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Multirate Multiscale Symbolic Models for Incrementally Stable Switched Systems

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Abstract— **This paper studies the construction of symbolic models for switched systems based on a multirate multiscale setting. We focus on switched systems with all incrementally stable subsystems. First, using multiple Lyapunov functions and mode-dependent average dwell-time, sufficient conditions are derived for incremental stability of switched systems, which lays a foundation for the construction of the symbolic models. Second, based on the multirate multiscale setting and bounded dwell-time constraints, multirate multiscale symbolic models are constructed for switched systems. The approximate bisimulation relation is established between the original system and the constructed symbolic model. Finally, the proposed construction method is illustrated via a numerical example from obstacle avoidance problems of robotic systems.**

I. Introduction

Switched systems are dynamic systems consisting of a family of finite subsystems and a switching signal that orchestrates the switching among them [1]. Switched systems can be used to model numerous engineering systems, such as networked control systems , mechanical systems and multi-agent systems; see [1]–[4] and references therein. Since some switching strategies may result in instability of the overall system and some subsystems may be unstable, many efforts have been made to investigate under which switching signals or strategies the stability and stabilization of the overall system are guaranteed. In the literature, numerous results can be found on stability analysis and controller design of switched systems; see, e.g., [1]–[4].

In recent years, considerable research has focused on characterization of dynamic systems that admit symbolic models, which are discrete approximations of these dynamics, resulting from replacing equivalent (sets of) continuous states by discrete symbols; see [5]–[7]. Using the symbolic models, one can deal with controller synthesis problems efficiently via techniques developed in the fields of supervisory control [8] or algorithmic game theory [9]. Since there exists an inclusion or equivalence relationship between the original system and the symbolic model, the synthesized controller is guaranteed to be correct by design and thus the formal verification can be reduced or neglected [10]. To construct the symbolic model, the key is to find an equivalence relation on the state space of dynamic systems. The equivalence relation leads to a new system, which is on the quotient space and shares the properties of interest with the original system. In the literature, many works can be found on symbolic models for switched systems; see [11]–[16]. However, some more general scenarios, including the case where the switching intervals are neither multiples nor factors of the sampling period and the case where some subsystems are unstable, have not been considered in previous works, which motivates us to study this topic further.

In this paper, we focus on symbolic models of switched systems with all incrementally stable subsystems. To this end, our first contribution is to establish sufficient conditions for incremental stability of switched systems. The stability conditions are established based on mode-dependent average dwell-time (MDADT) [17] and multiple Lyapunov functions. Note that for the first time, the MDADT is applied to address the incremental stability of switched systems, and thus reduces the conservatism caused by the average dwell-time (ADT) [4], [14] and bounded dwell-time [18]. In particular, the switching intervals are neither constant nor the same for all subsystems, it is necessary to use the MDADT to measure the switching intervals of each subsystem. Furthermore, the derived stability conditions lay a solid foundation for the abstraction construction afterwards.

With the incremental stability of switched systems, multirate multiscale symbolic model is constructed such that the approximate bisimulation relation is established between the original system and the symbolic model, which is the second contribution of this paper. The construction is based on the multirate multiscale setting and the dwell-times of each subsystem are bounded, which results in a novel abstraction construction method for switched systems. In comparison to previous works, the proposed construction method is more general from two perspectives. In terms of dwell-times, the commonly-used assumption, which claims that *the sampling period is a multiple or factor of the constant dwell-time* [12], [14], is not relaxed here, and thus the constructed symbolic model is general enough to recover those in [12], [14], [19] as special cases. In terms of the construction approach, the multirate multiscale setting in [12] is implemented, and thus the constructed symbolic model is more practical and can deal with some phenomena like fast/slow switching and asynchronous switching.

The remainder of this paper is organized below. Preliminaries are stated in Section III. Stability conditions are derived in Section IV. Symbolic model is constructed in Section V. A numerical example is given in Section VI. Conclusion and future work are presented in Section VII.

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II. Preliminaries

Let $\mathbb{R} := (-\infty, +\infty)$; $\mathbb{R}^+ := [0, +\infty)$; $\mathbb{N} := \{0, 1, 2, \ldots\}$; and $\mathbb{N}^+ := \{1, 2, \ldots\}$. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. Given $x \in \mathbb{R}^n$, x_i denotes the i-th element of x, and $\|x\|$ denotes the infinity norm of x. The closed ball centered at $x \in \mathbb{R}^n$ with radius $\varepsilon > 0$ is defined by $\mathbf{B}(x, \varepsilon) = \{y \in \mathbb{R}^n : \|x - y\| \leq \varepsilon\}$. $\lceil \cdot \rceil$ is the ceiling operator and $\lfloor \cdot \rfloor$ is the floor operator. Given a function $f : \mathbb{R}^+ \to \mathbb{R}^n$, $\|f\|$ is the supremum norm on $\mathbb{R}^+$; $f(t^-) := \limsup_{s \to 0^-} f(t+s)$; $f|_\tau$ means the restriction of f to $[0, \tau]$. $\mathcal{C}(\mathbb{R}, \mathbb{R}^n)$ is the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$. A function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing; $\alpha(t)$ is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{KL}$ if $\beta(s, t)$ is of class $\mathcal{K}$ for each fixed $t \in \mathbb{R}^+$ and decreases to zero as $t \to \infty$ for each fixed $s \in \mathbb{R}^+$. Given two sets $A, B \subset \mathbb{R}^n$, a relation $\mathcal{R} \subset A \times B$ is a map $\mathcal{R} : A \to 2^B$ defined by $b \in \mathcal{R}(a)$ if and only if $(a, b) \in \mathcal{R}$. $\mathcal{R}^{-1}$ denotes the inverse relation of $\mathcal{R}$, i.e., $\mathcal{R}^{-1} := \{(b, a) \in B \times A : (a, b) \in \mathcal{R}\}$.

Definition 1 ([12]): A *transition system* is a quintuple $\mathbf{T} = (\mathbb{X}, \mathbb{X}^0, \mathbb{U}, \Delta, \mathbb{Y})$ with: (i) a state set $\mathbb{X}$; (ii) a set of initial states $\mathbb{X}^0 \subseteq \mathbb{X}$; (iii) an input set $\mathbb{U}$; (iv) a transition relation $\Delta \subseteq \mathbb{X} \times \mathbb{U} \times \mathbb{X} \times \mathbb{Y}$; (v) a output set $\mathbb{Y}$. The system $\mathbf{T}$ is said to be *metric*, if the set of outputs $\mathbb{Y}$ is equipped with a metric $\mathbf{d}$, and *symbolic* if $\mathbb{X}$ and $\mathbb{U}$ are finite or countable.

A transition $(x, u, x', y) \in \Delta$ is denoted by $(x', y) \in \Delta(x, u)$. That is, the system $\mathbf{T}$ can evolve from the state x to the state x' under the input u and produce the output y. The *set of enabled inputs* at the state x is defined as enab$(x) := \{u \in \mathbb{U} : \Delta(x, u) \neq \varnothing\}$. For each $x \in \mathbb{X}$ and each $u \in$ enab(x), if $\Delta(x, u)$ has exactly one element, then $\mathbf{T}$ is *deterministic*. In this case, let $(x', y) = \Delta(x, u)$.

Definition 2 ([20]): Let $\mathbf{T}_i = (\mathbb{X}_i, \mathbb{X}_i^0, \mathbb{U}, \Delta_i, \mathbb{Y})$, $i = 1, 2$, be two metric transition systems and the output set $\mathbb{Y}$ be equipped with the metric $\mathbf{d}$. Let $\varepsilon > 0$, a relation $\mathcal{R} \subseteq \mathbb{X}_1 \times \mathbb{X}_2$ is said to be an *ε-approximate bisimulation relation (ε-ABR)* between $\mathbf{T}_1$ and $\mathbf{T}_2$, if for all $(x_1, x_2) \in \mathcal{R}$ and all $u \in \mathbb{U}$,

(i) for each $(x_1', y_1) \in \Delta_1(x_1, u)$, there exists $(x_2', y_2) \in \Delta_2(x_2, u)$ such that $\mathbf{d}(y_1, y_2) \leq \varepsilon$ and $(x_1', x_2') \in \mathcal{R}$.

(ii) for each $(x_2', y_2) \in \Delta_2(x_2, u)$, there exists $(x_1', y_1) \in \Delta_1(x_1, u)$ such that $\mathbf{d}(y_1, y_2) \leq \varepsilon$ and $(x_1', x_2') \in \mathcal{R}$.

We denote by $\mathbf{T}_1 \simeq_\varepsilon \mathbf{T}_2$ if there exists an ε-ABR $\mathcal{R}$ between $\mathbf{T}_1$ and $\mathbf{T}_2$ such that $\mathcal{R}(\mathbb{X}_1) = \mathbb{X}_2$ and $\mathcal{R}^{-1}(\mathbb{X}_2) = \mathbb{X}_1$.

III. Switched Systems

The class of switched systems to be studied and some related preliminaries are introduced in this subsection.

Definition 3 ([12]): A *switched system* is a quadruple $\Sigma = (\mathbb{R}^n, \mathfrak{L}, \mathcal{L}, F)$ with a state space $\mathbb{R}^n$; a finite set of modes $\mathfrak{L} = \{1, \ldots, \mathsf{L}\}$ with finite $\mathsf{L} \in \mathbb{N}$; a switching signal set $\mathcal{L} \subseteq \mathcal{S}(\mathbb{R}^+, \mathfrak{L})$ with $\mathcal{S}(\mathbb{R}^+, \mathfrak{L})$ as the set of piecewise constant functions from $\mathbb{R}^+$ to $\mathfrak{L}$, which are continuous from the right-hand side and have finite discontinuities on any bounded set in $\mathbb{R}^+$; and a collection of vector fields $F = \{f_1, \ldots, f_\mathsf{L}\}$ indexed from the mode set $\mathfrak{L}$. For each $l \in \mathfrak{L}$, $f_l : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous.

For the switched system Σ, a switching signal is a function $\sigma : \mathbb{R}^+ \to \mathfrak{L}$, whose discontinuities are called *switching time instants*. The switching time instant sequence is denoted by $\mathcal{T} := \{t_1, t_2, \ldots\}$, which is assumed to be strictly increasing. A piecewise continuously differential function $\mathbf{x} : \mathbb{R}^+ \to \mathbb{R}^n$ is said to be a *trajectory* of Σ, if it is continuous and there exists a switching signal $\sigma(t) \in \mathcal{L}$ such that for all $t \geq 0$, $\mathbf{x}$ is continuously differentiable and satisfies

$$\dot{\mathbf{x}}(t) = f_{\sigma(t)}(\mathbf{x}(t)). \tag{1}$$

We use $\mathbf{x}(t, x, \sigma)$ to denote the point reached at time $t \geq 0$ from the initial $x \in \mathbb{R}^n$ under the switching signal $\sigma \in \mathcal{L}$.

For the switched system Σ, Zeno phenomena are excluded due to the strict increase of the sequence $\mathcal{T}$. To measure the frequency of the discontinuities, a mode-dependent average dwell-time (MDADT) is introduced as follows.

Definition 4 ([17]): Consider a switching signal $\sigma(t)$ and any interval (t', t) with $t > t' > 0$. For the l-th subsystem, $l \in \mathbb{P}$, $N_{\sigma l}(t', t)$ is the number of the activation times in (t', t), and $\mathcal{T}_l(t', t)$ is the length of all time intervals which are in (t', t) and where the l-th subsystem is active. If there exist $N_{0l}, \tau_{al} > 0$ such that,

$$N_{\sigma l}(t', t) \leq N_{0l} + \tau_{al}^{-1} \mathcal{T}_l(t', t), \quad \forall t > t' > 0, \tag{2}$$

then N_{0l} is called the *mode-dependent chatter bound*, and τ_{al} is called the *mode-dependent average dwell-time (MDADT)*.

IV. Incremental Stability

To construct symbolic models, we need to guarantee that the system Σ is incrementally stable, which is defined below.

Definition 5 ([12]): The switched system Σ is *incrementally globally uniformly asymptotically stable (δ-GUAS)*, if there exists $\beta \in \mathcal{KL}$ such that for all $x_1, x_2 \in \mathbb{R}^n, \sigma \in \mathcal{L}$,

$$|\mathbf{x}(t, x_1, \sigma) - \mathbf{x}(t, x_2, \sigma)| \leq \beta(|x_1 - x_2|, t), \quad \forall t \geq 0. \tag{3}$$

If (3) holds for a given $l \in \mathfrak{L}$, then the l-th subsystem is *incrementally globally asymptotically stable (δ-GAS)*.

Definition 6 ([14]): The smooth functions $V_l : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$ are *multiple δ-GUAS Lyapunov functions* for Σ, if there exist $\alpha_{1l}, \alpha_{2l} \in \mathcal{K}_\infty$ and $\rho_l > 0$ such that for all $x_1, x_2 \in \mathbb{R}^n$ and all $l \in \mathfrak{L}$,

$$\alpha_{1l}(\|x_1 - x_2\|) \leq V_l(x_1, x_2) \leq \alpha_{2l}(\|x_1 - x_2\|), \tag{4}$$

$$\frac{\partial V_l(x_1, x_2)}{\partial x_1} f_l(x_1) + \frac{\partial V_l(x_1, x_2)}{\partial x_2} f_l(x_2) \leq -\rho_l V_l(x_1, x_2). \tag{5}$$

In addition, for each $l \in \mathfrak{L}$, V_l is the *δ-GAS Lyapunov function* for the subsystem Σ_l.

Theorem 1: Consider the switched system $\Sigma = (\mathbb{R}^n, \mathfrak{L}, \mathcal{L}, F)$. If for all $l \in \mathfrak{L}$,

(A.1) each subsystem Σ_l admits a δ-GAS Lyapunov function $V_l : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$;

(A.2) for each $p \in \mathfrak{L}$, there exists $\mu_l \geq 1$ such that

$$V_l(x_1, x_2) \leq \mu_{pl} V_p(x_1, x_2), \quad \forall x_1, x_2 \in \mathbb{R}^n; \quad (6)$$

(A.3) the MDADTs satisfy $\tau_{al} > \rho_l^{-1} \ln(\max_{p\in\mathfrak{L}}\{\mu_{pl}\})$,

then the switched system Σ is δ-GUAS.

Proof: We prove Theorem 1 by constructing a function $\beta \in \mathcal{KL}$ in (3). For all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}^+$, $\sigma(t)$ is a constant in $\mathfrak{L}$. From (A.1), all subsystems admit δ-GAS Lyapunov functions. Integrating (5) from t_k to any $t \in [t_k, t_{k+1})$ yields

$$\begin{aligned} &V_{\sigma(t)}(\mathbf{x}(t, x_1, \sigma), \mathbf{x}(t, x_2, \sigma)) \\ &\leq e^{-\rho_{\sigma(t_k)}(t-t_k)} V_{\sigma(t_k)}(\mathbf{x}(t_k, x_1, \sigma), \mathbf{x}(t_k, x_2, \sigma)). \quad (7) \end{aligned}$$

Taking $t \to t_{k+1}$, we have from (7) that

$$\begin{aligned} &V_{\sigma(t_{k+1}^-)}(\mathbf{x}(t_{k+1}^-, x_1, \sigma), \mathbf{x}(t_{k+1}^-, x_2, \sigma)) \\ &\leq e^{-\rho_{\sigma(t_k)}(t_{k+1}-t_k)} V_{\sigma(t_k)}(\mathbf{x}(t_k, x_1, \sigma), \mathbf{x}(t_k, x_2, \sigma)). \quad (8) \end{aligned}$$

At the switching time instant $t_k \in \mathcal{T}$, we obtain from (A.2), (8) and the continuity of the system state that

$$\begin{aligned} &V_{\sigma(t_{k+1})}(\mathbf{x}(t_{k+1}, x_1, \sigma), \mathbf{x}(t_{k+1}, x_2, \sigma)) \\ &\leq \mu_{\sigma(t_k)\sigma(t_{k+1})} V_{\sigma(t_{k+1}^-)}(\mathbf{x}(t_{k+1}^-, x_1, \sigma), \mathbf{x}(t_{k+1}^-, x_2, \sigma)) \\ &\leq \mu_{\sigma(t_k)\sigma(t_{k+1})} e^{-\rho_{\sigma(t_k)}(t_{k+1}-t_k)} \\ &\quad \times V_{\sigma(t_k)}(\mathbf{x}(t_k, x_1, \sigma), \mathbf{x}(t_k, x_2, \sigma)). \quad (9) \end{aligned}$$

Iterating (9) from 0 to any $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, we get

$$\begin{aligned} &V_{\sigma(t)}(\mathbf{x}(t, x_1, \sigma), \mathbf{x}(t, x_2, \sigma)) \\ &\leq \mu_{\sigma(t_{k-1})\sigma(t_k)} e^{-\rho_{\sigma(t_k)}(t-t_k)} \\ &\quad \times \prod_{0\leq t_i<t_k} \mu_{\sigma(t_{i-1})\sigma(t_i)} e^{-\rho_{\sigma(t_i)}(t_{i+1}-t_i)} \mathbf{V}_0 \\ &= \prod_{0\leq t_i \leq t} \mu_{\sigma(t_i)\sigma(t_{i+1})} e^{-\rho_{\sigma(t_i)}(t_{i+1}-t_i)} e^{-\rho_{\sigma(t)}(t-t_k)} \mathbf{V}_0 \\ &\leq \prod_{l=1}^{\mathsf{L}} \mu_l^{N_{\sigma l}(0,t)} e^{-\rho_l \mathcal{T}_l(0,t)} \alpha_{2\sigma(0)}(\|x_1 - x_2\|) \\ &\leq e^{\sum_{l=1}^{\mathsf{L}} (\ln \mu_l N_{\sigma l}(0,t) - \rho_l \mathcal{T}_l(0,t))} \alpha_{2\sigma(0)}(\|x_1 - x_2\|), \quad (10) \end{aligned}$$

where $\mu_l = \max_{p\in\mathfrak{L}}\{\mu_{pl}\}$, $\mathbf{V}_0 := V_{\sigma(0)}(x_1, x_2)$ and the second "$\leq$" holds due to (5). Moreover, for the right-hand side of (10), we have that

$$\begin{aligned} &\ln \mu_l N_{\sigma l}(0,t) - \rho_l \mathcal{T}_l(0,t) \\ &\leq \ln \mu_l (N_{0l} + \tau_{al}^{-1} \mathcal{T}_l(0,t)) - \rho_l \mathcal{T}_l(0,t) \\ &= N_{0l} \ln \mu_l + (\tau_{al}^{-1} \ln \mu_l - \rho_l) \mathcal{T}_l(0,t), \quad (11) \end{aligned}$$

where the "$\leq$" holds due to (2).

Let $\pi := \sum_{l=1}^{\mathsf{L}} \ln \mu_l N_{0l}$ and $\varpi_l := \tau_{al}^{-1} \ln \mu_l - \rho_l$. Hence, π is a constant and $\varpi_l < 0$ from (A.3). From (10) and (11), we have that for all $t \geq 0$, $V_{\sigma(t)}(\mathbf{x}(t, x_1, \sigma), \mathbf{x}(t, x_2, \sigma)) \leq \exp(\pi + \sum_{l=1}^{\mathsf{L}} \varpi_l \mathcal{T}_l(0,t)) \alpha_{2\sigma(0)}(\|x_1 - x_2\|)$, combining which with (A.1) yields that for all $\sigma(t) \in \mathcal{L}$, all $x_1, x_2 \in \mathbb{R}^n$ and all $t \geq 0$,

$$\|\mathbf{x}(t, x_1, \sigma) - \mathbf{x}(t, x_2, \sigma)\| \leq \beta(\|x_1 - x_2\|, t), \quad (12)$$

where $\beta(v, s) := \alpha_1^{-1}(e^{(\pi + \sum_{l=1}^{\mathsf{L}} \varpi_l \mathcal{T}_l(0,s))} \alpha_2(v))$, $\alpha_1(v) := \min_{l\in\mathfrak{L}} \alpha_{1l}(v)$ and $\alpha_2(v) := \max_{l\in\mathfrak{L}} \alpha_{2l}(v)$. As a result, the switched system Σ is δ-GUAS. ■

In Theorem 1, multiple Lyapunov functions are applied here, and condition (A.1) is a considerable relaxation to the cases in [12], [15] based on common Lyapunov function. Condition (A.2) is to measure the jumps of multiple Lyapunov functions caused by the switching. μ_{pl} is related to the currently-activated subsystem and the subsystem to be activated, and thus μ_{pl} depends on the system modes. Hence, condition (A.3) is mode-dependent, and each subsystem has its own ADT, which provides a flexibility for the design of the switching strategy. Condition (A.3) can be written equivalently as $\tau_{al} > \rho_l^{-1} \ln(\max_{p\in\mathfrak{L}}\{\mu_{lp}\})$. If μ_{pl} is only related to the currently-activated subsystem (or the subsystem to be activated), then we can change μ_{pl} to μ_p (or μ_l), and further condition (A.3) is reduced to $\tau_{al} > \rho_l^{-1} \ln \mu_l$ such that the δ-GUAS is guaranteed for the system Σ.

The following assumption is made on multiple Lyapunov functions [12], [14]. For each $l \in \mathfrak{L}$, assume that there exist $\gamma_l \in \mathcal{K}_\infty$ such that, for all $x_1, x_2, x_3 \in \mathbb{R}^n$,

$$|V_l(x_1, x_2) - V_l(x_1, x_3)| \leq \gamma_l(\|x_2 - x_3\|). \quad (13)$$

As shown in [14], this assumption is not restrictive if we are interested in dynamic systems on bounded subsets of $\mathbb{R}^n$, which is generally the case in practice.

V. Multirate Multiscale Symbolic Model

The main results of this paper are presented in this section, and multiscale symbolic models are constructed for switched systems in two different cases. For this purpose, we start with the discretization of switched systems.

A. Multirate Time Discretization of Switched Systems

To derive the time-discretization of the switched system Σ, the sampling technique is applied and the sampling period is assumed to be $\tau > 0$, which is a design parameter. Besides the sampling technique, we need to study the constraints on activation durations of all subsystems. To this end, we first assume that the switching is determined by a self-triggered controller [21], which selects the system mode and the corresponding activation duration. Next, we establish the set from which the self-triggered controller can choose the activation durations, and the following assumption is made.

Assumption 1: The dwell-times of all subsystem are bounded in $[\tau_{\min}, \tau_{\max}]$ with $\tau_{\max} \geq \tau \geq \tau_{\min} > 0$.

From Assumption 1, the dwell-times of each subsystem can be neither too short (which is impractical and may cause chattering or Zeno phenomena) nor too long (which is reduced to a single system [5], [22]), but is usually limited within a bounded interval. Hence, Assumption 1 relaxes the conditions in [11], [14], [18], [19]. From Assumption 1, the boundedness of dwell-times implies the boundedness of all MDADTs by considering the extreme cases. That is, $\tau_{\min}$ and $\tau_{\max}$ are respectively the lower and upper bounds of the MDADTs. In this way, we can impose the conditions in Theorems 1-2 on the choice of $\tau_{\min}$. In addition, $[\tau_{\min}, \tau_{\max}]$

is for all MDADTs, and the boundedness of all dwell-times are unknown and cannot be computed, which results in the difficulties in the abstraction construction.

Based on the sampling period, the interval $[\tau_{\min}, \tau_{\max}]$ is approximated via the following set

$$\Theta_1 := \{\theta_s = 2^{-s}\tau : s \in \{-N_2, \ldots, N_1\}\},$$

where $N_1, N_2 \in \mathbb{N}^+$ satisfy $2^{-(N_1+1)}\tau < \tau_{\min} \leq 2^{-N_1}\tau$ and $2^{N_2}\tau \leq \tau_{\max} < 2^{(N_2+1)}\tau$. Since the set Θ_1 is to approximate the continuous-time interval $[\tau_{\min}, \tau_{\max}]$ based on the sampling period, the activation durations from Θ_1 may be too coarse to extract enough information on the interval $[\tau_{\min}, \tau_{\max}]$. As a result, we need to further refine Θ_1 to obtain more admissible activation durations. To this end, the sampling period is approximated via the following set:

$$\Theta_2 := \{\theta_s = 2^{-s}\tau : s \in \mathcal{N}_{\mathfrak{a}} := \{0, \ldots, N_1\}\},$$

which consists of dyadic fractions of the sampling period $\tau > 0$ up to some scale parameter $N_1 \in \mathbb{N}$. With Θ_1 and Θ_2, the interval $[\tau_{\min}, \tau_{\max}]$ is approximated via the following set

$$\Theta_\tau := \{\theta_s = 2^{-N_1}k\tau : k \in \mathcal{N}_\tau\},$$

where $\mathcal{N}_\tau := \{1, \ldots, \lceil 2^{N_1}(\tau_{\max} - \tau_{\min})/\tau \rceil\}$.

With the sampling period τ and the set Θ_τ, the time discretization of the switched system Σ is described as a transition system $\mathbf{T}_\tau(\Sigma) := (\mathbb{X}_1, \mathbb{X}_1^0, \mathbb{U}_1, \Delta_1, \mathbb{Y}_1)$, where,

- the state set is $\mathbb{X}_1 = \mathbb{R}^n \times \mathfrak{L}$;
- the set of initial states is $\mathbb{X}_1^0 = \mathbb{R}^n \times \mathfrak{L}$;
- the input set is $\mathbb{U}_1 = \mathfrak{L} \times \Theta_\tau$;
- $(z', y) = \Delta_1(x, u)$ if and only if, for any $z = (x, l) \in \mathbb{X}_1$ and $u = (l', \theta_s) \in \mathbb{U}_1$,
  - for $\theta_s \leq \tau$, $z' = (x', l')$ and $y = \mathbf{x}|_{\theta_s}(\cdot, x, l')$ with $x' = \mathbf{x}(\theta_s, x, l')$;
  - for $\theta_s > \tau$, $z' = (x', l')$ with $x' = \mathbf{x}(\theta_s, x, l')$, and $y = (x, \mathbf{x}(\tau, x, l'), \ldots, \mathbf{x}(r\tau, x, l'))$,

  where $r := \lfloor (\theta_s - \omega)/\tau \rfloor \in \mathcal{N}_{\mathfrak{a}}$, and $\omega > 0$ is sufficiently small;
- the output set is $\mathbb{Y}_1 := \mathbb{Y}_{11} \cup \mathbb{Y}_{12}$, where $\mathbb{Y}_{11} := \cup_{\theta_s \in \Theta_2} \mathcal{C}([0, \theta_s], \mathbb{R}^n)$ and $\mathbb{Y}_{12} := \cup_{r \in \mathcal{N}_{\mathfrak{a}}} \mathbb{R}^{(r+1)\times n}$.

In the transition system $\mathbf{T}_\tau(\Sigma)$, the state is augmented as $(x, l) \in \mathbb{X}_1$ to include the state $x \in \mathbb{R}^n$ and the active mode $l \in \mathfrak{L}$. The constant $\omega > 0$ is introduced to avoid that x' is included in the output y when $\theta_s = k\tau$, $k \in \mathbb{N}$. The transition relation has two cases: $\theta_s \leq \tau$ and $\theta_s > \tau$. In these two cases, the evolution of the state is of the same mechanism, and the difference lies in the outputs. In the first case, $\theta_s \in \Theta_{\mathfrak{a}} := \Theta_\tau \cap \Theta_2$, and the output is a piecewise continuous function. In the second case, $\theta_s \in \Theta_{\mathfrak{b}} := \Theta_\tau \backslash \Theta_2$, and the output is a finite sequence of discrete-time states. As a result, we can distinguish short durations (i.e., $\theta_s \leq \tau$) from long durations (i.e., $\theta_s > \tau$) via the outputs. This setting is suitable in controller synthesis with continuous-time or hybrid specifications. Note that the dwell-time constraints are fulfilled by construction in these two cases.

The transition system $\mathbf{T}_\tau(\Sigma)$ is non-blocking and deterministic. Moreover, $\mathbf{T}_\tau(\Sigma)$ is metric when the set of outputs $\mathbb{Y}_1$ is equipped with the following metric:

- for $y \in \mathcal{C}([0, \theta_{s_1}], \mathbb{R}^n)$ and $y' \in \mathcal{C}([0, \theta_{s_2}], \mathbb{R}^n)$ with $\theta_{s_1}, \theta_{s_2} \in \Theta_2$,

$$\mathbf{d}(y, y') = \begin{cases} \|y - y'\|, & \text{if } \theta_{s_1} = \theta_{s_2}, \\ +\infty, & \text{otherwise}; \end{cases} \tag{14}$$

- for $y \in \mathbb{R}^{(r_1+1)\times n}$ and $y' \in \mathbb{R}^{(r_2+1)\times n}$,

$$\mathbf{d}(y, y') = \begin{cases} \max\limits_{1 \leq j \leq r_1} \|y_j - y'_j\|, & \text{if } r_1 = r_2, \\ +\infty, & \text{otherwise}. \end{cases} \tag{15}$$

B. Symbolic Model

For the time discretization $\mathbf{T}_\tau(\Sigma)$, a multiscale symbolic model is constructed in this subsection. To this end, we first approximate the state set $\mathbb{R}^n$ by a sequence of multiscale embedded lattices $[\mathbb{R}^n]_{2^{-s}\eta}$ with $s \in \mathcal{N}_{\mathfrak{a}}$, where $[\mathbb{R}^n]_{2^{-s}\eta} := \{q \in \mathbb{R}^n : q_i = k_i 2^{1-s}\eta, k_i \in \mathbb{Z}, i \in \{1, \ldots, n\}\}$, where $\eta > 0$ is called the state space sampling parameter. Note that $[\mathbb{R}^n]_\eta \subseteq \ldots \subseteq [\mathbb{R}^n]_{2^{-N_1}\eta}$. We associate a multiscale quantizer $\mathsf{Q}_\eta^s : \mathbb{R}^n \to [\mathbb{R}^n]_{2^{-s}\eta}$ such that $\mathsf{Q}_\eta^s(x) = q$ if and only if for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$q_i - 2^{-s}\eta \leq x_i \leq q_i + 2^{-s}\eta, \quad \forall i \in \{1, \ldots, n\}. \tag{16}$$

Based on geometrical considerations, we can verify that $\|x - \mathsf{Q}_\eta^s(x)\| \leq 2^{-s}\eta$ for a given $s \in \mathcal{N}_{\mathfrak{a}}$ and all $x \in \mathbb{R}^n$.

In the following, the multiscale symbolic model of the switched system $\mathbf{T}_\tau(\Sigma)$ is constructed as the transition system $\mathbf{T}_{\tau,\eta}(\Sigma) = (\mathbb{X}_2, \mathbb{X}_2^0, \mathbb{U}_2, \Delta_2, \mathbb{Y}_2)$, where,

- the state set is $\mathbb{X}_2 = [\mathbb{R}^n]_{2^{-N_1}\eta} \times \mathfrak{L}$;
- the set of initial states is $\mathbb{X}_2^0 = [\mathbb{R}^n]_\eta \times \mathfrak{L}$;
- the input set is $\mathbb{U}_2 = \mathfrak{L} \times \Theta_\tau$;
- $(w', y) = \Delta_2(w, u)$ if and only if, for $w = (q, l) \in \mathbb{X}_2$, $u = (l', \theta_s) \in \mathbb{U}_2$ and $y \in \mathbb{Y}_2$,
  - for $u \in \mathfrak{L} \times \Theta_{\mathfrak{a}}$, $w' = (q', l')$ with

$$q' = \mathsf{Q}_\eta^s(\mathbf{x}(\theta_s, q, l')), \quad y = \mathbf{x}|_{\theta_s}(\cdot, q, l'); \tag{17}$$

  - for $u \in \mathfrak{L} \times \Theta_{\mathfrak{b}}$, $w' = (q', l')$ with

$$q' = \mathsf{Q}_\eta^{s'}(\bar{x}'), \quad ((\bar{x}', l'), y) = \Delta_1(q, u), \tag{18}$$

  where $s' := \arg\{s' \in \mathcal{N}_{\mathfrak{a}} : \theta_{s'} = \mod(\theta_s - \omega, \tau) \in \Theta_2\}$ with sufficiently small $\omega > 0$;
- the output set is $\mathbb{Y}_2 = \mathbb{Y}_1$.

The system $\mathbf{T}_{\tau,\eta}(\Sigma)$ is symbolic, nonblocking, and deterministic. In addition, $\mathbf{T}_{\tau,\eta}(\Sigma)$ is metric when the set of outputs $\mathbb{Y}_2$ is equipped with the metric given in (14)-(15). Since the controller can choose a duration from the set Θ_τ, the control period is in a multirate setting. In addition, the abstract state is based on the multiscale quantizer $\mathbf{Q}_\eta^s$ with $s \in \mathcal{N}$. Hence, the symbolic model $\mathbf{T}_{\tau,\eta}(\Sigma)$ is multirate multiscale. Such a multirate multiscale setting can be applied to deal with both fast switching (i.e., short durations) and long durations. In particular, if the switched system is close to unsafe sets in a safety controller synthesis or when not all subsystems are δ-GAS, then the fast switching case is essential to be studied; long durations of stable subsystems can be used to reduce the effects of unstable subsystems (if exist) on the whole system.

C. Verification of Approximate Bisimulation Relation

The following theorem establishes the conditions to guarantee the ε-ABR between the systems $\mathbf{T}_\tau(\boldsymbol{\Sigma})$ and $\mathbf{T}_{\tau,\eta}(\boldsymbol{\Sigma})$.

Theorem 2: Consider the δ-GUAS switched system $\boldsymbol{\Sigma}$. Let Assumption 1 and (13) hold. Given the precision $\varepsilon > 0$, if there exist $\tau, \eta > 0$ such that

$$\eta \leq \min_{l\in\mathbb{P}} \Big\{ \min_{s'\in\mathcal{N}} \min_{\theta_s\in\Theta_\tau} 2^{s'}\gamma_l^{-1}((1-e^{-\rho_l\theta_s})\alpha_{1l}(\varepsilon)), \alpha_{2l}^{-1}(\alpha_{1l}(\varepsilon))\Big\}, \quad (19)$$

then $\mathbf{T}_\tau(\Sigma) \simeq_\varepsilon \mathbf{T}_{\tau,\eta}(\Sigma)$.

Proof: We define the relation $\mathcal{R} := \{((x,l^1),(q,l^2)) \in \mathbb{X}_1 \times \mathbb{X}_2 : l^1 = l^2 = l \in \mathfrak{L}, V_l(x,q) \leq \alpha_{1l}(\varepsilon)\}$. Next, we prove that $\mathcal{R}$ is an ε-ABR between $\mathbf{T}_\tau(\Sigma)$ and $\mathbf{T}_{\tau,\eta}(\Sigma)$. From (16), $\|x - q\| \leq 2^{-s}\eta$ for all $s \in \{0,\ldots,N_1\}$, and $V_l(x,q) \leq \alpha_{2l}(2^{-s}\eta) \leq \alpha_{1l}(\varepsilon)$, which implies that $\eta \leq \alpha_{2l}^{-1}(\alpha_{1l}(\varepsilon))$ as in (19). In the following, based on the relation between θ_s and τ, the proof is divided into two cases: the case $\theta_s \leq \tau$ and the case $\theta_s > \tau$.

Case 1: $\theta_s \in \Theta_\mathfrak{a}$. Let $(z,w) = ((x,l^1),(q,l^2)) \in \mathcal{R}$. Let $u = (l',\theta_s) \in \mathbb{U}_1$, $(z',y) = \Delta_1(x,u)$ and $(w',v) = \Delta_2(q,u)$, where $z' = (x',l')$ and $w' = (q',l')$. Since $\theta_s \in \Theta_\mathfrak{a}$, we have from (5) that for all $t \in [\tau_{\min},\theta_s]$,

$$\begin{aligned} V_{l'}(\mathbf{x}(t,x,l'),\mathbf{x}(t,q,l')) &\leq e^{-\rho_{l'}t}V_{l'}(x,q) \\ &\leq V_{l'}(x,q) \leq \alpha_{1l'}(\varepsilon). \end{aligned} \quad (20)$$

From (4) and (20), we yield that, for all $t \in [\tau_{\min},\theta_s]$,

$$\begin{aligned} \|y(t) - v(t)\| &= \|\mathbf{x}(t,x,l') - \mathbf{x}(t,q,l')\| \\ &\leq \alpha_{1l'}^{-1}(V_{l'}(\mathbf{x}(t,x,l'),\mathbf{x}(t,q,l'))) \\ &\leq \alpha_{1l'}^{-1}(V_{l'}(x,q)) \leq \varepsilon. \end{aligned} \quad (21)$$

That is, $\mathbf{d}(y,v) \leq \varepsilon$. Since $q' = \mathsf{Q}_\eta^s(\mathbf{x}(\theta_s,q,l'))$, we have from (13) that

$$\begin{aligned} |V_{l'}(x',q') - V_{l'}(x',\mathbf{x}(\theta_s,q,l'))| &\leq \gamma_{l'}(\|q' - \mathbf{x}(\theta_s,q,l')\|) \\ &\leq \gamma_{l'}(2^{-s}\eta), \end{aligned}$$

which implies

$$\begin{aligned} V_{l'}(x',q') &\leq V_{l'}(x',\mathbf{x}(\theta_s,q,l')) + \gamma_{l'}(2^{-s}\eta) \\ &= V_{l'}(\mathbf{x}(\theta_s,x,l'),\mathbf{x}(\theta_s,q,l')) + \gamma_{l'}(2^{-s}\eta) \\ &\leq e^{-\rho_{l'}\theta_s}V_{l'}(x,q) + \gamma_{l'}(2^{-s}\eta), \end{aligned}$$

where the second "$\leq$" holds from (20). Hence, we have $V_{l'}(x',q') \leq e^{-\rho_{l'}\theta_s}\alpha_{1l'}(\varepsilon) + \gamma_{l'}(2^{-s}\eta) \leq \alpha_{1l'}(\varepsilon)$, where, the first "$\leq$" holds due to $(x,q) \in \mathcal{R}$, and the second "$\leq$" holds from (19). We thus conclude that $(z',w') \in \mathcal{R}$.

Case 2: $\theta_s \in \Theta_\mathfrak{b}$. Let $(z,w) = ((x,l^1),(q,l^2)) \in \mathcal{R}$, we have $l^1 = l^2 = l$ and $V_l(x,q) \leq \alpha_{1l}(\varepsilon)$. Let $r := \lfloor \theta_s/\tau \rfloor \in \mathbb{N}$ and $\theta_{s'} = \theta_s - r\tau \in \Theta_2$. Similarly to (20), we obtain that for all $k \in \{0,\ldots,r\}$,

$$\begin{aligned} V_{l'}(\mathbf{x}(k\tau,x,l'),\mathbf{x}(k\tau,q,l')) &\leq e^{-k\tau\rho_{l'}}V_{l'}(x,q) \\ &\leq V_{l'}(x,q) \leq \alpha_{1l'}(\varepsilon). \end{aligned}$$

Similarly to (21), we conclude that for all $k \in \{0,\ldots,r\}$, $\mathbf{d}(y_k,v_k) \leq \varepsilon$ with $y_k := \mathbf{x}(k\tau,x,l')$ and $v_k := \mathbf{x}(k\tau,q,l')$, which further implies that $\mathbf{d}(y,v) \leq \varepsilon$.

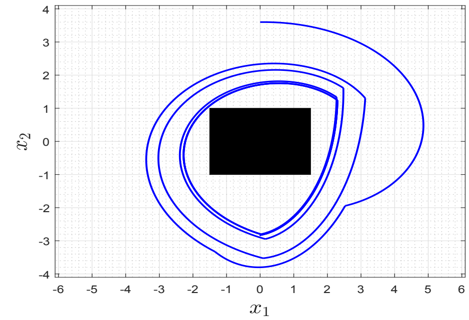

Fig. 1. The state trajectory of the system Σ with the initial state $x_0 = (0,3)$. The black region is the obstacle.

Since $\theta_s = \theta_{s'} + r\tau$, we have that

$$\begin{aligned} &V_{l'}(x',q') \\ &= V_{l'}(\mathbf{x}(\theta_{s'},\mathbf{x}(r\tau,x,l'),l'), \mathsf{Q}^{s'}(\mathbf{x}(\theta_{s'},\mathbf{x}(r\tau,q,l'),l'))) \\ &= V_{l'}(\mathbf{x}(\theta_{s'},\mathbf{x}(r\tau,x,l'),l'), \mathbf{x}(\theta_{s'},\mathbf{x}(r\tau,q,l'),l')) \\ &\quad + \gamma_{l'}(2^{-s'}\eta) \\ &\leq e^{-\theta_s\rho_{l'}}V_{l'}(x,q) + \gamma_{l'}(2^{-s'}\eta). \end{aligned}$$

Thus, we have from (19) that $(z',w') \in \mathcal{R}$.

From the above analysis for two different cases, we conclude that $\mathcal{R}$ is an ε-ABR for $\mathbf{T}_\tau(\Sigma)$ and $\mathbf{T}_{\tau,\eta}(\Sigma)$. ■

Theorem 2 implies that the state space sampling parameter η is upper bounded in (19) to achieve the ε-ABR. The upper bound is related to system modes, and exists since $1 - e^{-\rho_l\theta_s} > 0$ holds for all $\theta_s \in \Theta_\tau$. Hence, given the parameters $\tau > 0$ and $N_1, N_2 \in \mathbb{N}$, there exists $\eta > 0$ such that (19) holds for any desired precision $\varepsilon > 0$.

VI. Numerical Example

We here borrow the same example as in [14]. Consider a two-dimensional switched affine system with two modes (i.e. $n = 2$, $\mathsf{L} = 2$) given by

$$\Sigma : \ \dot{x}(t) = A_{p(t)}x(t) + B_{p(t)},$$

with $A_1 = \begin{bmatrix} -0.5 & 1 \\ -2 & -0.5 \end{bmatrix}$, $B_1 = \begin{bmatrix} -0.25 \\ -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.25 & 2 \\ -1 & -0.25 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 0.25 \\ 1 \end{bmatrix}$. These two subsystems are δ-GAS. For the switched system Σ, we consider a control design problem such that the following specification is satisfied: the trajectory of the switched system Σ is within the state space $\mathbb{X}$ while avoiding the obstacle $\mathbb{O} \subset \mathbb{X}$. Assume that the obstacle $\mathbb{O}$ contains the equilibrium points of all subsystems, and thus the specification cannot be achieved by neither the subsystem Σ_1 nor the subsystem Σ_2.

To deal with such a control problem, we aim to derive its symbolic model using the developed results in the previous sections. To this end, the δ-GUAS needs to be studied first for the switched system Σ. Note that the switched system Σ does not have a common δ-GUAS Lyapunov function but admits the multiple δ-GAS Lyapunov functions for two subsystems. Choose the multiple Lyapunov functions as $V_p(x,y) = \sqrt{(x-y)^\top M_p(x-y)}$ with $M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. We can compute that $\mu_1 = \mu_2 = \sqrt{2}$, $\dot{V}_1(x_1,x_2) \leq$

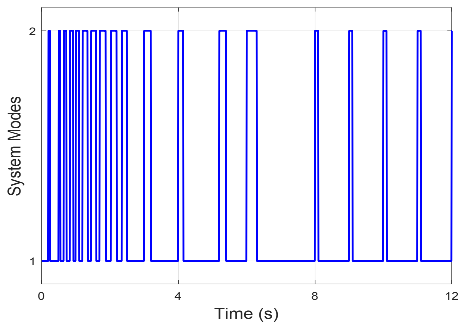

Fig. 2. Switching signal controlled with the symbolic controller and satisfying the MDADT conditions.

$-0.5V_1(x_1, x_2)$ and $\dot{V}_2(x_1, x_2) \leq -0.25V_2(x_1, x_2)$. Hence, both subsystems are δ-GAS. From Theorem 1, if $\tau_{a1} \geq 0.6931$ and $\tau_{a2} \geq 1.3863$, then the system Σ is δ-GUAS. The conditions (3) and (6) are satisfied with $\alpha_{11}(v) = \alpha_{21}(v) = v$, $\alpha_{12}(v) = \alpha_{22}(v) = \sqrt{2}v$ and $\gamma_1(v) = \gamma_2(v) = \sqrt{2}v$.

Let the dwell-times be bounded in $[0.2, 3.1]$ and the sampling period be $\tau = 0.5$. Hence, $N_1 = 1, N_2 = 2, \Theta_1 = \{0.25, 0.5, 1, 2\}$ and $\Theta_2 = \{0.25, 0.5\}$. Furthermore, $\Theta_\tau = \{0.25k : k \in \{1, \ldots, 12\}\}$. Compared with [12], [14] where the dwell-times of all subsystems are larger than the sampling period, we allow the dwell-times of all subsystems to be smaller than the sampling period. From Theorem 2, the desired precision is guaranteed by choosing the parameter η satisfying $\eta \leq 0.0428\varepsilon$, which is less conservative than $\eta \leq \varepsilon/48$ in [14] based on the constant dwell-time.

Let $\varepsilon = 0.34$ be the precision and $\eta = 0.0144$ is chosen. The state space is $\mathbb{X} = [-6, 6] \times [-4, 4]$ and the obstacle region is $\mathbb{O} = [-1.5, 1.5] \times [-1, 1]$. Using the procedure in Section V-B, the symbolic model $\mathbf{T}_{\tau,\eta}(\Sigma)$ is constructed with 7405694 abstract states, the number of which is smaller than that (i.e., 7696008) in [14]. Since the condition for η is less conservative than the one in [14], the number of symbolic states is reduced here. Note that the number of the transitions depends on the cardinality of Θ_τ. At each abstract state, there exists 12 dwell-times to be chosen from Θ_τ, and the computation times for both the transitions and the controller synthesis are related to the cardinality of Θ_τ. Hence, the number of the transitions and the computation times increase with the increase of the cardinality of Θ_τ, which is similar to the case in [12] but distinct from the case in [14]. With the symbolic model, we consider the switching design problem such that the obstacle $\mathbb{O}$ is avoided. If the second mode is activated first, the mode map of the symbolic controller is presented in Fig. 2.

VII. Conclusion

We studied symbolic models for switched systems using the multirate multiscale setting. We first addressed the incremental stability analysis of switched systems and established stability conditions, which provides alternative criteria for the stability analysis of switched systems. According to the multiscale setting and the boundedness of the dwell-times, we developed a novel multirate multiscale symbolic model for switched systems, and derive the approximate bisimulation relation between the symbolic model and the original system. Further researches are directed to the abstraction construction for more general switched systems and the applications of symbolic models.

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