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Eigenvalues of Time-invariant Max-Min-Plus-Scaling Discrete-Event Systems

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Abstract— **This paper proposes an approach to find the eigenvalues and eigenvectors of a class of autonomous max-min-plus-scaling (MMPS) systems. First we show that time-invariant, monotone and non-expansive MMPS systems with only time variables has a unique structural eigenvalue and eigenvector under some conditions. Then, we propose a mixed-integer linear programming (MILP) algorithm to calculate the eigenvalue and the corresponding eigenvector for such systems. Finally, we present a modified linear programming (LP) algorithm to find all the eigenvalues of a general time-invariant MMPS system.**

I. INTRODUCTION

Event-driven systems, such as large traffic networks, small latches in a digital circuit, and production units form a major class of dynamic systems. The dynamics of these systems evolve with respect to some events and hence are called discrete event systems (DES). Some examples of events are the arrival of a train at a station in a railway network, a voltage change at the input of a latch in a digital circuit, and the arrival of raw material at a production unit.

Max-plus linear systems model DES involving only synchronization [1]. Max-plus algebra and max-plus linear systems are thoroughly studied in [1], [2]. Max-plus linear systems can model only a limited number of real-life DES. Max-min-plus (MMP) systems can model synchronization and competition in DES. They employ functions involving $\max, \min,$ and plus operations. MMP systems represent a broader class of DES compared to max-plus linear systems.

Max-min-plus scaling (MMPS) systems introduced in [3] are better-suited models for complex applications such as a real-life railway network, production units, and clock schedule verification problems in digital circuits. An explanation of the applications of different operations (max, min, plus, scaling) in DES can be found in [3]. MMPS systems can be a model structure for approximating any DES (linear and nonlinear) in max-plus algebra. There are very few articles on MMPS systems in the literature and little research on the dynamics of these systems.

Studies on MMP systems can be found in [2], [4]–[7]. In the analysis of the dynamic behaviour of a stable MMP system, homogeneity and non-expansiveness play a key role [8]. Functions that are homogeneous, monotonic and non-expansive in the l_∞ norm are called topical functions [9]. MMP functions belong to the family of topical functions. Homogeneity implies that if we shift all the events by same amount of time, the dynamics of the system is not altered i.e. the system is time-invariant. Most practical DES are time-invariant. Monotonicity indicates that delaying some events cannot speed up any other event. By adding a scaling operation to an MMP function, we get an MMPS function. Scaling can possibly make the system non-homogeneous and expansive. Hence, a general MMPS function is not a topical function.

The cycle-time vector and eigenvector are two significant metrics of a DES. For example, in an asynchronous circuit, the cycle-time indicates the average speed of the circuit [10]. In a railway network, an eigenvector (when it exists) can act as a time-table [2]. The eigenvector can also be interpreted as the equilibrium point of the DES. It can be proved that a cycle-time vector of an MMP system (if it exists) is unique using non-expansiveness [8]. When the cycle time vector has identical components, the eigenvalue/asymptotic growth rate of an MMP system exists and is equal to the component value of the cycle-time vector [11].

Many algorithms exist in the literature that find the eigenvalues and corresponding eigenvectors of max-plus linear and MMP systems [2], [12], [13]. The power algorithm is the most popular among these. However, for MMPS systems, especially when they are non-monotone, the power algorithm can take a long time to converge. Moreover, it may not give all the eigenvalues (when multiple eigenvalues exist) of a general time-invariant MMPS system.

The main contributions of this article are as follows. We propose an ABC canonical form for MMPS systems and show that any MMPS system can be put in this form. We derive conditions for the MMPS system in ABC canonical form to be time-invariant, monotonic and non-expansive. Then, we prove that time-invariant, monotonic and non-expansive MMPS (topical MMPS) systems have a unique eigenvalue and eigenvector when the system is elementary. The following are the most important contributions of this paper. First, we propose a mixed integer linear programming (MILP) algorithm, which calculates the eigenvalue and eigenvector of a topical MMPS system. Second, we find a set of linear programming problems to calculate all the eigenvalues and eigenvectors of a general time-invariant MMPS system, which can be considered as a generalized approach.

Here is the outline of the paper. In Section II, the required

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mathematical preliminaries are presented. In Section III, the conditions on the MMPS system in the ABC canonical form to have the time-invariance, monotonicity and non-expansive properties are discussed. In Section IV, the proof for the existence of a unique structural eigenvector and eigenvalue for the topical MMPS systems is given. A novel MILP algorithm is proposed to find the eigenvalue and eigenvector via a transformation of MMPS system. In Section V, the MILP algorithm is modified to a collection of linear programming problems to find all the eigenvalues of a general time-invariant MMPS system. The paper is concluded in Section VI.

II. MATHEMATICAL PRELIMINARIES

In this paper we consider regular MMPS systems that are explicit, time-invariant, and autonomous with state $x(k)$ that have the dimension of time and k is an event counter. The states of the system keep growing linearly at each event as they are the time at which the k-th event occur (e.g. time of arrival of k-th train at a station).

Define $\top = \infty$, $\varepsilon = -\infty$, $\mathbb{R}_\top = \mathbb{R} \cup \{\infty\}$, $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$, and $\mathbb{R}_c = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$. Often we use the notation $\mathcal{R}$ to denote either $\mathbb{R}$, $\mathbb{R}_\varepsilon$, $\mathbb{R}_\top$ or $\mathbb{R}_c$. The notations $\mathbf{1}$ and $\mathbf{0}$ are used to denote the vector with all components equal to one and the zero vector of appropriate dimension, respectively.

From max-plus and min-plus algebra, we adopt the following notation for matrices $A, B \in \mathcal{R}^{m\times n}$ and $C \in \mathcal{R}^{n\times p}$:

$$[A\oplus B]_{ij} = \max([A]_{ij}, [B]_{ij}), [A\otimes C]_{ij} = \max_k([A]_{ik}+[C]_{kj})$$

$$[A\oplus' B]_{ij} = \min([A]_{ij}, [B]_{ij}), [A\otimes' C]_{ij} = \min_k([A]_{ik}+[C]_{kj})$$

Definition 1. Given the vector $v \in \mathbb{R}^n$, we define a max-plus diagonal matrix, $\mathrm{d}_\otimes(v)$ and the min-plus diagonal matrix, $\mathrm{d}_{\otimes'}(v)$

$$\mathrm{d}_\otimes(v) = \begin{bmatrix} v_1 & \varepsilon & \cdots & \varepsilon \\ \varepsilon & v_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \varepsilon & \cdots & \cdots & v_n \end{bmatrix}, \ \mathrm{d}_{\otimes'}(v) = \begin{bmatrix} v_1 & \top & \cdots & \top \\ \top & v_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \top & \cdots & \cdots & v_n \end{bmatrix}$$

Then the inverse max-plus diagonal matrix is $\mathrm{d}_\otimes(-v)$ and inverse min-plus diagonal matrix is $\mathrm{d}_{\otimes'}(-v)$. The n-dimensional max-plus identity matrix $E = \mathrm{d}_\otimes(\mathbf{0})$ and min-plus identity matrix is $T = \mathrm{d}_{\otimes'}(\mathbf{0})$.

Definition 2 ([2]). A matrix $A \in \mathcal{R}^{n\times m}$ is said to be regular if A has at least one finite element in each row.

Definition 3 ([6]). A general MMP system can be represented as the following canonical form.

$$y(k) = B \otimes' x(k-1), \ x(k) = A \otimes y(k) \tag{1}$$

where $B \in \mathbb{R}_\top^{m\times n}, A \in \mathbb{R}_\varepsilon^{n\times m}$, $x \in \mathcal{R}^n$, $y \in \mathcal{R}^m$.

Definition 4 ([3]). A max-min-plus-scaling (MMPS) function $f : \mathcal{R}^m \to \mathcal{R}$ of the variables $x_1, \dots, x_m \in \mathcal{R}$ is defined by the grammar

$$f := x_i|\alpha|\max(f_k, f_l)|\min(f_k, f_l)|f_k + f_l|\beta\cdot f_k,$$

where $\alpha \in \mathcal{R}$, $\beta \in \mathbb{R}$ are some scalars and f_k, f_l are MMPS functions. For vector-valued MMPS functions the above statements hold component wise. (Or alternatively, $\alpha_{i,j}$ are matrices, and $\beta_{i,j}$ are vectors). This definition is the 'Backus-Naur' form from computer science where the vertical bars separate the different ways by which the function can be recursively constructed.

Definition 5. *Max-min-plus-scaling system.* A max-min-plus-scaling (MMPS) system is described by a state-space model of the form

$$x(k) = f(x(k-1)),$$

where $x \in \mathcal{R}^n$ is the state and f is a vector-valued MMPS function in the variables x and $k \in \mathbb{Z}^+$.

Inspired from the canonical form of MMP systems (1), we propose the following definition for an MMPS system.

Definition 6. (ABC canonical form) Consider the following system:

$$x(k) = A \otimes (B \otimes' (C \cdot x(k-1))) \tag{2}$$

This system is an MMPS system in the ABC canonical form for some matrices $A \in \mathbb{R}_\varepsilon^{n\times m}, B \in \mathbb{R}_\top^{n\times p}$ and $C \in \mathbb{R}^{p\times n}$.

Proposition 1. Any MMPS system can be written in the ABC canonical form.

Proof. A general MMPS system can be written in a disjunctive canonical form [14] as follows.

$$x(k) = \max_{i=1,\dots,n} \min_{j=\{1,\dots,m_i\}} (\sigma_j^T x(k-1) + \rho_j) \tag{3}$$

for some integers $n, m_1, m_2, \dots, m_n$, vectors σ_j and real numbers ρ_j. Now define

$$z(k) = \begin{bmatrix} \sigma_1^T x(k-1) \\ \cdots \\ \sigma_P^T x(k-1) \end{bmatrix} = C \cdot x(k-1)$$

Here P is the total number of distinct σ_j vectors. Now we obtain $x(k) = F(z(k))$ where F is a max-min-plus function. This function can now be written [6] as

$$x(k) = A \otimes (B \otimes' z(k))$$

resulting in

$$z(k) = C\cdot x(k-1), \ y(k) = B\otimes' z(k), \ x(k) = A\otimes y(k) \tag{4}$$

Note that the ρ_j in (3) will appear as entries in the matrices A and B. □

Definition 7. (Homogeneous, monotone and non-expansive system) Consider a system $x(k+1) = f(x(k))$.
The system is called homogeneous if there holds:

$$f(x + \alpha\mathbf{1}) = f(x) + \alpha\mathbf{1}$$

for any $\alpha \in \mathbb{R}$.

A system is called monotone if there holds:

$$\text{if } x \le y \text{ then } f(x) \le f(y)$$

A system is called non-expansive in l-norm if there holds:

$$\|f(x) - f(y)\|_l \le \|x - y\|_l$$

A homogeneous, monotonic and non-expansive MMPS system will be referred as a topical MMPS system.

III. Topical MMPS system

Definition 8. A system $x(k+1) = f(x(k))$ where x is a time signal is time-invariant if for any $\tau \in \mathbb{R}$ there holds

$$x(k+1) + \tau\mathbf{1} = f(x(k) + \tau\mathbf{1})$$

This means that an MMPS system is time-invariant if and only if it is homogeneous.

A. Time-Invariant MMPS systems

Time-invariance in MMPS is defined with respect to state $x(k)$. Here k is only an event counter and has nothing to do with the time-invariance (unlike in discrete-time systems). An MMP function is always homogeneous [11]. Let f define an MMP system. Then from (1)

$$x(k) = f(x(k-1)) = A \otimes (B \otimes' x(k-1))$$

Homogeneity implies that

$$\begin{aligned} f(x(k-1) + h\mathbf{1}) &= f(x(k-1)) + h\mathbf{1} \\ A\otimes(B\otimes'(x(k-1)+h\mathbf{1})) &= A\otimes(B\otimes' x(k-1)) + h\mathbf{1}. \end{aligned} \tag{5}$$

Lemma 1. An MMPS system as in Definition 6 is time-invariant if and only if $\sum_j c_{ij} = 1, \quad \forall i$ where c_{ij} are the components of the matrix C.

Proof. Consider the MMPS system in the ABC canonical form (2). For the MMPS system we have

$$\begin{aligned} &A \otimes (B\otimes'(C \cdot (x(k-1) + h\mathbf{1}))) \\ &\quad= A \otimes (B \otimes' (C \cdot x(k-1) + C \cdot h\mathbf{1})) \end{aligned} \tag{6}$$

From (5), the equation (6) is equal to

$$A \otimes (B \otimes' (C \cdot x(k-1))) + C \cdot h\mathbf{1}$$

So the MMPS system is homogeneous when $C \cdot h\mathbf{1} = h\mathbf{1}$. This is true if and only if each row of C adds to 1. That is, $\sum_j c_{ij} = 1, \quad \forall i.$ □

B. Monotonicity of MMPS systems

Lemma 2. An MMPS system is monotonic if and only if $c_{ij} \geq 0 \ \forall i, j$.

Proof. The 'if' part can be directly proved from the monotonicity of MMP systems. Assume that $x(k-1) \leq y(k-1)$. As the MMP system is monotonic [11], we have

$$A \otimes (B \otimes' x(k-1)) \leq A \otimes (B \otimes' y(k-1))$$

An MMPS system is monotonic if

$$A \otimes (B \otimes' (C.x(k-1))) \leq A \otimes (B \otimes' (C.y(k-1)))$$

This is true if $C.x(k-1) \leq C.y(k-1)$. which is satisfied when $c_{ij} \geq 0 \quad \forall i, j$.

The 'only if' condition can be easily seen using any counter-example where $c_{ij} < 0$. □

C. Non-expansiveness of time-invariant MMPS systems

Lemma 3. A time-invariant MMPS function is non-expansive if and only if $|c_{ij}| \leq 1 \quad \forall i, j$.

Proof. The 'if' part can be proved using the result of the non-expansive property of an MMP system [11]. Consider the MMP system defined as in (1). Since the MMP system is non-expansive,

$$\begin{aligned} \|A \otimes (B \otimes' x(k-1)) - A \otimes (B \otimes' y(k-1))\| &\leq \\ \|x(k-1) - y(k-1)\| & \\ \|A \otimes (B \otimes' (x(k-1) - y(k-1))\| &\leq \\ \|x(k-1) - y(k-1)\| & \end{aligned}$$

Note that the operations $+$ and $-$ are distributive over $\otimes, \otimes'$ [2]. The $\|.\|$ here is the ∞- norm. Now consider the MMPS system as in (6). This system is non-expansive when

$$\begin{aligned} \|A \otimes (B \otimes' (C.(x(k-1) - y(k-1)))\| &\leq \\ \|x(k-1) - y(k-1)\| & \end{aligned} \tag{7}$$

Let $w(k-1) = x(k-1) - y(k-1)$. Equation (7) is true when $\|C.w(k-1)\| \leq \|w(k-1)\|$. Let us first take the case where all $|c_{ij}| \leq 1$. Then $\sum_j c_{ij} w_j(k-1) \leq \max(|w_i(k-1)|) \ \forall i$. The 'only if' condition can be easily seen using any counter-example where $|c_{ij}| > 1$. □

Note that if the MMPS system is homogeneous and monotonic, it is also non-expansive which can be deduced from Lemmas 1 and 2.

IV. Eigenvalue and eigenvector of a topical MMPS system

Definition 9. (Eigenvalue, eigenvector) The time-invariant DES, $x(k) = f(x(k-1))$, $x \in \mathcal{R}^n$ and $f : \mathcal{R}^n \to \mathcal{R}^n$ is said to have an additive eigenvalue if there exists a real number $\lambda \in \mathcal{R}$ and a vector $v \in \mathbb{R}^n$ such that

$$f(v) = v + \lambda\mathbf{1}.$$

The scalar λ is then called an eigenvalue and the vector v is called a corresponding eigenvector. Further, if v is an eigenvector, $v + h\mathbf{1}$ is also an eigenvector for any $h \in \mathbb{R}$. The eigenvalue of an MMPS system is the rate at which the system grows. If the existence of eigenvalue of the system depends only on the structure of the system matrices A, B, C, then it is called structural eigenvalue of the system [15]. This means that the existence of an eigenvalue is not affected by any finite numerical changes in the system matrices.

Definition 10. An MMPS system is called elementary, if for each $i \in \{1, \ldots, n\}$ and for each $j \in \{1, \ldots, m\}$, at least one of the two entries a_{ij}, b_{ji} is finite and $c_{ij} \neq 0$ if $a_{ij} = \varepsilon$.

Proposition 2. (Topical MMPS system) A topical MMPS system characterized by matrices A, B, C has a structural eigenvalue and eigenvector if and only if A, B are regular and elementary.

Proof. The proof is similar to that of *Theorem 15* in [16]. Even though the theorem in [16] is stated with respect to a bipartite MMP system, it can be seen that the proof is applicable for topical MMPS systems as well. The proof shows any system satisfying homogenity, monotonicity and non-expansiveness has a unique structural eigenvalue under the conditions stated in the above proposition. □

Remark 1. In [16], the term fixed-point is used in the proof instead of the term eigenvector.

For the computation of the eigenvalue and eigenvector of a topical MMPS system we discuss two algorithms, the power algorithm inspired from the power algorithms for max-plus and max-min-plus systems and the MILP algorithm.

Algorithm 1. (Power algorithm [15]) To compute the eigenvalue and eigenvector of a system we can use the power algorithm.

1) Take an arbitrary initial vector $x(0) = x_0 \neq \varepsilon\mathbf{1}$; that is, x_0 has at least one finite element.
2) Iterate $x(k) = f(x(k-1))$ until there are integers p, q with $p > q \geq 0$ and a real number c, such that $x(p) = x(q) + c\mathbf{1}$, i.e., until a periodic regime is reached.
3) Compute the eigenvalue as $\lambda = c/(p-q)$.
4) Compute the eigenvector as $v = \frac{1}{p-q}\sum_{j=q}^{p-1} x(j)$

Before we introduce the MILP to compute the eigenvalue and eigenvector we first introduce the normalized MMPS system.

A. Normalized MMPS representation

Given a time-invariant, monotone, non-expansive MMPS system

$$z(k)=C\cdot x(k-1),\ y(k)=B\otimes' z(k),\ x(k)=A\otimes y(k) \quad (8)$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^p$, $y \in \mathbb{R}^m$, $A \in \mathbb{R}_{\varepsilon}^{n\times m}$, $B \in \mathbb{R}_{\top}^{m\times p}$, and $C \in \mathbb{R}^{p\times n}$. Let the system (8) has an eigenvalue λ and eigenvector $(x_e^T, y_e^T, z_e^T)^T$. Then the system satisfies

$$z_e = C\cdot(x_\mathrm{e} - \lambda\mathbf{1}),\ y_e = B\otimes' z_e,\ x_\mathrm{e} = A\otimes y_e$$

Define $A_\lambda = [A]_{ij} - \lambda\ \ \forall i,j$ and $x_{e,\lambda} = x_\mathrm{e} - \lambda\mathbf{1}$, then

$$z_e = C\cdot x_{e,\lambda},\ y_e = B\otimes' z_e,\ x_{e,\lambda} = A_\lambda\otimes y_e. \quad (9)$$

Now we define

$$\begin{aligned} X &= \mathrm{d}_\otimes(x_{e,\lambda}), & X^{-1} &= \mathrm{d}_\otimes(-x_{e,\lambda}) \\ Y &= \mathrm{d}_\otimes(y_e), & Y^{-1} &= \mathrm{d}_\otimes(-y_e) \\ Y' &= \mathrm{d}_{\otimes'}(y_e), & (Y')^{-1} &= \mathrm{d}_{\otimes'}(-y_e) \\ Z' &= \mathrm{d}_{\otimes'}(z_e), & (Z')^{-1} &= \mathrm{d}_{\otimes'}(-z_e) \end{aligned} \quad (10)$$

Then we have,

$$\begin{aligned} X^{-1}\otimes x_{e,\lambda} &= \mathbf{0} & Y^{-1}\otimes y_e &= \mathbf{0} \\ (Y')^{-1}\otimes' y_e &= \mathbf{0} & (Z')^{-1}\otimes' z_e &= \mathbf{0} \end{aligned} \quad (11)$$

By applying the matrices (10) to equation (9), we get

$$\begin{aligned} X^{-1}\otimes x_{e,\lambda} &= X^{-1}\otimes A_\lambda\otimes y_e \\ &= \underbrace{X^{-1}\otimes A_\lambda\otimes Y}_{\tilde{A}}\otimes Y^{-1}\otimes y_e \\ Y^{-1}\otimes' y_e &= (Y')^{-1}\otimes' B\otimes' z_e \\ &= \underbrace{Y^{-1}\otimes' B\otimes' Z}_{\tilde{B}}\otimes'(Z')^{-1}\otimes' z_e \end{aligned} \quad (12)$$

From equation (11) and (12) we get the following:

$$\mathbf{0} = \tilde{B}\otimes'\mathbf{0}, \quad \mathbf{0} = \tilde{A}\otimes\mathbf{0} \quad (13)$$

Consider the normalized MMPS system,

$$\tilde{z}(k)=C\cdot\tilde{x}(k-1), \tilde{y}(k)=\tilde{B}\otimes'\tilde{z}(k), \tilde{x}(k)=\tilde{A}\otimes\tilde{y}(k) \quad (14)$$

This system has an eigenvalue $\tilde{\lambda} = 0$ and eigenvector, $\tilde{v}_e = (\tilde{x}_e^T, \tilde{y}_e^T, \tilde{z}_e^T)^T = (\mathbf{0}^T, \mathbf{0}^T, \mathbf{0}^T)^T$. When initialized at this eigenvector, $\tilde{v}_e$, the states of the normalized system stay at zero. Furthermore, there holds:

$$\begin{aligned} &x(k) = \tilde{x}(k) + (k\lambda)\mathbf{1} + x_\mathrm{e},\ y(k) = \tilde{y}(k) + (k\lambda)\mathbf{1} + y_e \\ &z(k) = \tilde{z}(k) + (k\lambda)\mathbf{1} + z_e \end{aligned}$$

Based on (13) we can conclude that

$$\min_l [\tilde{B}]_{jl} = 0\ \ \forall j,\ \ \max_j [\tilde{A}]_{ij} = 0\ \ \forall i \quad (15)$$

Hence, there exist variables $p_{jl} \in \{0,1\}$ and $q_{ij} \in \{0,1\}$ such that

$$[\tilde{B}]_{jl} \leq M\,(1-p_{jl})\ \ \forall j,l, \qquad \sum_l p_{jl} \geq 1\ \ \forall j \quad (16)$$

$$[\tilde{A}]_{ij} \geq -M\,(1-q_{ij})\ \ \forall i,j, \qquad \sum_j q_{ij} \geq 1\ \ \forall i \quad (17)$$

where M is a large positive number. Note that the inequalities (16) guarantee that in every row of $\tilde{B}$ there is at least one zero. Similarly, the inequalities (17) guarantee that in every row of $\tilde{A}$ there is at least one zero [17].

Let the variables λ and (x, y, z) stand for the unknown eigenvalue end eigenvector respectively, then from (12)

$$[\tilde{B}]_{jl} = B_{jl} - y_j + z_l,\ [\tilde{A}]_{ij} = A_{ij} - \lambda - x_i + y_j$$

The unknown values λ and (x, y, z) can be computed by solving the mixed-integer linear programming problem (MILP).

Algorithm 2. (Eigenvalues and eigenvectors of a topical MMPS system - MILP)

$$\begin{aligned} &\min_{x,y,z,p,q} \lambda \\ \text{subject to}\quad & y_j - z_l \leq B_{jl}\ \ \forall j,l \\ & -y_j + z_l + M\,p_{jl} \leq -B_{jl} + M\ \ \forall j,l \\ & -\lambda - x_i + y_j \leq -A_{ij}\ \ \forall i,j \\ & \lambda + x_i - y_j + M\,q_{ij} \leq A_{ij} + M\ \ \forall i,j \\ & -\sum_l p_{jl} \leq -1\ \ \forall j, \quad -\sum_j q_{ij} \leq -1\ \ \forall i \\ & z = C\cdot x \end{aligned}$$

In case any $A_{ij} = \varepsilon$ (or any $B_{jl} = \top$), the corresponding constraint can be omitted from the MILP and then the p_{jl} (or q_{ij}) related to that constraint should be set to zero.

Remark 2. The above MILP algorithm can also be used for computing the eigenvalue and eigenvector of an MMP system by choosing C as an identity matrix making $z(k) = x(k)$.

Example 1. Given a topical MMPS system in ABC canonical form with

$$A = \begin{bmatrix} 10 & 5 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 8 \\ 5 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$$

By solving the MILP we obtain

$$\lambda = 13.8 \quad v_e = \begin{bmatrix} 7 & 0 & 10.8 & 6.8 & 5.6 & 2.8 \end{bmatrix}^T$$

By solving power algorithm, we get the same eigenvalue and a shifted eigenvector,

$$v_{ep} = \begin{bmatrix} 87 & 80 & 90.8 & 86.8 & 85.6 & 82.8 \end{bmatrix}^T = v_e + (80)\mathbf{1}$$

By using (10), we get

$$X = \mathrm{d}_{\otimes}(x_{e,\lambda}) = \begin{bmatrix} 7 & \varepsilon \\ \varepsilon & 0 \end{bmatrix}, \qquad Y = \mathrm{d}_{\otimes}(y_e) = \begin{bmatrix} 10.8 & \varepsilon \\ \varepsilon & 6.8 \end{bmatrix}$$
$$Y' = \mathrm{d}_{\otimes'}(y_e) = \begin{bmatrix} 10.8 & \top \\ \top & 6.8 \end{bmatrix}, \; Z' = \mathrm{d}_{\otimes'}(z_e) = \begin{bmatrix} 5.6 & \top \\ \top & 2.8 \end{bmatrix}$$

Then from (12), we get

$$\tilde{B} = \begin{bmatrix} 2.8 & 0 \\ 3.8 & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} 0 & -9 \\ 0 & -5 \end{bmatrix}$$

It can be verified that $\tilde{B} \otimes' \mathbf{0} = \mathbf{0}$, and $\tilde{A} \otimes \mathbf{0} = \mathbf{0}$. Note that every row of $\tilde{A}$ and every row of $\tilde{B}$ contains at least one zero.

V. Eigenvalues of general Time invariant MMPS systems

Multiple eigenvalues might exist for a general time-invariant MMPS system. Then, for every eigenvalue λ_s with eigenvector $v_s = (x_s^T, y_s^T, z_s^T)^T$, we can compute a corresponding normalized system $(\tilde{A}_s, \tilde{B}_s, C)$ such that

$$\mathbf{0} = \tilde{B}_s \otimes' \mathbf{0}, \quad \mathbf{0} = \tilde{A}_s \otimes \mathbf{0}.$$

We define the matrices $F_{A_s} \in \mathbb{R}_{\varepsilon}^{n\times m}$, $F_{B_s} \in \mathbb{R}_{\top}^{m\times p}$ as follows:

$$[F_{A_s}]_{ij} = \begin{cases} 0 & \text{if } [\tilde{A}_s]_{ij} = 0 \\ \varepsilon & \text{if } [\tilde{A}_s]_{ij} < 0 \end{cases}, \; [F_{B_s}]_{jl} = \begin{cases} 0 & \text{if } [\tilde{B}_s]_{jl} = 0 \\ \top & \text{if } [\tilde{B}_s]_{jl} > 0 \end{cases}$$

The matrices F_{A_s} and F_{B_s} are called the structure matrices for the eigenvalue λ_s. Different pairs of structure matrices may give rise to different eigenvalues. Let there be S eigenvalues $\lambda_s, \; s = 1, \ldots, S$ with structure matrices F_{A_s} and F_{B_s}. Every structure matrix, F_{A_s} consists of n rows of an $m \times m$ max-plus identity matrix, E and every structure matrix F_{B_s} consists of m rows of an $p \times p$ min-plus identity matrix, T. So the number of possible structure matrices F_A is less than or equal to m^n. Similarly, the number of possible structure matrices F_B is less than or equal to p^m. Therefore the number of eigenvalues is always smaller or equal to $m^n p^m$. Typically many entries of A and B will be ε or $\top$ respectively. This will decrease the number of possible structure matrices dramatically. As a consequence, the maximum number of eigenvalues will also decrease.

The algorithm for finding an eigenvalue and the corresponding eigenvector for a general time-invariant MMPS reduces to a linear programming problem (LPP). Based on the location of zeros in the structure matrices F_A and F_B, some of the inequalities change to equalities:

$$\begin{aligned} y_j - z_l = B_{jl} \quad \forall j, l \;\; \text{s.t.} \;\; [F_B]_{jl} = 0 \\ \lambda + x_i - y_j = A_{ij} \quad \forall i, j \;\; \text{s.t.} \;\; [F_A]_{ij} = 0 \end{aligned}$$

Algorithm 3. (Eigenvalues & eigenvectors of a general time-invariant MMPS system using a set of LPP)

$$\begin{aligned} & \min_{x,y,z} \; \lambda \\ \text{subject to} \quad & y_j - z_l \leq B_{jl} \quad \forall j, l \;\; \text{s.t.} \;\; [F_B]_{jl} = \top \\ & -y_j + z_l = B_{jl} \quad \forall j, l \;\; \text{s.t.} \;\; [F_B]_{jl} = 0 \\ & -\lambda - x_i + y_j \leq -A_{ij} \quad \forall i, j \;\; \text{s.t.} \;\; [F_A]_{ij} = \varepsilon \\ & \lambda + x_i - y_j = A_{ij} \quad \forall i, j \;\; \text{s.t.} \;\; [F_A]_{ij} = 0 \\ & z = C \cdot x \end{aligned}$$

For a topical MMPS system, only one pair of structure matrices gives a feasible solution for the MILP. For the eigenvalue in example 1, the structure matrices F_A and F_B are given by

$$F_A = \begin{bmatrix} 0 & \varepsilon \\ 0 & \varepsilon \end{bmatrix}, \quad F_B = \begin{bmatrix} \top & 0 \\ \top & 0 \end{bmatrix}$$

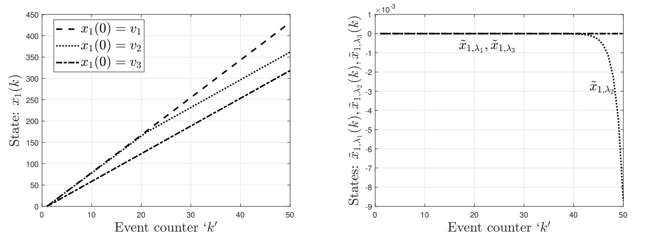
Note that the structure of structure matrices F_A, F_B is similar to that of the normalized system matrices $\tilde{A}, \tilde{B}$. For time-invariant MMPS systems, distinct structure matrix pairs might give rise to different eigenvalues. It is possible that some pairs are invalid i.e. the linear programming problem corresponding to these matrices is infeasible.

Example 2. Consider the time-invariant MMPS system in ABC canonical form with system matrices

$$A = \begin{bmatrix} 9 & 5 \\ 2 & 6 \\ 2 & 10 \end{bmatrix}, B = \begin{bmatrix} 8 & 3 & 9 \\ 5 & 8 & 2 \end{bmatrix}, C = \begin{bmatrix} -0.75 & 1.75 & 0 \\ 1.2 & 0.8 & -1 \\ -0.4 & -0.4 & 1.8 \end{bmatrix}$$

Note that the system is non-monotone and expansive as some of the elements in the C matrix are negative and some are greater than one. By solving the linear programming problems (in this case $2^3 \times 3^2 = 72$ LPPs) for all possible pairs of the structure matrices, we found three distinct eigenvalues and associated eigenvectors

$$\begin{aligned} & \lambda_1 = 8.8 \quad \lambda_2 = 8.6316 \quad \lambda_3 = 6.5 \\ & v_1 = \begin{bmatrix} 0 & 1 & 5 & -1.2 & 3.8 & 1.75 & -4.2 & 8.6 \end{bmatrix}^T \\ & v_2 = [0.5263 \; -2.6316 \; 1.3684 \; \cdots \\ & \qquad \cdots 0.1579 \; 0 \; -5 \; -2.8421 \; 3.3053]^T \\ & v_3 = [0 \; -6 \; -2 \; -2.5 \; -5.5 \; -10.5 \; -2.8 \; -1.2]^T \end{aligned}$$



(a) Original MMPS system (b) Normalized MMPS systems

Fig. 1: Growth rate of state x_1 and $\tilde{x}_1$ of MMPS system from the example 2 initialized at different eigenvectors, v_1, v_2, v_3.

The structure matrices associated with eigenvalues, $\lambda_1, \lambda_2, \lambda_3$ are

$$F_{A_1} = \begin{bmatrix} \varepsilon & 0 \\ \varepsilon & 0 \\ \varepsilon & 0 \end{bmatrix}, \quad F_{B_1} = \begin{bmatrix} \top & 0 & \top \\ \top & 0 & \top \end{bmatrix}$$

$$F_{A_2} = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 0 \\ \varepsilon & 0 \end{bmatrix}, \quad F_{B_2} = \begin{bmatrix} \top & 0 & \top \\ 0 & \top & \top \end{bmatrix}$$

$$F_{A_3} = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 0 \\ \varepsilon & 0 \end{bmatrix}, \quad F_{B_3} = \begin{bmatrix} 0 & \top & \top \\ 0 & \top & \top \end{bmatrix}$$

The MMPS system in above example can be normalized with different eigenvalues. The normalized system, corresponding to an eigenvalue (say λ_1), initialized at zero will be equivalent to the original MMPS system initialized at the corresponding eigenvector (v_1).

Figure 1a shows the three different growth rates of state $x_1(k)$ that corresponds to the original MMPS system. Figure 1b shows the states of three normalized systems, $\tilde{x}_{1,\lambda_1}(k)$, $\tilde{x}_{1,\lambda_2}(k)$, and $\tilde{x}_{1,\lambda_3}(k)$ that corresponds to the eigenvalues λ_1, λ_2, and λ_3 respectively. The original MMPS system is initialized at three eigenvectors (associated to λ_1, λ_2, λ_3), v_1, v_2 and v_3 respectively. All three normalized systems are initialized at zero.

It can be observed from Figure 1a that the state $x_1(k)$ continues to grow with the same rates λ_1 and λ_3 when initialized at v_1 and v_3, respectively. When started at v_2, the state grows at rate λ_2 for some time and jumps to rate λ_3. This might be because the eigenvalue, λ_2 is unstable. Figure 1b supports this premise. As discussed in section IV, when the normalized system is initialized at zero, the states stay at zero. We see that the state of normalized systems that are derived with eigenvalues λ_1 and λ_3 stays at zero while the state of the normalized system, which is derived from λ_2 is growing. Due to the small numerical error in MATLAB calculations, the normalized system derived with λ_2 deviates from its eigenvector, $\mathbf{0}$. This causes the state $\tilde{x}_{1,\lambda_2}$ to grow due to the instability of the eigenvalue, λ_2. The stability of eigenvalues of a general time-invariant MMPS system will be studied in detail in our future research.

Note that power algorithm for this example cannot find the unstable eigenvalue, λ_2. Also, multiple iterations of power algorithm with random initial conditions are required to find all the stable eigenvalues.

VI. Conclusions

In conclusion, this study is aimed at understanding the properties of MMPS systems via the analysis of the eigenvalues and eigenvectors of the system. Our analysis has shown a method to find the eigenvalues and eigenvectors for different classes of MMPS systems. The insights from this study are important to establish a framework of stability for these classes of systems.

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