\mathbf{b}_n^{\mathrm{T}}KX(t) \\ &\quad + \textstyle\sum_{n=1}^{N} 2w_n(t)f_n(t) + |G^N(t)|^2. \end{aligned} \tag{28}$$

From the well-posedness analysis (see Sec. II-B), we see that $w(t)$ is a strong solution to the following stochastic evolution equation:

$$\begin{aligned} &\mathrm{d}w(t) = [-\mathscr{A}w(t) + qw(t) - \psi^{\mathrm{T}}(\cdot)\mathbf{v}(t) \\ &+ f(w(t) + \psi^{\mathrm{T}}(\cdot)\mathbf{u}(t))]\mathrm{d}t + g(w(t) + \psi^{\mathrm{T}}(\cdot)\mathbf{u}(t))\mathrm{d}\mathscr{W}(t). \end{aligned} \tag{29}$$

For $V_2(w)$, calculating the generator $\mathscr{L}$ along (29) (see [25, P. 228]) we obtain

$$\begin{aligned} &\mathscr{L}V_2(w(t)) = \langle \tfrac{\partial V_2(w(t))}{\partial w}, -\mathscr{A}w(t) + qw(t) - \psi^{\mathrm{T}}\mathbf{v}(t)\rangle_{\mathscr{O}} \\ &\quad + \langle \tfrac{\partial V_2(w(t))}{\partial w}, f(w(t) + \psi^{\mathrm{T}}\mathbf{u}(t))\rangle_{\mathscr{O}} \\ &\quad + \tfrac{1}{2}\langle \tfrac{\partial^2 V_2(w(t))}{\partial w^2} g(w(t) + \psi^{\mathrm{T}}u(t)), g(w(t) + \psi^{\mathrm{T}}\mathbf{u}(t))\rangle_{\mathscr{O}} \\ &\overset{(2)}{\le} 2\langle w(t), -\mathscr{A}w(t) + qw(t) - \psi^{\mathrm{T}}\mathbf{v}(t)\rangle_{\mathscr{O}} \\ &\quad + 2\langle w(t), f(w(t) + \psi^{\mathrm{T}}\mathbf{u}(t))\rangle_{\mathscr{O}} + \sigma_g^2\|w(t) + \psi^{\mathrm{T}}\mathbf{u}(t)\|_{L^2(\mathscr{O})}^2 \\ &\le \textstyle\sum_{n=1}^{\infty} 2(-\lambda_n + q)w_n^2(t) + \sum_{n=1}^{\infty} 2w_n(t)\mathbf{b}_n^{\mathrm{T}}KX(t) \\ &\quad + \textstyle\sum_{n=1}^{\infty} 2w_n(t)f_n(t) + 2\sigma_g^2|X(t)|_{\Lambda_2}^2 + 2\sigma_g^2\sum_{n=N+1}^{\infty} w_n^2(t), \end{aligned} \tag{30}$$

where $\Lambda_2 = \operatorname{diag}\{\Psi^N, I_N\}$ with Ψ^N defined in Assumption 2. From (26), (28), and (30), it follows

$$\begin{aligned} &\mathscr{L}V_{\text{tail}}(t) + 2\delta V_{\text{tail}}(t) = 2\sigma_g^2|X(t)|_{\Lambda_2}^2 - |G^N(t)|^2 \\ &+ \textstyle\sum_{n=N+1}^{\infty} 2(-\lambda_n + q + \delta + \sigma_g^2)w_n^2(t) \\ &+ \textstyle\sum_{n=N+1}^{\infty} 2w_n(t)f_n(t) + \sum_{n=N+1}^{\infty} 2w_n(t)\mathbf{b}_n^{\mathrm{T}}KX(t). \end{aligned} \tag{31}$$

By the Young inequality, we have for $\alpha_1, \alpha_2 > 0$,

$$\begin{aligned} &\textstyle\sum_{n=N+1}^{\infty} 2w_n(t)\mathbf{b}_n^{\mathrm{T}}KX(t) \\ &\le \alpha_1\textstyle\sum_{n=N+1}^{\infty} w_n^2(t) + \frac{\|\psi\|_N^2}{\alpha_1}|KX(t)|^2, \\ &\|\psi\|_N^2 = \textstyle\sum_{i=1}^{N}\|\psi_i\|_N^2, \quad \|\psi_i\|_N^2 = \sum_{n=N+1}^{\infty}|\langle\psi_i, \phi_n\rangle_{\mathscr{O}}|^2. \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\textstyle\sum_{n=N+1}^{\infty} 2w_n(t)f_n(t) \\ &\le \alpha_2\textstyle\sum_{n=N+1}^{\infty} w_n^2(t) - \frac{1}{\alpha_2}|F^N(t)|^2 + \frac{1}{\alpha_2}\sum_{n=1}^{\infty} f_n^2(t). \end{aligned} \tag{33}$$

From Parseval's equality, we obtain

$$\begin{aligned} &\tfrac{1}{\alpha_2}\textstyle\sum_{n=1}^{\infty} f_n^2(t) = \frac{1}{\alpha_2}\|f(w(\cdot,t) + \psi^{\mathrm{T}}(\cdot)\mathbf{u}(t))\|_{L^2(\mathscr{O})}^2 \\ &\overset{(2)}{\le} \tfrac{1}{\alpha_2}\sigma_f^2\|\|w(\cdot,t) + \psi^{\mathrm{T}}(\cdot)\mathbf{u}(t)\|_{L^2(\mathscr{O})}^2 \\ &\le \tfrac{2}{\alpha_2}\sigma_f^2|X(t)|_{\Lambda_2}^2 + \tfrac{2}{\alpha_2}\sigma_f^2\textstyle\sum_{n=N+1}^{\infty} w_n^2(t). \end{aligned} \tag{34}$$

Combination of (33) and (34) implies

$$\begin{aligned} \textstyle\sum_{n=N+1}^{\infty} 2w_n(t)f_n(t) &\le (\alpha_2 + \tfrac{2\sigma_f^2}{\alpha_2})\textstyle\sum_{n=N+1}^{\infty} w_n^2(t) \\ &\quad - \tfrac{1}{\alpha_2}|F^N(t)|^2 + \tfrac{2\sigma_f^2}{\alpha_2}|X(t)|_{\Lambda_2}^2. \end{aligned} \tag{35}$$

Besides, from Parseval's equality and (2) we have

$$\begin{aligned} &|G^N(t)|^2 = \textstyle\sum_{n=1}^{N} g_n^2(t) \le \sum_{n=1}^{\infty} g_n^2(t) \\ &\le \sigma_g^2\|w(\cdot,t) + \psi^{\mathrm{T}}(\cdot)u(t)\|_{L^2}^2 \\ &\le 2\sigma_g^2|X(t)|_{\Lambda_2}^2 + 2\sigma_g^2\textstyle\sum_{n=N+1}^{\infty} w_n^2(t). \end{aligned} \tag{36}$$

Let $\eta(t) = \operatorname{col}\{X(t), F^N(t)\}$. Combining (27), (31), (32), (35), and using (36) and S-procedure, we obtain for $\beta > 0$,

$$\begin{aligned} &\mathscr{L}V(t) + 2\delta V(t) \\ &\quad + \beta[2\sigma_g^2|X(t)|_{\Lambda_2}^2 + 2\sigma_g^2\textstyle\sum_{n=N+1}^{\infty} w_n^2(t) - |G^N(t)|^2] \\ &\le [G^N(t)]^{\mathrm{T}}\Phi_1 G^N(t) + \eta^{\mathrm{T}}(t)\Phi_2\eta(t) \\ &\quad + \textstyle\sum_{n=N+1}^{\infty} 2\Upsilon_n w_n^2(t) < 0 \end{aligned} \tag{37}$$

provided $\Upsilon_n := -\lambda_n + q + \delta + \frac{\alpha_1+\alpha_2}{2} + \frac{\sigma_f^2}{\alpha_2} + (\beta+1)\sigma_g^2 < 0$, $n > N$, and

$$\Phi_1 = P - (\beta+1)I < 0, \tag{38a}$$

$$\Phi_2 = \begin{bmatrix} \phi + \frac{2\sigma_f^2}{\alpha_2}\Lambda_2 + \frac{\|\psi\|_N^2}{\alpha_1}K^{\mathrm{T}}K & P \\ * & -\frac{1}{\alpha_2}I \end{bmatrix} < 0, \tag{38b}$$

$$\phi = P(\tilde{A} - \tilde{\mathbf{B}}K) + (\tilde{A} - \tilde{\mathbf{B}}K)^{\mathrm{T}}P + 2\delta P + 2\sigma_g^2(\beta+1)\Lambda_2.$$

From the monotonicity of λ_n, $n \in \mathbb{N}$, we have $\Upsilon_n < 0$, $n > N$ iff

$$-\lambda_{N+1} + q + \delta + \tfrac{\alpha_1+\alpha_2}{2} + \tfrac{\sigma_f^2}{\alpha_2} + (\beta+1)\sigma_g^2 < 0. \tag{39}$$

To obtain equivalent LMIs for the design of the gain K, we denote

$$Q = P^{-1}, \; Y = P^{-1}K^{\mathrm{T}} = QK^{\mathrm{T}}, \; \hat{\beta} = \tfrac{1}{\beta+1} \in (0,1). \tag{40}$$

Multiplying Φ_2 from the left and right by $\operatorname{diag}\{P^{-1}, I\}$, and applying Schur complement, we find that (38b) holds iff

$$\begin{bmatrix} \Theta & \sigma_f Q & Y & \sigma_g Q \\ * & -\frac{\alpha_2}{2}\Lambda_2^{-1} & 0 & 0 \\ * & * & -\alpha_1\|\psi\|_N^{-2} & 0 \\ * & * & * & -\frac{\hat{\beta}}{2}\Lambda_2^{-1} \end{bmatrix} < 0, \tag{41}$$

$$\Theta = \tilde{A}Q + Q\tilde{A}^{\mathrm{T}} - \tilde{\mathbf{B}}Y^{\mathrm{T}} - Y\tilde{\mathbf{B}}^{\mathrm{T}} + 2\delta Q + \alpha_2 I.$$

Multiplying Φ_1 from the left and right by $(1+\beta)^{-\frac{1}{2}}P^{-\frac{1}{2}}$, we find that (38a) holds iff

$$\hat{\beta}I - Q < 0. \tag{42}$$

Moreover, by Schur complement, (39) is equivalent to

$$\begin{bmatrix} -\lambda_{N+1} + q + \delta + \frac{\alpha_2}{2} + \frac{\alpha_1}{2} & \sigma_g & \sigma_f \\ * & -\hat{\beta} & 0 \\ * & * & -\alpha_2 \end{bmatrix} < 0 \tag{43}$$

In particular, (41), (42) and (43) are LMIs in Q, Y, α_1, α_2 and $\hat{\beta}$. If (41), (42), and (43) are feasible, the controller gain is obtained by $K = Y^{\mathrm{T}}Q^{-1}$. Summarizing, we have

Theorem 1: Consider system (15) with a globally Lipschitz f,g satisfying (2) for some $\sigma_f, \sigma_g > 0$ and the control law (22). Assume that $z_0 \in \mathscr{D}(\mathscr{A})$ almost surely and $z_0 \in L^2(\Omega; L^2(0,1))$. Let $\delta > 0$ be a desired decay rate and $N \in \mathbb{N}$ satisfy (5). Let there exist scalars $\alpha_1, \alpha_2, \hat{\beta} > 0$, matrices $Y \in \mathbb{R}^{2N\times N}$, $0 < Q \in \mathbb{R}^{2N\times 2N}$ such that LMIs (41), (42), and (43) are feasible. Then the proportional-integral control law (16), (22) with $K = Y^{\mathrm{T}}Q^{-1}$ exponentially stabilizes (15) in the mean square with a decay rate δ, i.e. the following inequality holds

$$\mathbb{E}[u^2(t) + \|w(\cdot,t)\|^2_{L^2(\mathscr{O})}] \le M\mathrm{e}^{-2\delta t}\mathbb{E}\|z_0(\cdot)\|^2_{L^2(\mathscr{O})}, \quad (44)$$

for $t \ge 0$ and some $M > 0$. The LMIs (41), (42), and (43) are always feasible provided $\sigma_f, \sigma_g > 0$ are small enough.

Proof: First, employ Itô's formula for $\mathrm{e}^{2\delta t}V_P(t)$ and $\mathrm{e}^{2\delta t}V_1(t)$ along stochastic ODE (23a) (see [26, Theorem 4.18]) and infinite-dimensional Itô's formula for $\mathrm{e}^{2\delta t}V_2(w(t))$ along (29) (see [25, Theorem 7.2.1]), respectively. Then by taking expectation on both sides and using (25), (26) (see arguments similar to (2.66)-(2.68) in [6]), we can obtain (44).

We claim next that (41), (42), and (43) (i.e., (38) and (39)) are feasible for small enough $\sigma_f, \sigma_g > 0$ and large enough N. Choose $\alpha_1 = \lambda_{N+1}$, $\alpha_2 = \sigma_f$, and $\rho = 1$. From $\|\psi\|_N^2$ defined below (33) and the fact that $\|\psi_i\|^2_{L^2} = \rho = 1$, we obtain $\|\psi\|_N^2 \le N$. By Lemma 1, we have $\frac{1}{\alpha}\|\psi\|_N^2 < \frac{2N}{\lambda_{N+1}} \to \frac{|\mathscr{O}|}{2\pi}$, $N \to \infty$. Therefore, there exists N-independent $\chi_0 > 0$ such that $\frac{1}{\alpha}\|\psi\|_N^2 \le \chi_0$ for all N. Fix N_0 such that

$$-\lambda_n + q + \delta < 0, \ -\mu_n + q < 0, \quad n > N_0.$$

Design $K \in \mathbb{R}^{N\times 2N}$ and $P \in \mathbb{R}^{2N\times 2N}$ be of the form

$$K = \left[\begin{array}{cc|cc} K_u & 0 & K_w & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \ P = \left[\begin{array}{cc|cc} P_{11} & 0 & P_{12} & 0 \\ * & p_u I & 0 & 0 \\ \hline * & * & P_{22} & 0 \\ * & * & * & p_w I \end{array}\right],$$

where $p_u, p_w > 0$, $K_u, K_w \in \mathbb{R}^{N_0\times N_0}$ and $0 < P_0 = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} \in \mathbb{R}^{2N_0\times 2N_0}$ are to be determined later. Let

$$\hat{B}_0 = \left[\begin{array}{c} I_{N_0} \\ \hline -\begin{bmatrix} \langle\psi_1,\phi_1\rangle_{\mathscr{O}} & \cdots & \langle\psi_{N_0},\phi_1\rangle_{\mathscr{O}} \\ \vdots & \ddots & \vdots \\ \langle\psi_1,\phi_{N_0}\rangle_{\mathscr{O}} & \cdots & \langle\psi_{N_0},\phi_{N_0}\rangle_{\mathscr{O}} \end{bmatrix} \end{array}\right], \ K_0 = [K_u, K_w],$$

$$\hat{A}_0 = \mathrm{diag}\{-\mu_1, \ldots, -\mu_{N_0}, -\lambda_1, \ldots, -\lambda_{N_0}\} + qI_{2N_0}.$$

By using Schur complement and choosing $p_u, p_w = \frac{1}{N}$, we find that (38b) holds for $N \to \infty$ if

$$P_0(\hat{A}_0 - \hat{B}_0K_0) + (\hat{A}_0 - \hat{B}_0K_0)^{\mathrm{T}}P_0 + 2\delta P_0 + \chi_0 K_0^{\mathrm{T}}K_0 < 0. \quad (45)$$

Since the pair $(\tilde{A}_0, \mathbf{B}_0)$ is stabilizable, the pair $(\hat{A}_0, \hat{B}_0)$ is also stabilizable. We can choose $K_0 = [K_u, K_w] \in \mathbb{R}^{N_0\times 2N_0}$ such that $\hat{A}_0 - \hat{B}_0K_0 + \delta I$ is Hurwitz. Let $P_0 \in \mathbb{R}^{2N_0\times 2N_0}$ be such that

$$P_0(\hat{A}_0 - \hat{B}_0K_0 + \delta I) + (\hat{A}_0 - \hat{B}_0K_0 + \delta I)^{\mathrm{T}}P_0 = -\chi I, \quad (46)$$

where $\chi > 0$ is independent of N and satisfies $-\chi I + \chi_0 K_0^{\mathrm{T}}K_0 < 0$. Then $P_0 = O(1)$, $N \to \infty$. Substituting (46) into (45), we find that (45) is feasible. Since we take $p_u, p_w = \frac{1}{N} < 1$, we obtain $P = O(1)$, $N \to \infty$. Take $\beta = N$. It is obvious that (38a) and (39) hold true for $N \to \infty$. By continuity, (38) and (39) are feasible for small enough $\sigma_f, \sigma_g = \frac{1}{N^2}$ and large enough $N \to \infty$. ■

***Remark** 5:* For fixed σ_g, we find that comparatively smaller ρ leads to larger σ_f. However, we cannot let $\rho \to 0^+$, otherwise we have open-loop control with $u(x,t) \to 0$. To optimize the value of ρ we can choose a grid of ρ and solve LMIs (41)-(43) to find maximum σ_f,

D. Discussion on 1D case

For 1D case, we consider system (1) in $\mathscr{O} = (0,a)$, $a > 0$ with the boundary $\partial\mathscr{O} = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 = \{0\}$, $\Gamma_2 = \{a\}$. Consider the operator (3). The eigenvalues and corresponding eigenfunctions of $\mathscr{A}$ are as follows:

$$\lambda_n = \frac{n^2\pi^2}{a^2}, \ \phi_n(x) = \frac{\sqrt{2}}{\sqrt{a}}\sin(\frac{n\pi}{a}x), \ n \ge 1. \quad (47)$$

The eigenfunctions form a complete orthonormal system in $L^2(0,a)$. Following [9], [22] for 1D deterministic case, we take

$$\psi_i(x) = \frac{(-1)^{i+1}\sqrt{2}\rho}{\sqrt{a}}\sin(\sqrt{\mu_i}x), \quad \mu_i = \frac{(i-\frac{1}{2})^2\pi^2}{a^2}, \quad i \in \mathbb{N}, \ (48)$$

where $\rho > 0$ is the tuning parameter. It can be easily verified that $\{\psi_i\}_{i=1}^{\infty}$ satisfy

$$\psi_i''(x) + \mu_i\psi_i(x) = 0, \quad \|\psi_i\|_{L^2(\mathscr{O})} = \rho,$$
$$\psi_i(0) = \psi_i'(a) = 0, \quad \psi_i(a) = \sqrt{2}\rho.$$

In particular, $\{\psi_i\}_{i=1}^{\infty}$ is an orthogonal family. We have $\langle\psi_i,\phi_n\rangle_{\mathscr{O}} = \frac{\sqrt{2}\rho\phi_n'(a)}{\mu_i-\lambda_n}$. Consider the control input $u(t) = \sum_{i=1}^{N} u_i(t)$. Since the eigenvalues λ_n are simple, the pair $(\tilde{A}, \tilde{\mathbf{B}})$ (see (21)) is controllable (see Lemma 2.1 in [22]). Following the arguments similar to (14)-(43), we find that if LMIs (41), (42), and (43) (where λ_n, ϕ_n are replaced by (47) and ψ_i, μ_i are replaced by (48)) are feasible, the considered systems in domain $\mathscr{O} = (0,a)$, $a > 0$ is mean-square L^2 exponentially stable with decay rate $\delta > 0$.

***Remark** 6:* The 1D case in this section corresponds to [9], [22] for the deterministic case with $\rho = \frac{1}{\sqrt{2}}$. In our example, we show that an appropriate choice of ρ leads to larger Lipschitz constants.

III. Numerical Examples

In this section, we first consider system (17) in the square domain $\mathscr{O} = (0,a_1) \times (0,a_2)$ with $a_1 = a_2 = 1$ and the boundary (6). The nonlinear functions f and g satisfy (2). We fix either $\sigma_g = 0.1$ or $\sigma_g = 0.2$. We take $q = 49.4$, which results in an unstable open-loop system for $f(z) \equiv 0$ with 3 unstable modes. Choose $\delta = 10^{-3}$ and $\rho = 0.01, 0.1, \frac{1}{\sqrt{2}}$, respectively. The LMIs (41), (42), and (43) were verified for $N \in \{3, \ldots, 10\}$ to obtain $\sigma_f^{\max}$ (the maximal value of σ_f) which preserves the feasibility. The results are given in Table I. From Table I, we see that $\rho = 0.01$ and $\rho = 0.1$ lead to larger $\sigma_f^{\max}$ than $\rho = 1/\sqrt{2}$ (here $\rho = 1/\sqrt{2}$ corresponds to [9]). Note that for the multiple λ_{N+1}, $\sigma_f^{\max}$ is affected by $\|\psi\|_N^2$ which is increasing for larger N.

TABLE I

2D CASE: $\sigma_f^{\max}$ FOR $N \in \{3,\dots,10\}$.

N	λ_{N+1}	$\frac{\|\psi\|_N^2}{\rho^2}$	$\sigma_f^{\max}$ $(\sigma_g = 0.2)$			$\sigma_f^{\max}$ $(\sigma_g = 0.1)$		
			$\rho = \frac{1}{\sqrt{2}}$	$\rho = 10^{-1}$	$\rho = 10^{-2}$	$\rho = \frac{1}{\sqrt{2}}$	$\rho = 10^{-1}$	$\rho = 10^{-2}$
3	$8\pi^2$	0.65	0.33	1.12	1.13	–	0.38	0.39
4	$10\pi^2$	0.75	0.53	1.45	1.46	–	0.71	0.72
5	$10\pi^2$	1.03	0.46	1.34	1.35	–	0.60	0.61
6	$13\pi^2$	1.09	1.08	2.40	2.41	0.32	1.65	1.67
7	$13\pi^2$	1.18	1.05	2.36	2.38	0.30	1.62	1.63
8	$17\pi^2$	1.24	1.23	2.65	2.67	0.48	1.91	1.93
9	$17\pi^2$	1.52	1.18	2.58	2.60	0.44	1.84	1.86
10	$18\pi^2$	1.55	1.41	2.98	3.00	0.67	2.24	2.26

We next consider the 1D case with $q = 3\pi^2$ in domain $\mathscr{O} = (0,1)$ with boundary $\partial\mathscr{O} = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 = \{0\}$, $\Gamma_2 = \{1\}$. We fix either $\sigma_g = 0.2$ or $\sigma_g = 0.4$. Choose $\delta = 10^{-3}$ and $\rho \in \{\frac{1}{\sqrt{2}}, 0.1, 0.01\}$. The LMIs (41), (42), and (43) (where λ_n, ϕ_n are replaced by (47) and ψ_i, μ_i are replaced by (48)) were verified for $N \in \{2,\dots,8\}$ to obtain $\sigma_f^{\max}$ which preserves the feasibility. The results are given in Table II. From Table II, we see that $\rho = 0.1$ and $\rho = 0.01$ lead to larger $\sigma_f^{\max}$ than $\rho = 1/\sqrt{2}$ (here $\rho = 1/\sqrt{2}$ corresponds to [9], [22]).

TABLE II

1D CASE: $\sigma_f^{\max}$ FOR $N \in \{2,\dots,8\}$.

N	$\sigma_f^{\max}$ $(\sigma_g = 0.2)$			$\sigma_f^{\max}$ $(\sigma_g = 0.4)$		
	$\rho = \frac{1}{\sqrt{2}}$	$\rho = 10^{-1}$	$\rho = 10^{-2}$	$\rho = \frac{1}{\sqrt{2}}$	$\rho = 10^{-1}$	$\rho = 10^{-2}$
2	1.97	2.67	2.69	0.71	1.42	1.44
3	2.95	3.87	3.89	1.70	2.62	2.65
4	3.48	4.51	4.54	2.24	3.28	3.31
5	3.83	4.93	4.96	2.59	3.70	3.73
6	4.06	5.22	5.25	2.84	4.00	4.03
7	4.24	5.44	5.47	3.02	4.22	4.25
8	4.38	5.60	5.64	3.16	4.39	4.42

IV. CONCLUSIONS

In this paper, we considered the state-feedback global stabilization of stochastic semilinear 2D parabolic PDEs with nonlinear multiplicative noise. Improvements and extension of current results to various high-dimensional PDEs may be topics for future research.

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