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Gain-scheduling control synthesis with inescapability conditions for nonlinear systems under input saturation

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Abstract— **This paper proposes a novel methodology to design a gain-scheduled state-feedback control law for nonlinear systems subjected to input saturation constraints and bounded external disturbances. The overall goal is disturbance rejection understood as determining the smallest possible inescapable set starting from zero initial conditions. The control design is addressed via iterative algorithms based on Linear Matrix Inequalities, which are obtained from the application of $\mathscr{H}_1$ star norm together with small gain argumentations. Numerical simulations are provided to show the effectiveness of the proposed method.**

I. INTRODUCTION

Stability analysis and stabilization is an essential cornerstone in the study of dynamical systems due to its relevance in control engineering applications. An important notion in this framework is the notion of inescapable set, which can be defined as a certain subset of the state space region such that all trajectories starting from the origin remain inside. If disturbances are present, the above-mentioned trajectories must consider worst-case disturbance trajectories, converting the problem statement to a disturbance rejection one.

The estimation of inescapable sets is strongly related to peak-to-peak bounding [1]. As discussed in [2], the optimal controller minimising the $\mathscr{L}_1$-induced norm (induced peak to peak) of continuous-time linear systems cannot be expressed in terms of LMIs, but a suboptimal controller can however be obtained with LMIs (well, BMIs including an auxiliary decay rate). Hence, the inescapability of some ellipsoids can be ensured for any disturbance signal with the proper control synthesis, provided that its maximum value is bounded. Non-LMI set-based manipulations (shooting) may also be used to determine inescapable sets [3] for uncertain nonlinear systems under some quasi-convexity assumptions (which include polytopic ones), but they scale poorly to higher dimensions.

In the case of nonlinear dynamical systems, under mild assumptions (continuous first derivatives) they can modeled as a quasi-LPV (q-LPV) system by embedding the nonlinearity into an uncertainty ball $\Delta(x)$.

The basic idea of q-LPV modeling is extracting the nonlinearities $\eta = \phi(x,z)$ as algebraic equations $\eta = \Delta(x)z$ if such factorisation can be actually carried out (maybe it is non-unique and different performance can be eked out from different q-LPV models, see [4]). If the rest of model equations are linear, they can be expressed in LFT form [5], [6], [7]. In order to bound the gain of Δ (which is usually needed in stability analysis LMIs) we must assume that $x \in \Omega$, where Ω is a predefined compact modeling region. If Δ is comprised of several nonlinearities in block-diagonal form, then individual bounds to each of the blocks end up building a polytopic bound on the whole nonlinearity structure. Note that the larger Ω is, the wider the bounds of Δ will be, evidently.

Note, however, that such q-LPV model is, in general, only locally valid in Ω. Therefore, it must be ensured that the trajectories do not exit the modeling region Ω when discussing quasi-LPV stability or disturbance rejection problems.

The key advantage of resorting to linear parameter-varying (LPV) framework is that the use of linear methods can be applied to the control design of nonlinear systems or time-varying systems [8], but at the expense of some degree of conservatism. Quasi-LPV models with no rational dependence on Δ are also called Takagi-Sugeno (TS) models [9], [10] or plainly 'polytopic' ones [11]. Those with rational dependence can be consider in descriptor [12], [13] or LFT form, equivalent if differentiability index of the descriptor representation is unity. The use of rational dependence can diminish the number of vertices of polytopic boundings and reduce some of the conservatism in the modeling phase with respect to non-rational TS options.

The objective of this paper is to provide a state-feedback control methodology based on Linear Fractional Transformation (LFT) modeling with the objective of achieving the smallest inescapable set around the origin without reaching the input saturation level [14], [15], for a given nonlinear system. To this end, an iterative algorithm based on LMIs is proposed to address the control synthesis. As a large Ω results in conservatisms, the iterations include a varying-size modeling region, so that the final modeling region is coincident with the proven inescapable ellipsoid. So, the contribution of this paper is the adaptation of the sector nonlinearity bound, together with the presence of actuator saturation in a gain-scheduled state feedback.

The structure of the paper is as follows: next section discusses preliminary definitions, notation and problem statement; Section III discusses the main theorem stating some matrix inequality conditions for inescapability in a disturbance-rejection problem; Section IV discusses the use of that theorem to find the 'smallest' inescapable set; examples are provided in Section V; last, a conclusion section closes the paper.

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II. Preliminaries and problem statement

Let us consider a nonlinear time-invariant system such that it can be expressed as the LFT interconnection of some linear dynamics with a block-diagonal nonlinear operator, in the form:

$$\begin{aligned} \dot{x} &= Ax + Bu + Fw + G\eta, \\ \eta &= \Delta(x)z, \\ z &= H_A x + H_B u + H_F w + H_G \eta, \end{aligned} \tag{1}$$

where $\Delta(x) = diag(\delta_1(x)I_1, ..., \delta_p(x)I_p)$ with $\delta_j(x) : \mathscr{R}^n \to \mathscr{R}, j = 1, ..., p$ being individual nonlinearities, and $I_1, \ldots I_p$ being identity matrices of appropriate dimensions. Vector $x \in \mathscr{R}^n$ denotes the state of the system, $u \in \mathscr{R}^m$ will be a vector of manipulated inputs, and $w \in \mathscr{R}^q$ stands for a disturbance input. There is no loss of generality in assuming the equilibrium point under zero inputs to be $x = 0$, and $\Delta(0) = 0$, as any other value for $\Delta(0)$ can be embedded in the linear equations. The system matrices $A, B, F, G, H_A, H_B, H_F, H_G$ are assumed to be time-constant and known.

We will consider that there exists input saturation so $u^T(t)u(t) \leq 1$ must be fulfilled, and that the disturbance input is bounded so $w^T(t)w(t) \leq 1$ for all $t \geq 0$.

We will assume that functions δ_j are continuous, thus bounded in a given compact set Ω of the state-space region containing an open neighborhood of the setpoint $x = 0$. Denoting such bound $\gamma_j = max_{x\in\Omega}|\delta_j(x)|, j = 1, ..., p$, system (1) can be rewritten as:

$$\begin{aligned} \dot{x} &= Ax + Bu + Fw + G\Gamma\bar{\eta}, \\ \bar{\eta} &= \bar{\Delta}(x)z, \\ z &= H_A x + H_B u + H_F w + H_G \Gamma\bar{\eta}, \end{aligned} \tag{2}$$

where $\Gamma = diag(\gamma_1 I_1, ..., \gamma_p I_p)$. Note that $\bar{\Delta}(x) = diag(\delta_1(x)/\gamma_1 I_1, ..., \delta_p(x)/\gamma_p I_p)$ attains a maximum $\mathscr{L}_2$ gain of unity, because it is a diagonal matrix gain whose elements range in $[-1, 1]$ for $x \in \Omega$. This kind of construction is denoted as LPV-embedding in literature [16] when the actual "shape" of $\bar{\Delta}(x)$ is disregarded, and robust stability and performance are proven for all Δ such that $\|\Delta\|_{2\to2} \leq 1$. Note that the scaling matrix Γ depends on the modeling region Ω.

Example: consider the system:

$$\dot{x} = \underbrace{-x + \left(\frac{sin(x)}{1 - 0.1sin(x)}\right)}_{f(x)} x + \underbrace{\left(1 + \frac{x}{1 + 0.3x}\right)}_{g(x)} u + w. \tag{3}$$

Given $\Omega : x/|x| \leq \pi$, the above system can be written in the form (2) as:

$$\begin{aligned} &A = -1, \quad B = 1, \quad F = 1, \\ &G = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad H_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ &H_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H_F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad H_G = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.3 \end{bmatrix}, \\ &\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & \pi \end{bmatrix}, \quad \Delta(x) = \begin{bmatrix} sin(x) & 0 \\ 0 & \frac{1}{\pi}x \end{bmatrix}. \end{aligned} \tag{4}$$

A. Problem statement

Our goal will be a disturbance-rejection one, from this definition below.

Definition 1. A set $W \subset \Omega$ will be said to be inescapable under a nonlinear state-feedback control law $u = \nu(x)$ if $x(0) \in W$ implies $x(t) \in W$ for all t for all admissible disturbance trajectories $w(t)$ if said closed-loop control law is used in (2).

Ideally, we wish to obtain the inescapable set of 'minimum size' containing the origin. Given the inherent conservatism of LMIs, it will just be an ellipsoidal estimate of it with, say, minimum length of largest semiaxis.

In particular, the objective of this paper is presenting a methodology based on bilinear matrix inequalities (BMI) to design a gain-scheduled controller such that, given the disturbance constraints, there exists an inescapable set around the origin in closed loop such that input does not saturate; we will try to minimise the radius of a sphere containing it.

III. Main Result: NONLINEAR STATE-FEEDBACK CONTROL

Now, let us first assume that a given, fixed, modeling region Ω has been used to express a nonlinear system in the form (2). Let us propose the following non-linear state-feedback control law for system (2) on the form:

$$\begin{aligned} u &= K_1 x + K_2 \eta_u, \\ \eta_u &= \Delta(x) z_u, \\ z_u &= K_3 x + K_4 \eta_u, \end{aligned} \tag{5}$$

where $\Delta(x)$ is defined in (2). Note that (5) can be expressed as the following rational expression on Δ:

$$u = \left(K_1 + K_2 (I - \Delta(x)K_4)^{-1} \Delta(x) K_3\right) x. \tag{6}$$

Theorem 1. The closed-loop system (2) with controller (5) renders the ellipsoid $x^T Q^{-1} x < 1$ inescapable with $u^T u \leq 1$, $\forall t \geq 0$ if there exist $\alpha > 0$ and matrices $Q > 0, \mathscr{E}_1, \mathscr{E}_2$, Y_1, Y_3, and K_2, K_4 such that the following matrix inequalities hold:

$$\begin{bmatrix} \Pi_1 & F & \Pi_2\mathscr{E}_1 & \Pi_3 \\ (*) & -\alpha I & 0 & \Pi_4 \\ (*) & (*) & -\mathscr{E}_1 & \mathscr{E}_1\Pi_5 \\ (*) & (*) & (*) & -\mathscr{E}_1 \end{bmatrix} < 0, \tag{7}$$

$$\begin{aligned} &\begin{bmatrix} Q & 0 & Y_3^T & Y_1^T \\ (*) & \mathscr{E}_2 & \mathscr{E}_2 K_4^T & \mathscr{E}_2 K_2^T \\ (*) & (*) & \mathscr{E}_2 & 0 \\ (*) & (*) & (*) & I \end{bmatrix} \geq 0, \\ &\Pi_1 = He(AQ + BY_1) + \alpha Q, \\ &\Pi_2 = \begin{bmatrix} G\Gamma & BK_2\Gamma \end{bmatrix}, \\ &\Pi_3 = \begin{bmatrix} QH_A^T + Y_1^T H_B^T & Y_3^T \end{bmatrix}, \\ &\Pi_4 = \begin{bmatrix} H_F^T & 0 \end{bmatrix}, \\ &\Pi_5 = \begin{bmatrix} \Gamma^T H_G^T & 0 \\ 0 & K_4^T \end{bmatrix}, \end{aligned} \tag{8}$$

where $\mathscr{E}_i$ are suitable multipliers satisfying $\mathscr{E}_i\bar{\Delta} = \bar{\Delta}\mathscr{E}_i, i = 1,2$ being $\bar{\Delta} = I_2 \otimes \Delta$. Moreover, the control gains K_1, K_3 in (5) are obtained as $K_1 = Y_1 Q^{-1}$, $K_3 = Y_3 Q^{-1}$.

Remark 1: For the above theorem to be valid, suitable matrix inequality conditions must be added to the above inequalities to ensure that the ellipsoid in the statement of Theorem 1 is contained in Ω, depending on the shape of Ω, omitted for brevity.

Proof: Given any matrix $P = P^T > 0$, consider the Lyapunov function:

$$V = x^T P x > 0 \tag{9}$$

Let $\eta = [\eta_1^T,\ \eta_2^T]$ and $z = [z_1^T,\ z_2^T]$. For any w such that $w^T w \le 1$, if the condition

$$\begin{aligned} &\dot{V} + \lambda_1\left(x^T P x - 1\right) + \lambda_2\left(1 - w^T w\right) \\ &+ z^T \mathscr{E}_1^{-1} z - \eta^T \mathscr{E}_1^{-1} \eta < 0 \end{aligned} \tag{10}$$

holds for some scalars $\lambda_1, \lambda_2 > 0$, then the closed-loop system is inescapable for any x satisfying $x^T P x < 1$ and any time-varying operator Δ satisfying $||\Delta||_\infty < 1$ for some scaling matrix of approppriate dimensions $\mathscr{E}_1$ such that $\mathscr{E}_1\Delta = \Delta\mathscr{E}_1$ (scaled small gain theorem). Choosing $\lambda_1 = \lambda_2 = \alpha$, one has that (10) is true $\forall x, w$ if and only if

$$\begin{bmatrix} \tilde{\Pi}_1 & PF & P\Pi_2 & \tilde{\Pi}_3 \\ (*) & -\alpha I & 0 & \Pi_4 \\ (*) & (*) & -\mathscr{E}_1^{-1} & \Pi_5 \\ (*) & (*) & (*) & -\mathscr{E}_1 \end{bmatrix} < 0, \tag{11}$$

$$\tilde{\Pi}_1 = He\left(PA + PBK_1\right) + \alpha P,$$
$$\tilde{\Pi}_3 = \begin{bmatrix} H_A^T + K_1^T H_B^T & K_3^T \end{bmatrix}.$$

Pre-and post multiplying the above matrix inequality by $diag\left(P^{-1}, I, \mathscr{E}_1, I\right)$, and denoting $Q = P^{-1}$, $Y_1 = K_1 Q$, $Y_3 = K_3 Q$ the equivalent condition given in the first matrix inequality in (7) is obtained.

The second matrix inequality in (7) is obtained from the input saturation condition, which comes from the constraint:

$$\lambda_3\left(x^T P x - 1\right) + \left(1 - u^T u\right) - z^T \mathscr{E}_2^{-1} z + \eta^T \mathscr{E}_2^{-1} \eta \ge 0, \tag{12}$$

for some $\mathscr{E}_2$ such that $\mathscr{E}_2\Delta = \Delta\mathscr{E}_2$, which means that $\forall x$ satisfying $x^T P x < 1$ the control input u verifies $u^T u \le 1$. Choosing $\lambda_3 = 1$, the above inequality is true if

$$\begin{bmatrix} P & 0 & K_3^T & K_1^T \\ (*) & \mathscr{E}_2^{-1} & K_4^T & K_2^T \\ (*) & (*) & \mathscr{E}_2 & 0 \\ (*) & (*) & (*) & I \end{bmatrix} \ge 0. \tag{13}$$

Pre-and post multiplying the above matrix inequality by $diag\left(P^{-1}, \mathscr{E}_2, I, I\right)$, and denoting $Q = P^{-1}$, $Y_1 = K_1 Q$, $Y_3 = K_3 Q$ the equivalent condition given in the second matrix inequality in (7) is obtained. ■.

The following corollary allows to design a non-scheduled feedback controler $u = K_1 x$, as a particular case of the above theorem.

Corollary 1. Given a scalar $\alpha > 0$, the closed-loop system (2) with (5) is inescapable in the ellipsoid $x^T Q^{-1} x < 1$ $\forall w\ /\ w^T w \le 1$, $\forall t \ge 0$ satisfying $u^T u \le 1$, $\forall t \ge 0$ if there exist matrices $Q > 0, \mathscr{E}, Y$ such that the following LMIs hold:

$$\begin{bmatrix} \Pi_1 & F & G\Gamma\mathscr{E} & \Pi_3 \\ (*) & -\alpha I & 0 & H_F^T \\ (*) & (*) & -\mathscr{E} & \mathscr{E}\Gamma^T H_G^T \\ (*) & (*) & (*) & -\mathscr{E} \end{bmatrix} < 0, \quad \begin{bmatrix} Q & Y^T \\ (*) & I \end{bmatrix} \ge 0, \tag{14}$$

$$\Pi_1 = He\left(AQ + BY\right) + \alpha Q, \qquad \Pi_3 = QH_A^T + Y^T H_B^T,$$

where $\mathscr{E}$ is a scaling satisfying $\mathscr{E}\Delta = \Delta\mathscr{E}$. Moreover, the control gain is obtained as $K = YQ^{-1}$. ■

The root of the algorithm in next section will be using Theorem 1; in order to numerically find a solution we can resort to a BMI solver or, if K_2 and K_4 and α are fixed, to an LMI solver. Implementation may involve, hence, iterated LMIs. Nevertheless, BMI implementation details are out of the scope of this work.

IV. Minimising inescapable set size for nonlinear system

There may be two specific problems to be solved with the above theorem, by appending extra LMIs and posing objective functions as follows:

a) Stabilization: Maximising ρ subject to $Q \ge \rho I$ would obtain the inescapable ellipsoid $\mathscr{E}$ with largest minimum semiaxis.

b) Disturbance rejection from $x = 0$: Minimizing ρ subject to $Q \le \rho I$ would obtain the "minimum radius" inside where there exists an inescapable set if initial conditions are at the origin. (Problem 1)

Variations of the above problems using volume as objetive function to maximise/minimise may be also thought of, details omitted for brevity.

However, if Ω were 'fixed', as required by Theorem 1, we would have a single q-LPV model and the result of the above optimizations, even if valid in a q-LPV setup, would be conservative from the nonlinear systems point of view, as Ω must forcedly be larger (or equal) than the proven ellipsoids, but if it is larger it means that the bounds for $\Delta(x)$ are conservative. Ideally, the modeling region Ω should be "equal" to the proven inescapable ellipsoid.

In order to solve these problems, Theorem 1 assumes a fixed modeling region Ω. However, the matrix inequality conditions may render infeasible due to:

- Excessively small Ω, so minimum inescapable sets (it size is not zero due to disturbance inputs) are too large to fit in it.
- Excessively large Ω so q-LPV bound Γ increases too much and/or saturation conditions for robust stabilization get too restrictive.

So, we may think in changing the modeling region size to better approach the "true" minimum and maximum inescapable sets of the original nonlinear systems via q-LPV techniques.

Regarding the first 'stabilization' problem, well, if a q-LPV model in Ω renders Theorem 1 feasible, then the estimated largest inescapable ellipsoid will be the one such that it is tangent to the boundary of Ω. This problem is somehow well studied in literature. Thus, in this work, we will delve into the details of how to address the second (disturbance rejection) problem with changing modeling region.

A. Minimising inescapable set from origin: iterative algorithm

Starting from a large modeling region Ω may include alternate equilibrium points or attractors so the "minimal" inescapable set from the origin may not be obtained. On the other hand, as commented above, disturbance variation may render infeasible a too small modeling region.

So, our proposal for this disturbance-rejection problem will be removing the geometric conditions of $x^T Q^{-1} x \leq 1$ being inside Ω and starting with $\Omega = \{0\}$, the origin, i.e., $\Gamma = 0$; then, feasibility of Theorem 1 should be checked.

As trajectories with $\Delta = 0$ will be a subset of those of the q-LPV system (2), the minimum inescapable ellipsoid for a certain modeling region Ω_i such that $\Omega_i \supseteq \Omega_{i-1}$ will be larger than that from Ω_{i-1} because Γ will be more restrictive.

Thus, starting from $\Omega_0 = \{0\}$, the following iterative algorithm is proposed, where the control synthesis problem will be addressed by iteratively modifying the modeling region in multiple steps, since $\bar{\gamma}$ depends on the modeling state-space region Ω.

The proposed algorithm reads as follows:

[Algorithm 1]

- **Step 1**: Set $i = 0$. Solve Problem $a)$ choosing $\Gamma^{(i)} = \Gamma = 0$. Set $Q_{i+1} = Q$.
- **Step 2**: Set i=i+1. Obtain $\gamma_j^{(i)} = max_{x \in \Omega_i} \delta_j(x), \;\; j = 1, ..., p$ where $\Omega_i = \{x \in \mathscr{R}^n / x^T Q_i^{-1} x \leq 1\}$.
- **Step 3**: Solve Problem $a)$ with the constraint $Q > Q_i$ choosing $\Gamma^{(i)} = \Gamma = diag(\gamma_1^i, .., \gamma_p^{(i)})$.
- **Step 4**: If an unfeasible solution is obtained in Step 3, stop iterating without a feasible control. Otherwise, go to Step 5.
- **Step 5**: If $max_{1 \leq j \leq p} |\gamma_j^{(i)} - \gamma_j^{(i-1)}| \leq \varepsilon$ with a prescribed tolerance $\varepsilon > 0$, then go to step 6. Otherwise, go to Step 2.
- **Step 6**: Expand modeling region Ω by a small enough factor $\xi >= 1$, and check feasibility of conditions in Theorem 1.

Note that, in the above algorithm, the geometric conditions discussed in Remark 1 must be omitted in step 3, in order for the ellipsoids to increase size until convergence. Indeed, condition $Q > Q_i$ just indicates that the ellipsoid in next iteration must be larger than the one in previous iteration, to generate a sequence of nested modeling regions Ω_i for the q-LPV approximation of the underlying nonlinear system. However, the last step (step 6) is the only one in which these geometric conditions must be present, in order to find an actually valid solution of Theorem 1.

Note also that no solution is guaranteed: the actual system may be unstable or LMI/BMI might not be able to prove its stability as they are only sufficient conditions. Indeed, increasing the size of Ω increases the bounds $\gamma_j, j = 1, ..., p$, thus inescapable sets get larger and, well, algorithm may not converge if the system is more and more difficult to control as we get far away from the origin. Thus, we are not leaving the realm of 'sufficient but not necessary' inescapability conditions, as obviously expected.

V. SIMULATION EXAMPLE

Consider system (2) with matrices

$$A = \begin{bmatrix} -1 & 0.5 \\ -0.9 & -0.02 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0.15 \\ 0.3 & -0.15 \end{bmatrix},$$
$$G = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.1 \end{bmatrix}, \quad H_A = \begin{bmatrix} 1 & 0.2 \\ 0.1 & 0 \end{bmatrix}, \qquad (15)$$
$$H_B = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}, \quad H_F = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix}, \quad H_G = \begin{bmatrix} 0.3 & 0.15 \\ -0.45 & 0 \end{bmatrix},$$

where $\Delta(x) = \sigma \cdot (diag\,(sin(0.1x_1), 0.9x_2))$ and σ is a fixed scalar that determines the "size" of nonlinearities.

Of course, $\sigma = 0$ would mean considering just an LTI system. In order to show the behavior of the proposed algorithm with respect to the size of nonlinearities we will consider three cases: $\sigma = 2.0$, $\sigma = 5.5$ and $\sigma = 6.8$, from 'more linear' to 'more nonlinear', informally speaking. Let us analyse the results of each of them.

a) Case 1 ($\sigma = 2.0$): By means of Algorithm 1 we obtain the smallest inescapable set which corresponds to the dash-dotted black ellispoid depicted in Fig 1. The rest of inner ellispoids corresponds to the evolution at each iteration as long as the modeling region is updated with the obtained inescapable set in previous steps, with the following styles: solid-red line for iteration 1, solid-blue line for iteration 2, solid-magenta line for iteration 3, solid-black line for iteration 4. It can be seen that the smallest inescapable set is not significatively increased after iterations. This fact can be expected due to the small size of nonlinear terms in this case. The external radius of the inescapable ellipsoids (namely ρ) is also depicted as a function of α in Figure 2 for each iteration (with the same styles used in Fig. 1), where the search range in α has been set in the interval $1.9 \leq \alpha \leq 3$. Of course, our goal is minimising such external radius (largest semiaxis) so we chose the value of α that makes such radius minimum, indicated with the markers in the referred figure. Thus, the actually proven evolution of ρ at each iteration of Algorithm 1 has been depicted in Figure 7 (solid red line), which summarises the evolution of all three cases.

b) Case 2 ($\sigma = 5.5$): In this case, the smallest inescapable set (dash-dotted black ellispoid depicted in Figure 3) is obtained after 8 iterations by Algorithm 1. In comparison to Case 1, it can be appreciated that more iterations are necessary due to the influence of the existing nonlinearities, which increase the difficulty of the control problem as modeling region gets larger. The rest of inner ellispoids corresponds to the evolution at each iteration (legend omitted for brevity and illustration clarity). The parameter ρ is also depicted as a function of α in Figure 4, for each iteration

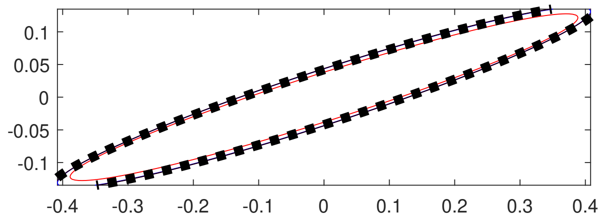

Fig. 1: Evolution of the inescapable set at each iteration by Algorithm 1 with different style lines to denote said evolution at each iteration of Algorithm 1. The black-dashed line denotes the inescapable set with $\sigma = 2.0$.

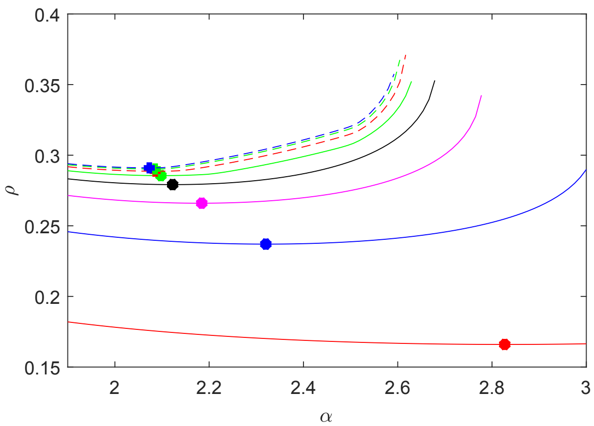

Fig. 2: Evolution of ρ as a function of α in Algorithm 1 for the case $\sigma = 2.0$. Each coloured line denotes said evolution at each iteration of Algorithm 1. The dash marks denote the minimum ρ at each iteration, which is the objective of our optimization.

(with the same styles and search range of α as used in Fig. 2). The evolution of ρ at each iteration of Algorithm 1 has been depicted in Figure 7 in solid green line, where the reached value is bigger than Case 1, as well as the size of the smallest inescapable set because, evidently, the larger family of models arising from the q-LPV embedding forcedly makes the optimal solution set to be larger. For further illustration, the size bound on the two nonlinearities as iterations progress appears in Figure 5.

c) Case 3 ($\sigma = 6.8$): In this case, the inescapable ellipsoids obtained after each iteration do not converge, as can be seen in Figure 6. Indeed, after 4 iterations no feasible solution is found for any value of α. This fact reveals that Algorithm 1 is unable to find a stabilizing controller on the form (5) due to the influence of large nonlinear terms and, possibly, saturation constraints. As no valid solution is obtained, the equivalent to figures 2 and 4 is not shown for brevity. The evolution of ρ at each iteration of Algorithm 1 has been depicted in Fig. 7 (solid blue line), but the value of ρ increases as iterations probress until no feasible solution is found so blue line there can be thought to be 'infinite' from iterations 5 onwards.

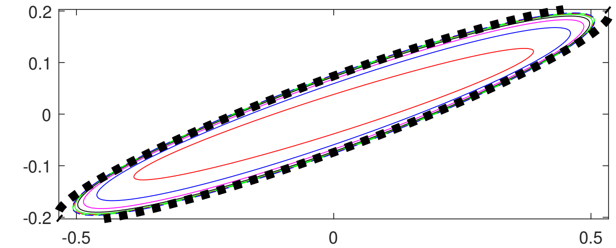

Fig. 3: Evolution of the inescapable set at each iteration by Algorithm 1. The black-dashed line denotes the inescapable set with $\sigma = 5.5$.

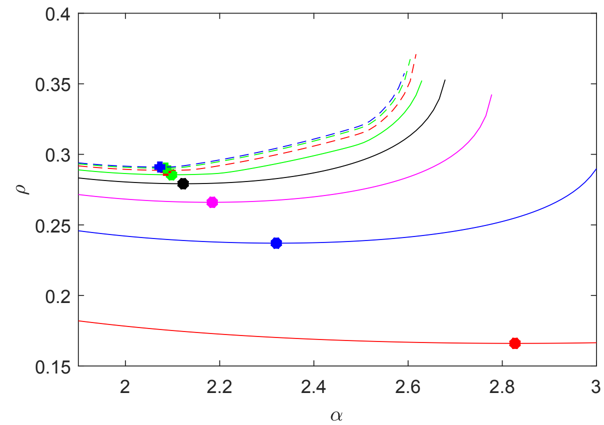

Fig. 4: Evolution of ρ as a function of α in Algorithm 1 for the case $\sigma = 5.5$. Each coloured line denotes said evolution at each iteration of Algorithm 1. The dash marks denote the minimum ρ at each iteration.

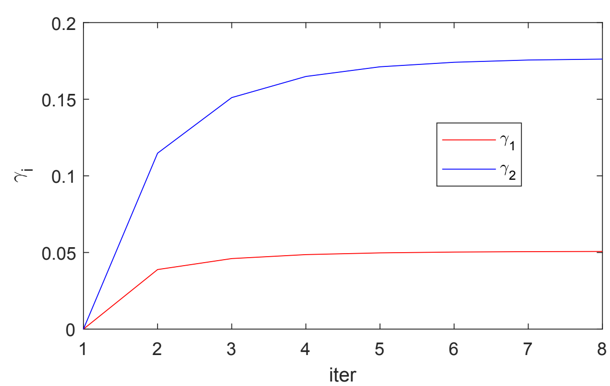

Fig. 5: Evolution of γ_1, γ_2 inside the modeling region Ω_i with respect to each iteration $i = 1, 2, \ldots$ in Algorithm 1 (Example 1) for the case $\sigma = 5.5$.

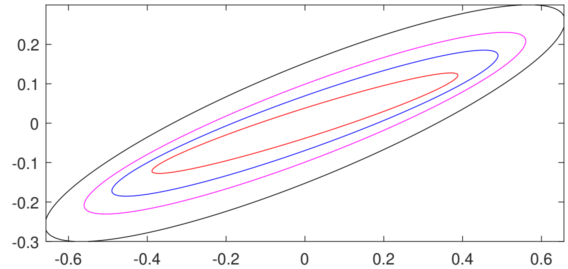

Fig. 6: Evolution of the inescapable set at each iteration by Algorithm 1 with different style lines to denote said evolution at each iteration of Algorithm 1. No convergence was achieved with $\sigma = 6.8$: system was 'too non-linear'.

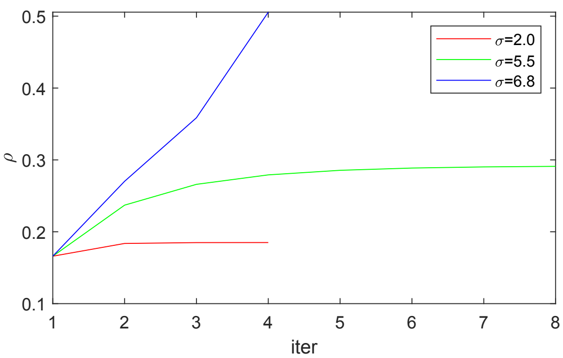

Fig. 7: Evolution of ρ inside the modeling region Ω_i with respect to each iteration $i = 1, 2, \ldots$ in Algorithm 1 for the three cases: $\sigma = 2.0$, $\sigma = 5.5$ and $\sigma = 6.8$.

Detail/discussion on the evolution of Algorithm 1: In all cases, the first iteration of Algorithm 1 is performed assuming $\Omega = 0$, that is to say, forcing all nonlinear terms to be zero ($\gamma_1 = 0$, $\gamma_2 = 0$). As a result, the inescapable set with minimum external radius ρ is obtained (solid-red line in Fig. 1) corresponding to the linear system. Thus, first iteration is coincident in all three cases as only the linearised model at the origin is used in the matrix inequality conditions.

The second step has been performed assuming that Ω is the inescapable ellipsoid (solid-red line) obtained in the previous step. In Case 2, for instance, we obtain $\gamma_1 = 0.0389, \gamma_2 = 0.1148$ (see Fig. 5). Algorithm 1 is executed until the increment between the external radius ρ of the two last consecutive inescapable ellipsoids is less than a prescribed tolerance (Case 1, 2) or no feasible solution is found for any value of α (Case 3) in the search range.

VI. CONCLUSIONS

This paper has presented a BMI methodology to estimate the 'minimum-size' inescapable set from the origin for a given nonlinear system under bounded disturbances and saturation. Matrix inequalities are stated from a quasi-LPV model; scaled-small-gain and norm-bound considerations have been discussed instead of polytopic vertex enumeration.

As conservatism of q-LPV models depends on modeling region size, an iterative algorithm adapting the said modeling region size as iterations progress has been presented. An academic example illustrates the results of the proposed methodology.

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