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Bilinear Controllability of a Simple Reparable System

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***Abstract*— Reparable systems are systems that are characterized by their ability to undergo maintenance actions when failures occur. These systems are often described by transport equations, all coupled through an integro-differential equation. In this paper, we address the understudied aspect of the controllability of reparable systems. In particular, we focus on a two-state reparable system and our goal is to design a control strategy that enhances the system availability- the probability of being operational when needed. We establish bilinear controllability, demonstrating that appropriate control actions can manipulate system dynamics to achieve desired availability levels. We provide theoretical foundations and develop control strategies that leverage the bilinear structure of the equations.**

I. INTRODUCTION

In the realm of reliability and maintenance, reparable systems play a pivotal role, where maintenance strategies are carefully devised to promptly address failures and minimize total breakdown. Reparable systems occur naturally in problems of product design, inventory systems, computer networking, electrical power system and complex manufacturing processes. These systems are characterized by their ability to undergo repair actions when failures occur, ensuring their continuous functionality. Extensive research has been conducted in this domain, particularly focusing on systems with arbitrarily distributed repair times, often governed by complex systems of coupled partial and integro-differential hybrid equations (e.g., [2], [7], [8], [9], [10], [11], [13], [18]).

The primary emphasis in much of this prior research was concerned with the well-posedness of mathematical models and its the asymptotic behavior. While these endeavors have contributed significantly to our understanding of the reparable systems, our current work takes a distinct perspective. We seek to shed light on the critical issue of controllability concerning maintenance strategies with a specific aim: enhancing the availability of reparable systems with prescribed demands.

Availability, in this context, refers to the probability that a reparable system remains operational and free from failure or undergoing repair actions when it is needed for use [15], [16]. To address this concern, this work mainly focuses on a particular class of reparable multi-state systems with arbitrarily distributed repair time, initially introduced by Chung [2]. The system is characterized by coupled transport and integro-differential equations, which proposing a challenging yet realistic model for various real-world applications.

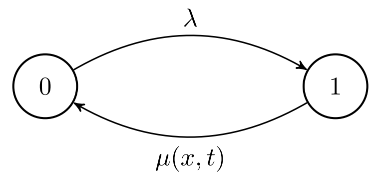

Fig. 1. Transition diagram of the reparable two-state system

In [2], it is assumed that there are M distinct failure modes associated with a device and initially, the device is in the good mode, denoted by 0. Transitions between states 0 and j are allowed, with $j = 1, 2, \ldots, M$, which are determined by failure rates and repair rates. Repair times, on the other hand, follow arbitrary distributions, adding an element of unpredictability to the maintenance process. The repair actions undertaken in our model are naturally likened to corrective maintenance, which serves to restore a failed system to operational status. In fact, optimal repair rate design of such systems has been discussed in our previous work [1], [14]. It gave rise to a bilinear open-loop control problem. In contrast, our current is aimed at constructing a space-time dependent repair rate design for achieving exact controllability of the system under certain conditions.

In this work, we focus on a simplified scenario where the reparable system has only one failure mode and the repair rate is allowed to depend on the system running time which is more realistic (see Fig. 1). While one failure mode may seem restrictive, it captures the essence of the original model and serves the purpose to convey the core principles of our maintenance strategy design.

The precise model characterizing the two-state reparable system reads

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \int_0^L \mu(x) p_1(x,t)\, dx, \qquad (1)$$

$$\frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -\mu(x) p_1(x,t), \qquad (2)$$

with boundary condition

$$p_1(0,t) = \lambda p_0(t), \qquad (3)$$

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and nonnegative initial conditions

$$p_0(0) = p_{0_0}, \quad p_1(x, 0) = p_{1_0}(x). \tag{4}$$

Here

1) $p_0(t)$: probability that the device is in good mode 0 at time $t > 0$;
2) $p_1(x, t)$: probability density (with respect to repair time x) that the failed device is in failure mode 1 at time $t > 0$ and has an elapsed repair time of $x \in [0, L]$ for $0 < L < \infty$. Let $\hat{p}_1(t)$ denote the probability of the device in failure mode 1 at time t, then $\hat{p}_1(t)$ is given by
$$\hat{p}_1(t) = \int_0^L p_1(x, t)\, dx; \tag{5}$$
3) $\lambda > 0$: constant failure rate of the device for failure mode 1;
4) $\mu(x) \geq 0$: repair rate of the device with an elapsed repair time x. Assume that for $0 < l < L$,
$$\int_0^l \mu(x)\, dx < \infty \quad \text{and} \quad \int_0^L \mu(x)\, dx = \infty. \tag{6}$$
5) The initial probability distributions of the system in good and failure modes satisfy
$$p_{0_0} + \int_0^L p_{1_0}(x)\, dx = 1. \tag{7}$$

The following assumptions are associated with the device:

1) The failure rates are constant;
2) All failures are statistically independent;
3) All repair time of failed devices are arbitrarily distributed;
4) There is only one mode of failure denoted by 1, and 0 implies the good state;
5) Repair is to like-new and it does not cause damage to any other part of the system.
6) Transitions are permitted only between states 0 and 1;
7) The repair process begins soon after the device is in failure state;
8) The repaired device is as good as new;
9) No further failure can occur when the device has been down.

A. Well-posedness and stability of the Model

The well-posedness and stability issues of system (1)–(4) for given failure and time-independent repair rates have been well studied in [11], [12], [18] by using C_0-semigroup theory in a nonreflexive Banach space $X = \mathbb{R} \times L^1(0, L)$ equipped with the norm $\| \cdot \|_X = | \cdot | + \| \cdot \|_{L^1}$. It is proven in [18] that system operator generates a positive C_0-semigroup of contraction. Thus the solution to (1)–(4) is nonnegative if the initial data are nonnegative. In fact, using the method of characteristics we get

$$p_0(t) = p_{0_0} e^{-\lambda t} + \int_0^t e^{-\lambda(t-\tau)} \int_0^\tau \mu(x) p_1(x, \tau) dx d\tau.$$

and

$$p_1(x, t) = \begin{cases} \lambda p_0(t - x) e^{-\int_0^x \mu(s) ds}, & x < t, \\ p_{1_0}(x - t) e^{-\int_{x-t}^x \mu(s)\, ds}, & x \geq t. \end{cases}$$

It is easy to verify that $p_0 \in W^{1,\infty}(0, \infty)$ and $p_1 \in L^\infty(0, \infty; W^{1,1}(0, L))$ for $t > x$, which is independent the regularity of the initial data. For $x \geq t$, the same regularity results hold when $p_{1_0} \in W^{1,1}(0, L)$. Moreover,

$$\int_0^L \mu(x)\, dx = \infty \implies p_1(L, t) = 0, \tag{8}$$

which indicates that the probability density distribution of the system in failure mode becomes zero once the repair time reaches its maximum. As a result, one can show that $\frac{dp_0}{dt} + \frac{\partial \int_0^L p_1(x,t)\, dx}{\partial t} = 0$, and hence from (7)

$$p_0(t) + \int_0^L p_1(x, t)\, dx = 1, \tag{9}$$

for $\forall t > 0$. Furthermore, it can be shown that zero is a simple eigenvalue of the system operator and also a unique spectrum on the imaginary axis. The C_0-semigroup generated by the system operator is eventually compact. As a result, the time-dependent solution exponentially converges to the its steady-state solution, which is the eigenfunction associated with the zero eigenvalue given by

$$P_{ss}(x) = \left(p_{0_{ss}}, \lambda p_{0_{ss}} e^{-\int_0^x \mu(s)\, ds}\right)^T. \tag{10}$$

The detailed proof can be found in [11], [12].

The present work will investigate the bilinear controllability of the system via a space-time dependent repair rate $\mu(x, t)$. Due to the properties of nonnegativity and conservation of the system, the desired states should also satisfy these attributes described by (8)–(9) and the boundary condition (3). Moreover, we assume that the desired probability density distribution of the system in failure mode is a strictly decreasing function of the same regularity of the system solution. In other words, while under repair, it is not expected that the desired density distribution of the failure rate increases.

B. Problem Statment

Given $t_f > 0$ and a nonnegative initial datum $P_0(x) = (p_{0_0}, p_{1_0}(x))^T \in X$ satisfying (7), let $P^*(x) =$

$(p_0^*, p_1^*(x))^T \in X$ be a desired nonnegative distribution satisfying $\|P_0\|_X = \|P^*\|_X = 1$, that is,

$$p_0(0) + \int_0^L p_1(x,0)\mathrm{d}x = p_0^* + \int_0^L p_1^*(x)\mathrm{d}x = 1. \tag{11}$$

Moreover, assume

$$p_1^*(0) = \lambda p_0^*,\ p_1^*(L) = 0,\ p_1^* \in W^{1,1}(0,L), \tag{12}$$

$$p_1^*(x) > 0 \quad \text{and} \quad \frac{dp_1^*(x)}{dx} < 0,\ \forall x \in (0,L). \tag{13}$$

Determine whether there exists a space-time dependent repair rate $\mu(x,t)$ such that the solution $P(x,t) = (p_0(t), p_1(x,t))^T$ to (1)–(2) satisfies $P(\cdot, t_f) = P^*(\cdot)$.

II. Bilinear Control Design

First of all, observe that if the repair rate is time-independent, we set the steady-state solution in (10) to be the desired distribution $(p_0^*, p_1^*(x))$, i.e., $p_{0_{ss}} = p_0^*$ and $\lambda p_{0_{ss}} e^{-\int_0^x \mu(s)\,ds} = p_1^*(x)$. We obtain

$$\mu(x) = (-\ln p_1^*(x))' = -\frac{p_{1_x}^*}{p_1^*}, \tag{14}$$

which satisfies (6). This also implies that if the repair rate $\mu(x)$ satisfies (14), then the system solution converges to $(p_0^*, p_1^*(x))$ exponentially.

Next we investigate the bilinear controllability of the repair rate when it is allowed to depend on system running time t. Note that for any $t_f > 0$ we can always choose a constant $c_0 > 0$ such that $c_0 \sum_{k=1}^{\infty} \frac{1}{k^2} = t_f$. In fact, $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ and hence $c_0 = \frac{6t_f}{\pi^2}$. Inspired by [3], [4], we consider the space-time dependent repair rate $\mu(x,t)$ given by

$$\mu(x,t) = -\frac{p_{1_x}}{p_1} + \alpha i \frac{(g(x)p_1)_x}{p_1}, \tag{15}$$

for $t \in [c_0 \sum_{k=1}^{i-1} \frac{1}{k^2}, c_0 \sum_{k=1}^{i} \frac{1}{k^2})$ and $i \in \mathbb{Z}^+$, where $g(x) = \frac{1}{p_1^*(x)}$ and $\alpha > 0$ is a constant to be properly chosen. Here we set $\sum_{k=1}^{i} \frac{1}{k^2} = 0$ if $i = 0$ and let $t_i = c_0 \sum_{k=1}^{i} \frac{1}{k^2}, i \in \mathbb{Z}^+$, in the rest of our discussion. In this case, μ is piecewise defined in time t and it is straightforward to verify that μ satisfies (6) if $\alpha \geq \sup_{x\in[0,L]} |p_1^*(x)| = p_1^*(0)$.

Our main result of this work is stated as follows.

Theorem 2.1: Given a nonnegative initial datum $P_0 = (p_{0_0}, p_{1_0})^T$ with $\|P_0\|_X = 1$, for any nonnegative $P^* = (p_0^*, p_1^*)^T \in X$ satisfying conditions (11)–(13), there exists a space-time dependent repair rate μ defined by (15), such that the solution $P(\cdot,t) = (p_0(t), p_1(\cdot,t))^T$ to (1)–(4) satisfies $P^*(\cdot, t_f) = (p_0^*, p_1^*(\cdot))^T$.

The proof of Theorem 2.1 contains several components. We first establish the well-posedness and stability analysis of the closed-loop system.

A. Well-posedness of the Closed-Loop System

Replacing μ by (15) in (1)–(2) leads to the following closed-loop system

$$\frac{dp_{0,i}(t)}{dt} = \alpha i \int_0^L \frac{\partial (g(x)p_{1,i})}{\partial x}\mathrm{d}x, \tag{16}$$

$$\frac{\partial p_{1,i}(x,t)}{\partial t} = -\alpha i \frac{\partial (g(x)p_{1,i})}{\partial x}, \tag{17}$$

where $p_{0,i}(t) := p_0(t)$ and $p_{1,i}(x,t) := p_1(x,t)$ for $(x,t) \in (0,L) \times (t_{i-1}, t_i)$ and $i \in \mathbb{Z}^+$, with boundary condition

$$p_{1,i}(0,t) = \lambda p_{0,i}(t) \tag{18}$$

and the initial conditions

$$p_{0,i}(0) = p_{0,i-1}(t_{i-1}), \tag{19}$$

$$p_{1,i}(x,0) = p_{1,i-1}(x, t_{i-1}). \tag{20}$$

We first solve the solution to the closed-loop system (16)–(17) using the method of characteristics.

Let $\phi_{0,i}(t) = p_{0,i}(t)$ and $\phi_{1,i}(x,t) = \alpha i g(x) p_{1,i}(x,t)$ for $t \in (t_{i-1}, t_i)$ and $i \in \mathbb{Z}^+$. Then (16)–(17) become

$$\frac{d\phi_{0,i}(t)}{dt} = \int_0^L \frac{\partial \phi_{1,i}}{\partial x}\mathrm{d}x, \tag{21}$$

$$\frac{\partial \phi_{1,i}}{\partial t} = -\alpha i g(x) \frac{\partial \phi_{1,i}}{\partial x}, \tag{22}$$

with boundary condition

$$\phi_{1,i}(0,t) = \alpha i g(0) p_{1,i}(0,t) = \alpha i g(0) \lambda \phi_{0,i}(t) \tag{23}$$

and initial conditions

$$\phi_{0,i}(0) = p_{0,i}(0), \quad \phi_{1,i}(x,0) = \alpha i g(x) p_{1,i}(x,0). \tag{24}$$

Let $\frac{dx}{dt} = \alpha i g(x), \quad x(0) = x_0$. Then $\frac{dx}{\alpha i g(x)} = \frac{1}{\alpha i} p_1^*(x)\mathrm{d}x = dt$. Let $\tilde{p}_{1,i}^*(x) = \frac{1}{\alpha i}\int_0^x p_1^*(s)\,ds$. Then $\tilde{p}_{1,i}^*(x) = t + \frac{1}{\alpha i}\int_0^{x_0} p_1^*(s)\,ds$. Since $\frac{d\tilde{p}_{1,i}^*}{dx} = \frac{1}{\alpha i} p_1^* > 0$ for $x \in (0,L)$ by (13), this implies that $\tilde{p}_{1,i}^*(x)$ is a strictly monotonically increasing function for $x \in [0,L]$, and hence invertible. Let $\xi = \tilde{p}_{1,i}^*(x) - t$. Then $x = (\tilde{p}_{1,i}^*)^{-1}(\xi + t)$. Define $\Psi_{1,i}(t) = \phi_{1,i}((\tilde{p}_{1,1}^*)^{-1}(t+\xi), t)$. Then

$$\frac{d\Psi_{1,i}}{dt} = \alpha i g(x) \frac{\partial \phi_{1,i}}{\partial x} + \frac{\partial \phi_{1,i}}{\partial t} = 0. \tag{25}$$

For $\xi < 0$, i.e., $\tilde{p}_{1,i}(x) < t$, the solution is determined by the boundary condition, so we integrate (25) from some t such that $x = (\tilde{p}_1^*)^{-1}(\xi + t) = 0$, i.e., $\xi + t = 0$, and hence $t = -\xi$. Integrating (25) from $-\xi$ to t follows

$$\begin{aligned}\Psi_{1,i}(t) &= \phi_{1,i}(0, -\xi) = \phi_{1,i}(0, -\tilde{p}_{1,i}^*(x) + t)) \\ &= \alpha i g(0) \lambda \phi_{0,i}\left(t - \tilde{p}_{1,i}^*(x)\right).\end{aligned} \tag{26}$$

For $\xi \geq 0$, i.e., $\tilde{p}_{1,i}^*(x) \geq t$, the solution is determined by the initial condition. So we integrate (25) from 0 to t and obtain

$$\begin{aligned}\Psi_{1,i}(t) &= \phi_{1,i}((\tilde{p}_{1,i}^*)^{-1}(\xi), 0)\\ &= \phi_{1,i}((\tilde{p}_{1,i}^*)^{-1}(\tilde{p}_{1,i}^*(x) - t), 0). \end{aligned} \tag{27}$$

To simplify the notation, we let

$$\psi(x,t) = (\tilde{p}_{1,i}^*)^{-1}(\tilde{p}_{1,i}^*(x) - t), \quad t_{i-1} < t \leq \tilde{p}_{1,i}^*(x).$$

Therefore,

$$\phi_{1,i}(x,t) = \begin{cases} \phi_{1,i}(\psi(x,t), 0), \quad t \leq \tilde{p}_{1,i}^*(x); \\ \alpha i g(0)\lambda \phi_{0,i}(t - \tilde{p}_{1,i}^*(x)), \ t > \tilde{p}_{1,i}^*(x). \end{cases}$$

Solving $\phi_{0,i}$ from (21) yields

$$\begin{aligned}\phi_{0,i}(t) &= \int_0^t \big(\phi_{1,i}(L,\tau) - \phi_{1,i}(0,\tau)\big)\, d\tau + \phi_{0,i}(0)\\ &= \begin{cases} \displaystyle\int_0^t \big(\phi_{1,i}(\psi(L,\tau),0) - \alpha i g(0)\lambda\phi_{0,i}(\tau)\big)\, d\tau \\ \qquad +\phi_{0,i}(0), \quad t \leq \tilde{p}_{1,i}^*(L); \\ \alpha i g(0)\lambda \int_0^t \big(\phi_{0,i}(\tau - \tilde{p}_{1,i}^*(L)) - \phi_{0,i}(\tau)\big)\, d\tau \\ \qquad +\phi_{0,i}(0), \quad t > \tilde{p}_{1,i}^*(L). \end{cases}\end{aligned}$$

Therefore, for any initial datum $P_0 = (p_{0_0}, p_{1,0})^T \in X$, if the solution $P_i(x,t) = (p_{0,i}(t), p_{1,i}(x,t))^T$ to (16)–(20) exists, then it is given by

$$p_{0,i}(t) = \begin{cases} \alpha i \displaystyle\int_0^t \big(g(\psi(L,\tau))p_1(\psi(L,\tau),0) \\ -g(0)\lambda p_0(\tau)\big)\, d\tau + p_{0,i}(0), \ t \leq \tilde{p}_{1,i}^*(L); \\ \alpha i g(0)\lambda \int_0^t \big(p_0(\tau - \tilde{p}_{1,i}^*(L)) - p_0(\tau)\big)\, d\tau \\ +p_{0,i}(0), \ t > \tilde{p}_{1,i}^*(L), \end{cases} \tag{28}$$

and

$$p_{1,i}(x,t) = \begin{cases} \dfrac{1}{g(x)} g(\psi(x,t))p_1(\psi(x,t),0), \\ \qquad t \leq \tilde{p}_{1,i}^*(x); \\ \frac{1}{g(x)} g(0)\lambda p_0(t - \tilde{p}_{1,i}^*(x)), \\ \qquad t > \tilde{p}_{1,i}^*(x). \end{cases} \tag{29}$$

To focus on our discussion, we first assume that $0 < p_1^*(0) \leq 1$ and investigate the properties of the solution when $\alpha = 1$ and $i = 1$. In this case, $\tilde{p}_{1,1}^*(x) = \int_0^x p_1^*(s)\, ds$. We can rewrite (16)–(20) as an abstract Cauchy problem in X

$$\begin{cases} \dot{P}(t) = \mathcal{A}P(t), \quad t > 0, \\ P(0) = (p_{0_0}, p_{1_0})^T, \end{cases} \tag{30}$$

where $\mathcal{A}$ is defined by

$$\mathcal{A} = \begin{pmatrix} 0 & \int_0^L \frac{\partial(g(x)\cdot)}{\partial x} \mathrm{d}x \\ 0 & -\frac{\partial(g(x)\cdot)}{\partial x} \end{pmatrix} \tag{31}$$

with domain

$$\begin{aligned}D(\mathcal{A}) = \big\{ P \in X \big| g p_1 \in W^{1,1}(0,L) \\ \text{and } p_1(0) = \lambda p_0 \big\}.\end{aligned}$$

Note that $gp_1 \in W^{1,1}(0,L)$ implies $gp_1 \in C[0,L]$ by Sobolev imbedding, and hence

$$\int_0^L \frac{\partial(g(x)p_1(x))}{\partial x}\, dx = g(L)p_1(L) - g(0)p_1(0) < \infty. \tag{32}$$

Moreover, since $\lim_{x\to L} g(x) = \lim_{x \to L} \frac{1}{p_1^*(x)} = \infty$ based on (12), we must have $p_1(L) = 0$.

Theorem 2.2: The system operator $\mathcal{A}$ defined by (31) generates a positive C_0-semigroup of contraction, denoted by $T(t), t \geq 0$, on X.

The proof follows the similar procedure as in that of [18, Thm. 2.1]. The details are omitted here due to space limit.

B. Exponential Stability

To establish the exponential stability of (30), we first note that when $\mu(x,t) = -\frac{p_{1_x}}{p_1} + \frac{(g(x)p_1)_x}{p_1}$, the desired distribution $P^*(x) = (p_0^*, p_1^*(x))^T$ is the only steady-state solution of the closed-loop system (30) and it is the eigenfunction associated with eigenvalue zero of $\mathcal{A}$. One can further show that zero is a simple eigenvalue and the only spectrum on the imaginary axis following the proof of [12, Prop. 2.1]. In addition, we can show that $T(t)$ is eventually compact.

Theorem 2.3: The C_0-semigroup $T(t)$ is compact when $t > \max\{L, \|p_1^*\|_{L^1}\}$.

Proof. First of all, we can show that the resolvent operator $\mathcal{R}(r, \mathcal{A})$ is compact for any $r \in \rho(\mathcal{A})$ as in [12]. According to [17, Cor. 3.4, p. 50], it suffices to show that $T(t)$ is continuous in the uniform operator topology for $t > \max\{L, \|p_1^*\|_{L^1}\}$, that is,

$$\sup_{\|P_0\|_X \leq 1, P_0 \neq 0} \|T(t+h)P_0 - T(t)P_0\|_X \xrightarrow{h \to 0} 0,$$

uniformly, where

$$\begin{aligned}\|T(t+h)P_0 - T(t)P_0\|_X = |p_0(t+h) - p_0(t)| \\ + \int_0^L |p_1(x,t+h) - p_1(x,t)|\, dx.\end{aligned}$$

Since $\tilde{p}_{1,1}^*(x)$ is strictly monotonically increasing for $x \in [0,L]$, we have $\|p_1^*\|_{L^1} = \tilde{p}_{1,1}^*(L) \geq \tilde{p}_{1,1}^*(x)$. Thus when $t > \max\{L, \|p_1^*\|_{L^1}\}$, we get $t > x$ and $t > \tilde{p}_{1,1}^*(x)$. With the help of (28)–(29), we obtain

$$\begin{aligned}&|p_0(t+h) - p_0(t)| \\ &\quad = \left| \alpha g(0)\lambda \int_t^{t+h} \big(p_0(\tau - \tilde{p}_{1,i}^*(L)) - p_0(\tau)\big)\, d\tau \right| \\ &\quad \leq 2\alpha g(0)\lambda h \sup_{t \geq 0} |p_0(t)| \end{aligned} \tag{33}$$

and

$$\int_0^L |p_1(x,t+h)-p_1(x,t)|\,dx$$
$$= \int_0^L \Big|\frac{1}{g(x)}g(0)\lambda\big(p_0(t+h-\tilde{p}^*_{1,1}(x)) - p_0(t-\tilde{p}^*_{1,1}(x))\big)\Big|\,dx$$
$$= g(0)\lambda\int_0^L \Big|p_0(t+h-\tilde{p}^*_{1,1}(x)) - p_0(t-\tilde{p}^*_{1,1}(x))\Big| p_1^*(x)\,dx. \tag{34}$$

Let $\tilde{t} = t - \tilde{p}^*_{1,1}(x) > 0$, then $d\tilde{t} = -p_1^*(x)dx$, and hence (34) becomes

$$g(0)\lambda\int_0^L \Big|p_0(t+h-\tilde{p}^*_{1,1}(x)) - p_0(t-\tilde{p}^*_{1,1}(x))\Big| p_1^*(x)\,dx$$
$$= g(0)\lambda\int_{t-\tilde{p}^*_{1,1}(L)}^{t} \left|p_0(\tilde{t}+h)-p_0(\tilde{t})\right|\,d\tilde{t}. \tag{35}$$

Moreover, in light of (33) we get

$$\int_0^L |p_1(x,t+h)-p_1(x,t)|\,dx$$
$$= g(0)\lambda\int_{t-\tilde{p}^*_{1,1}(L)}^{t} \left|p_0(\tilde{t}+h)-p_0(\tilde{t})\right|\,d\tilde{t}$$
$$\le 2(g(0)\lambda)^2\tilde{p}^*_{1,1}(L)h\sup_{\tilde{t}\ge 0}|p_0(\tilde{t})|. \tag{36}$$

Thus, by (33) and (36) we have

$$\|T(t+h)P_0 - T(t)P_0\|_X$$
$$\le (1+g(0)\lambda\tilde{p}^*_{1,1}(L))2g(0)\lambda h\sup_{t\ge 0}|p_0(t)|$$
$$\le C(h)\|P_0\|_X,$$

where $C(h) := (1+g(0)\lambda\tilde{p}^*_{1,1}(L))2g(0)\lambda h$ and the last inequality follows from the fact that $\sup_{t\ge 0}|p_0(t)| \le \sup_{t\ge 0}\|T(t)P(0)\|_X$ and $\|T(t)\|_{\mathcal{L}(X)} \le 1$ for all $t \ge 0$. Finally, since $\lim_{h\to 0} C(h) = 0$, we get

$$\lim_{h\to 0}\|T(t+h)-T(t)\|_X = 0,$$

which completes the proof. □

According to [5, p. 331, Cor. 3.3] and (10), the following result holds immediately.

Corollary 2.4: For $P(0) \in X$, the time-dependent solution $P(x,t) = T(t)P(0)$ to (30) converges to its steady-state solution $P^*(x) = (p_0^*, p_1^*(x))^T$ exponentially, that is,

$$\|P(\cdot,t) - P^*(\cdot)\|_X \le M_0 e^{-\varepsilon_0 t}, \tag{37}$$

for some constants $\varepsilon_0 > 0$ and $M_0 \ge 1$.

III. Bilinear Controllability

In this section, we present the proof of our main Theorem 2.1. Now consider the weighted closed-loop system (16)–(18) for $\alpha \ge p_1^*(0)$ and $i \in \mathbb{Z}^+$.

Proof of Theorem 2.1. Based on (16)–(17), the closed-loop system is now weighted by αi for $t \in [t_{i-1}, t_i), i \in \mathbb{Z}^+$, and hence the decay rate of the system solution to its steady-state becomes $\alpha i\varepsilon_0$ for $t \in [t_{i-1}, t_i)$. Further note that $t_i - t_{i-1} = \frac{c_0}{i^2}$. Consequently, by Cor. 2.4 we have

$$\|P(\cdot, c_0\sum_{k=1}^{i}\frac{1}{k^2}) - P^*(\cdot)\|_X$$
$$\le M_0 e^{-c_0\sum_{k=1}^{i}\alpha k\varepsilon_0\frac{1}{k^2}} = M_0 e^{-\alpha\varepsilon_0 c_0\sum_{k=1}^{i}\frac{1}{k}}. \tag{38}$$

Since $t_f = c_0\sum_{k=1}^{\infty}\frac{1}{k^2}$ and $\lim_{i\to\infty}\sum_{k=1}^{i}\frac{1}{k}$ diverges, (38) immediately implies

$$P(\cdot, t_f) = P^*(x),$$

which completes the proof. □

A. Boundedness of μ for $x \in [0,l]$ with $0 < l < L$

Finally, if $p_1^* \in W^{1\,\infty}(0,L)$, then one can show that the repair rate μ defined by (15) is bounded for $x \in [0,l]$ with $0 < l < L$, when $t_f > 2\|p_1^*\|_{L^1}$. Since $\frac{p_{1_x}}{p_1}$ converges to $\frac{p^*_{1_x}}{p_1^*}$ as $i \to \infty$, which is in $L^\infty(0,l)$ for $0 < l < L$. It suffices to show that $\sup_{x\in[0,l]}|\alpha i g(x)p_1(x,t)|$ is finite for $i \in \mathbb{Z}^+$ large enough.

Proposition 3.1: Let $\alpha \ge \max\{p_1^*(0), \frac{1}{c_0}, \frac{1}{c_0\varepsilon_0}\}$. For $t_f > 2\|p_1^*\|_{L^1}$, the solution $p_1(x,t)$ to (16)–(18) satisfies

$$\lim_{i\to\infty}\sup_{x\in[0,l]}\left|\frac{\partial(\alpha i g(x)p_1(x,t))}{\partial x}\right| < \infty.$$

Further if $p_1^* \in W^{1\,\infty}(0,L)$, then the repair rate $\mu(x,t)$ defined by (15) is bounded for $x \in [0,l]$ with $0 < l < L$.

Proof. For simplicity, we denote $(p_{0,i}, p_{1,i}(x,t))^T$ by $(p_0, p_1(x,t))^T$ in the rest of the proof. For $t_f > 2\|p_1^*\|_{L^1}$, there exists $i \in \mathbb{Z}^+$ large enough such that $t_{i-1} > 2\|p_1^*\|_{L^1}$, where $\|p_1^*\|_{L^1} \ge \tilde{p}^*_{1,i}(L)$ for any $i \in \mathbb{Z}^+$ and $\alpha \ge 1$. Thus for $t \ge t_{i-1}$ we have $t > 2\tilde{p}^*_{1,i}(L)$, and hence by (29),

$$p_1(x,t) = \frac{1}{g(x)}g(0)\lambda p_0(t-\tilde{p}^*_{1,i}(x)).$$

Therefore,

$$\frac{\partial(g(x)p_1(x,t))}{\partial x} = g(0)\lambda\frac{dp_0(t-\tilde{p}^*_{1,i}(x))}{dt}\left(-\frac{p_1^*(x)}{\alpha i}\right). \tag{39}$$

With the help of (28), we get

$$\frac{dp_0(t-\tilde{p}^*_{1,i}(x))}{dt} = \alpha i g(0)\lambda\big(p_0(t-\tilde{p}^*_{1,i}(x)-\tilde{p}^*_{1,i}(L)) - p_0(t-\tilde{p}^*_{1,i}(x))\big). \tag{40}$$

Combining (39) with (40) and (38) yields

$$\begin{aligned}
&\sup_{x\in[0,l]}\left|\frac{\partial(\alpha i g(x)p_1(x,t))}{\partial x}\right| \\
&=\alpha i(g(0)\lambda)^2\sup_{x\in[0,l]}\big|\big(p_0\big(t-\tilde p^*_{1,i}(x)-\tilde p^*_{1,i}(L)\big) \\
&\quad -p_0\big(t-\tilde p^*_{1,i}(x)\big)\big)p_1^*(x)\big| \\
&\le \alpha i(g(0)\lambda)^2\sup_{x\in[0,l]}\big(\big|p_0(t-\tilde p^*_{1,i}(x)-\tilde p^*_{1,i}(L))-p_0^*\big| \\
&\quad +\big|p_0(t-\tilde p^*_{1,i}(x))-p_0^*\big|\big)\cdot\sup_{x\in[0,l]}|p_1^*(x)| \\
&\le \alpha^2 i(g(0)\lambda)^2 \\
&\qquad \cdot\Big(\sup_{\tau\in[t-\tilde p^*_{1,i}(l)-\tilde p^*_{1,i}(L),t-\tilde p^*_{1,i}(L)]}\big|p_0(\tau)-p_0^*\big| \\
&\quad +\sup_{\tau\in[t-\tilde p^*_{1,i}(l),t]}\big|p_0(\tau)-p_0^*\big|\Big) \\
&\le 2\alpha^2 i(g(0)\lambda)^2\sup_{\tau\in[t-2\tilde p^*_{1,i}(L),t]}\big|p_0(\tau)-p_0^*\big| \\
&\le 2\alpha^2 i(g(0)\lambda)^2\|P(\cdot,t-2\tilde p^*_{1,i}(L))-P^*\|_X.
\end{aligned}$$

Recall that $\int_0^L p_1^*(x)\,dx\le 1$ and $\alpha\ge\frac{1}{c_0}$. We have

$$\tilde p^*_{1,i}(L)=\frac{1}{\alpha i}\int_0^L p_1^*(x)\,dx\le\frac{c_0}{i}.$$

Then $t-2\tilde p^*_{1,i}(L)\ge t_{i-1}-\frac{2c_0}{i}$ for $i\ge 2$. In light of Corollary 2.4 and (38) we get

$$\begin{aligned}
&\|P(\cdot,t-2\tilde p^*_{1,i}(L))-P^*\|_X \\
&\le\|P(x,t_{i-1}-\frac{2c_0}{i})-P^*\|_X \\
&\le M_0e^{-\alpha\varepsilon_0c_0\sum_{k=1}^{i-1}\frac1k}\cdot e^{\frac{2c_0}{i}\alpha\varepsilon_0(i-1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\frac{\partial(\alpha i g(x)p_1(x,t))}{\partial x}\right| \\
&\le 2\alpha^2 i(g(0)\lambda)^2M_0e^{-\alpha\varepsilon_0c_0\sum_{k=1}^{i-1}\frac1k}\cdot e^{\frac{2c_0}{i}\alpha(i-1)\varepsilon_0},
\end{aligned}\tag{41}$$

where

$$\lim_{i\to\infty}e^{\frac{2c_0}{i}\alpha\varepsilon_0(i-1)}=e^{2c_0\alpha\varepsilon_0}.\tag{42}$$

It remains to analyze the property of $ie^{-\alpha\varepsilon_0c_0\sum_{k=1}^{i-1}\frac1k}$ when i is sufficiently large. Let $j=i-1$. Then

$$\begin{aligned}
&ie^{-\alpha\varepsilon_0c_0\sum_{k=1}^{i-1}\frac1k}=(j+1)e^{-\alpha\varepsilon_0c_0\sum_{k=1}^{j}\frac1k} \\
&=je^{-\alpha\varepsilon_0c_0\sum_{k=1}^{j}\frac1k}+e^{-\alpha\varepsilon_0c_0\sum_{k=1}^{j}\frac1k} \\
&=e^{-(-\ln j+\alpha\varepsilon_0c_0\sum_{k=1}^{j}\frac1k)}+e^{-\alpha\varepsilon_0c_0\sum_{k=1}^{j}\frac1k}.
\end{aligned}$$

Since $\lim_{j\to\infty}(-\ln j+\sum_{k=1}^{j}\frac1k)=\gamma>0$ is the Euler-Mascheroni constant [6, Sec.1.5], we have for $\alpha\ge\frac{1}{\varepsilon_0c_0}$,

$$\lim_{j\to\infty}\big(e^{-(-\ln j+\alpha\varepsilon_0c_0\sum_{k=1}^{j}\frac1k)}+e^{-\alpha\varepsilon_0c_0\sum_{k=1}^{j}\frac1k}\big)\le e^{-\gamma}.\tag{43}$$

Finally, combining (41) with (42) and (43) follows

$$\lim_{i\to\infty}\sup_{x\in[0,l]}\left|\frac{\partial(\alpha i g(x)p_1(x,t))}{\partial x}\right|<\infty.$$

This completes the proof. □

IV. Conclusion

Bilinear controllability of a simple reparable system via system repair rate is addressed in this work. A specific control law in feedback form is constructed. The construction essentially makes use of the exponential convergence of the system solution to its steady-state. In fact, there are many other ways of choosing the control weight in (15) as long as the series in (38) diverges. Our approach is generic and can be applied to a broad family of reparable systems of similar attributes.

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