2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

Pontryagin's Minimum Principle for output-feedback systems: A compact overview

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Abstract— **Pontryagin's Maximum Principle is one of two famous results towards characterization of solutions of optimal control problems. Due to a shift in the desire from maximizing gain or profit to minimizing costs it is nowadays more often and henceforth in this publication referred to as Minimum Principle. The theory behind it utilizes variational calculus and provides necessary conditions. Many versions of the minimum principle exist. Among them variations exist that consider constraints, sufficient conditions, whereas other realize time-discrete formulations of the problem. Nevertheless, so far a generalization towards output-feedback systems is not found in the literature. The goal of this publication is to extend the existing theory via including an output function and variables. Additionally, general equality and inequality constraints as well as terminal constraints and sufficient conditions will be incorporated. The original Pontryagin's Minimum Principle can then be seen as a special case of the derived criteria.**

I. Introduction

Optimal control (OC) [1], [2] has become a very important topic in the 1950s and is nowadays indispensable in the fields of e.g. robotics, aviation, and financial economics. In OC an objective function, representing costs or the deviation from a target state, shall be minimized while the development of the system's states is given via ordinary differential equations. Two famous results, the Hamilton-Jacobi-Bellman equation (HJBE) which is based on Bellman's principle of optimality [3] regarding end pieces of optimal trajectories and Pontryagin's Minimum Principle (PMP) which characterizes optimal solutions using small deviations, have been developed. The nowadays so-called Pontryagin's Minimum Principle was first derived by Lev Semenovich Pontryagin and his students [4], [5], [6]. Its concept is to consider small variations of an optimal trajectory and investigate the resulting change in the objective [7], [1], i.e. lifting the idea of differential calculus from $\mathbb{R}^n$ to function spaces. The idea behind the method is illustrated in Fig. 1. Shown are the optimal solution trajectory $x(\cdot)$ in the state space, variations $(x + \Delta x)(\cdot)$ of that trajectory and the reachable region for a given initial vector $x(0)$ and target vector $x(t_f)$ at a final time t_f. The advantage of PMP over the HJBE is the possibility of consideration of a free final time and free final state, compare Fig. 2. Unfortunately, PMP can not be used to derive a feedback-law but rather for numerical solutions.

Over the decades, the theory behind PMP has been extended to allow the consideration of constraints regarding the state and input variables [8], [9], [10] as well as the terminal state [11], [12], [13]. A discrete-time version has been developed [8], [14] and PMP has been adapted to other types of differential equations, e.g. semilinear and quasiliniear parabolic equations [15]. The minimum principle itself only delivers necessary conditions. However, sufficient conditions using convexity conditions are given in the literature [16], [17], [11]. Furthermore, PMP can be used to derive the HJBE and vice versa [18]. Despite all of the mentioned extensions and applications, to the best knowledge of the author a version of PMP which is applicable to output-feedback systems can not be found in the literature except for one special case in the appendix of [19]. The focus of this work is to generalize the minimum principle to allow systems that include an output function, see Section II. Furthermore, in Section III the new results will be combined with the constraint case including time-dependent equality and inequality constraints as well as terminal constraints for the input and output. An illustrative application example, the incline phase of a rocket launch, will be given in Section IV. In Section V, the equations derived in Section II are used to obtain the output-feedback version of the HJBE and sufficient conditions are stated. An outlook and concluding remarks will be given in Section VI.

II. Pontryagin's Minimum Principle for output-feedback systems

To obtain a version of PMP which is applicable to output-feedback systems we will focus on a proof that is based on the classical PMP equations and application of the chain rule. Instead of this very simple and short calculation one could, e.g., also follow the derivations in [1] using variation

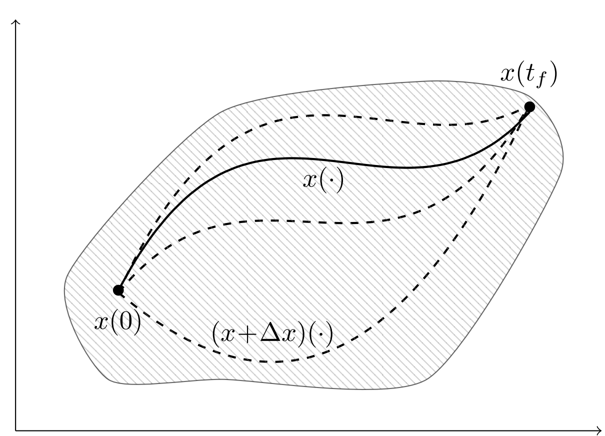

Fig. 1. Optimal solution (solid line) and suboptimal solutions (dashed lines) in state space as well as the reachable region (hatched area) when starting in $x(0)$

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methods. A proof using the variation approach can be found in the appendix of [19] for the special case of a time-independent system with an infinite horizon or in [20] for the same setup as considered in this paper. With more effort this proof can be extended to the cases that will be discussed throughout this section. Nevertheless, the in this paper will be on the derivation using the chair rule approach.

To prepare for the output-feedback version we first recap the classical version which was derived for the following class of optimal control problems.

$$\min_{u(\cdot)} \int_0^{t_f} \ell\big(\tau, x(\tau), u(\tau)\big)\,\mathrm{d}\tau + L\big(t_f, x(t_f)\big) \tag{1a}$$

$$\text{s.t. } \forall t \in \mathbb{R}_{\geq 0}: \; \dot{x}(t) = f\big(t, x(t), u(t)\big) \tag{1b}$$

$$x(0) = x_0 \in \mathbb{R}^{n_x} \tag{1c}$$

Here $f \in C^1\big(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}; \mathbb{R}^{n_x}\big)$ is the function representing the systems dynamics, $\ell \in C^1\big(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}; \mathbb{R}\big)$ is the stage cost, and $L \in C^1\big(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x}; \mathbb{R}\big)$ the terminal cost function. The numbers $n_x, n_u \in \mathbb{N}$ are the dimensions of the state and input space, respectively, while the final time is denoted with t_f. For this setup Lev Semenovich Pontryagin and his students derived the well known Pontryagin's Minimum Principle, which can be split up into four cases letting the final time t_f and the final state $x(t_f) = x_f$ be free or fixed.

Theorem 1: (Pontryagin's Minimum Principle)
Given an optimal control problem (OCP) as in (1) with ℓ, L, and f continuously differentiable. Then for an optimal solution (x, u, λ) the following conditions must hold.

$$\begin{aligned}
\dot{\lambda}^\top(t) &= -\lambda^\top(t) \cdot \nabla_x f\big(t, x(t), u(t)\big) - \nabla_x \ell\big(t, x(t), u(t)\big),\\
0 &= \nabla_u \ell\big(t, x(t), u(t)\big) + \lambda^\top(t) \cdot \nabla_u f\big(t, x(t), u(t)\big),\\
\dot{x}(t) &= f\big(t, x(t), u(t)\big),\\
x(0) &= x_0
\end{aligned}$$

Hereby, $\lambda \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}^{n_x}\big)$ is a Lagrange multiplier which in the field of control theory is often called co-state. If in addition the final time and/or final state is free further conditions apply.

a) Final time t_f and final state x_f are fixed: $x(t_f) = x_f$
b) Final time t_f is fixed and final state x_f is free:

$$\lambda^\top(t_f) = \nabla_x L\big(t_f, x(t_f)\big) \tag{2}$$

c) Final time t_f is free and final state x_f is fixed: $x(t_f) = x_f$ and

$$\begin{aligned}
0 = {} & \ell\big(t_f, x(t_f), u(t_f)\big) + \lambda^\top(t_f) \cdot f\big(t_f, x(t_f), u(t_f)\big)\\
& + \frac{\partial}{\partial t} L\big(t_f, x(t_f)\big)
\end{aligned} \tag{3}$$

d) Final time t_f and final state x_f are free: (2) and (3)

Now we consider the output-feedback case, i.e. the OCP

$$\min_{u(\cdot)} \int_0^{t_f} \ell\big(\tau, y(\tau), u(\tau)\big)\,\mathrm{d}\tau + L\big(t_f, y(t_f)\big) \tag{4a}$$

$$\text{s.t. } \forall t \in \mathbb{R}_{\geq 0}: \; \dot{x}(t) = f\big(t, x(t), u(t)\big) \tag{4b}$$

$$\forall t \in \mathbb{R}_{\geq 0}: \; y(t) = h\big(t, x(t)\big) \tag{4c}$$

$$x(0) = x_0 \in \mathbb{R}^{n_x} \tag{4d}$$

In addition to the OCP (1) the output function $h \in C^1\big(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x}; \mathbb{R}^{n_y}\big)$ is added. The stage cost function ℓ as well as the terminal cost L now depend on the output values $y \in \mathbb{R}^{n_y}$ instead of the states x but remain of class C^1. Furthermore, $n_y \in \mathbb{N}$ is the dimension of the output.
If we now substitute (4c) into ℓ and L and define

$$\begin{aligned}
\tilde{\ell}\big(t, x(t), u(t)\big) &:= \ell\big(t, h\big(t, x(t)\big), u(t)\big),\\
\tilde{L}\big(t_f, x(t_f)\big) &:= L\big(t_f, h\big(t_f, x(t_f)\big)\big),
\end{aligned}$$

the OCP (4) is of the same type as (1). Applying the chain rule now leads to the following:

Corollary 2: (PMP for output-feedback systems)
Suppose an OCP as in (4) with ℓ, L, f, and h continuously differentiable. Then the following conditions for an optimal solution (x, y, u, λ) must hold.

$$\begin{aligned}
\dot{\lambda}^\top(t) = {} & -\lambda^\top(t) \cdot \nabla_x f\big(t, x(t), u(t)\big)\\
& - \nabla_y \ell\big(t, y(t), u(t)\big) \cdot \nabla_x h\big(t, x(t)\big)
\end{aligned} \tag{5a}$$

$$0 = \nabla_u \ell\big(t, y(t), u(t)\big) + \lambda^\top(t) \cdot \nabla_u f\big(t, x(t), u(t)\big) \tag{5b}$$

$$\dot{x}(t) = f\big(t, x(t), u(t)\big) \tag{5c}$$

$$y(t) = h\big(t, x(t)\big) \tag{5d}$$

$$x(0) = x_0$$

Depending on whether or not the final time and final state are fixed or free additional conditions apply.

a) Final time t_f and final state x_f are fixed: $x(t_f) = x_f$
b) Final time t_f is fixed and final state x_f is free:

$$\lambda^\top(t_f) = \nabla_y L\big(t_f, y(t_f)\big) \cdot \nabla_x h\big(t_f, x(t_f)\big) \tag{6}$$

c) Final time t_f is free and final state x_f is fixed: $x(t_f) = x_f$ and

$$\begin{aligned}
0 = {} & \ell\big(t_f, y(t_f), u(t_f)\big) + \lambda^\top(t_f) \cdot f\big(t_f, x(t_f), u(t_f)\big)\\
& + \frac{\partial}{\partial t} L\big(t_f, y(t_f)\big)\\
& + \nabla_y L\big(t_f, y(t_f)\big) \cdot \frac{\partial}{\partial t} h\big(t_f, x(t_f)\big)
\end{aligned} \tag{7}$$

d) Final time t_f and final state x_f are free: (6) and (7)

To obtain (7) it is crucial to notice that

$$\begin{aligned}
\frac{\partial}{\partial t} \tilde{L}\big(t_f, y(t_f)\big) = {} & \frac{\partial}{\partial t} L\big(t_f, h\big(t_f, x(t_f)\big)\big)\\
& + \nabla_y L\big(t_f, h\big(t_f, x(t_f)\big)\big) \cdot \frac{\partial}{\partial t} h\big(t_f, x(t_f)\big)
\end{aligned}$$

instead of $\frac{\partial}{\partial t} L(t_f, h(t_f, x(t_f)))$. The additional term - compared to the state-feedback version - appears due to the

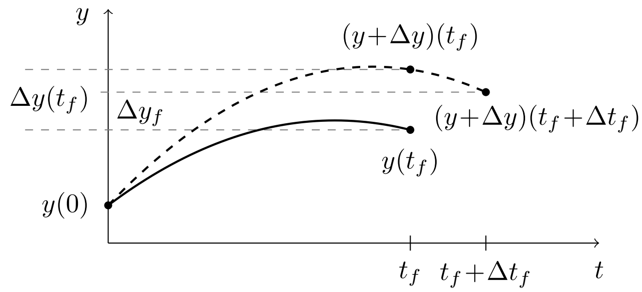

Fig. 2. Variation $(y + \Delta y)(t_f + \delta t_f)$ of the final output vector $y(t_f)$

explicit dependency of the output function h on the time. Using the chain rule is a very elegant way to derive Equations (6) and (7) based on (2) and (3). The complexity of considering variations $t_f + \Delta t_f$ of the final time t_f and variations $(y+\Delta y)(t_f+\delta t_f)$ of the final output $y_f = y(t_f)$ at the same time remains well hidden. To be more precise, Equation (6) is used to determine

$$\begin{aligned}\Delta y_f &= (y + \Delta y)(t_f + \Delta t_f) - y(t_f)\\ &= h\big(t_f + \Delta t_f, (x + \Delta x)(t_f + \Delta t_f)\big) - h\big(t_f, y(t_f)\big)\\ &\overset{!}{=} 0,\end{aligned}$$

while (7) characterized t_f. A visualisation of the different end points is given in Fig. 2.

Having a PMP for systems with output-feedback of the form (4c) one naturally has the idea to try the same procedure with input dependent output, e.g.

$$y(t) = h\big(t, x(t), u(t)\big).$$

Using this setup one problem, namely, the presence of $u(t_f)$ in the terminal cost L, arises. This issue could be solved by keeping $u(t_f)$ fixed and restricting the input space using terminal constraints very similar to what will be shown in the next section. On the other hand if the final time and the final state are fixed the terminal cost L is unnecessary and can be removed from (4). Thus, the minimum principle simplifies to (5) with $h\big(t, x(t)\big) \to h\big(t, x(t), u(t)\big)$.

Before investigating how constraints can be incorporated it shall be noted that in Corollary 2 all functions could also depend on some parameters $p \in \mathbb{R}^{n_p}$ ($n_p \in \mathbb{N}$) leading to a parametric version of PMP and a family of solutions $(x_p, y_p, u_p, \lambda_p)$. The parametric version of PMP for an infinite horizon can be found in the appendix of [19].

III. Constraints

The goal in this section is to extend the OCP (4) by the following constraints and again derive necessary conditions for a minimum.

$$\begin{aligned}\forall t \in \mathbb{R}_{\geq 0}: \quad & g_1\big(t, y(t), u(t)\big) \leq 0 && \text{(8a)}\\ & g_2\big(t, y(t), u(t)\big) = 0 && \text{(8b)}\\ & G_1\big(t_f, y(t_f)\big) \leq 0 && \text{(8c)}\\ & G_2\big(t_f, y(t_f)\big) = 0 && \text{(8d)}\end{aligned}$$

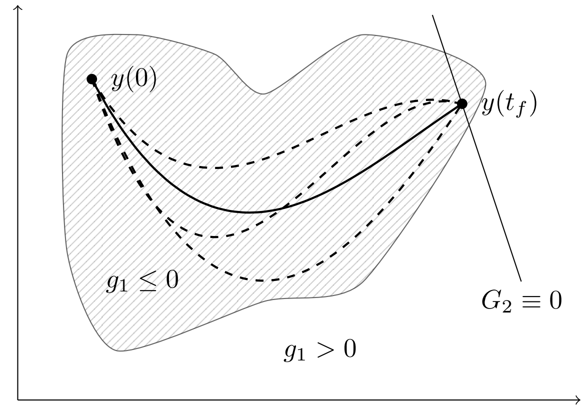

Fig. 3. Optimal solution (solid line) and suboptimal solutions (dashed lines) in output space as well as the feasible region (hatched area, $g_1 \leq 0$) and terminal equality constraints $G_2 \equiv 0$

As all functions in (4), $g_i\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_{g_i}}$, $G_i\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_{G_i}}$ ($i \in \{1,2\}$) need to be at least continuously differentiable. The numbers of constraints of each type are denoted with n_{g_i} and n_{G_i} ($i \in \{1,2\}$). Such rather general inequality and equality constraints have been already considered for state-feedback systems, see [16], [9] and others. Nevertheless, mostly constraints containing only the states or input or final state are considered, see [5], [11], [13]. An illustration of the feasible region in which all possible output trajectories must be contained is shown in Fig. 2. Additionally, the final output vector $y(t_f)$ has to be in a hyperplane defined by terminal equality constraints β_2.

To incorporate the constraints (8) Lagrange multipliers $\alpha_i \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}^{n_{g_i}}\big)$ with $\alpha_1(\cdot) \geq 0$ as well as $\beta_i \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}^{n_{G_i}}\big)$ ($i \in \{1,2\}$) with $\beta_1 \geq 0$ are introduced. To state the final theorem of this section the constraints g_1 and g_2 as well as G_1 and G_2 need one more property. A more general version of this property is stated in [9].

Definition 3: (Uniformly positively linear independence) Functions $A_i, B_j \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}\big)$ ($i \in \{1, \ldots, n_a\}$, $j \in \{1, \ldots, n_b\}$, $n_a, n_b \in \mathbb{N}$) are called uniformly in t positively linear independent if there exists $\delta > 0$ such that for any $a_i, b_j \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}\big)$ with $a_i(\cdot) \geq 0$ and

$$\forall t \geq 0: \sum_{i=1}^{n_a} a_i(t) + \sum_{j=1}^{n_b} |b_j(t)| = 1$$

it holds:

$$\left| \sum_{i=1}^{n_a} a_i(t) \cdot A_i(t) + \sum_{j=1}^{n_b} b_j(t) \cdot B_j(t) \right| \geq \delta.$$

Having this PMP for output-feedback including input and output as well as terminal equality and inequality constraints can be obtained using the results stated in [9] and the procedure outlined in Section II.

Corollary 4: (PMP for output-feedback systems with equality and inequality constraints)
Suppose an OCP as in (4) together with the constraints (8), where all functions are at least continuously differentiable and $t \mapsto g_1\big(t, y(t), u(t)\big)$ and $t \mapsto g_2\big(t, y(t), u(t)\big)$ as well as $t \mapsto G_i\big(t, y(t)\big)$ and $t \mapsto G_i\big(t, y(t)\big)$ $(i \in \{1,2\})$ are uniformly in t positively linear independent. Then for $(x, y, u, \lambda, \alpha_1, \alpha_2, \beta_1, \beta_2)$ to be an optimal solution the following conditions must hold for all $t \geq 0$.

$$\begin{aligned}
\dot{\lambda}^\top(t) &= -\lambda^\top(t) \cdot \nabla_x f\big(t, x(t), u(t)\big) \\
&\quad - \nabla_y \ell\big(t, y(t), u(t)\big) \cdot \nabla_x h\big(t, x(t)\big) && \text{(9a)} \\
&\quad - \sum_{i=1}^{2} \alpha_i^\top(t) \cdot \nabla_y g_i\big(t, y(t), u(t)\big) \cdot \nabla_x h\big(t, x(t)\big) \\
0 &= \nabla_u \ell\big(t, y(t), u(t)\big) + \lambda^\top(t) \cdot \nabla_u f\big(t, x(t), u(t)\big) \\
&\quad + \sum_{i=1}^{2} \alpha_i^\top(t) \cdot \nabla_u g_i\big(t, y(t), u(t)\big) && \text{(9b)} \\
\dot{x}(t) &= f\big(t, x(t), u(t)\big) && \text{(9c)} \\
y(t) &= h\big(t, x(t)\big) && \text{(9d)} \\
0 &= \alpha_1^\top(t) \cdot g_1\big(t, y(t), u(t)\big) && \text{(9e)} \\
0 &\geq g_1\big(t, y(t), u(t)\big) && \text{(9f)} \\
\alpha_1(t) &\geq 0, \quad \beta_1 \geq 0 && \text{(9g)} \\
0 &= g_2\big(t, y(t), u(t)\big) && \text{(9h)} \\
x(0) &= x_0 && \text{(9i)}
\end{aligned}$$

Depending on whether or not the final time and final state are fixed or free again additional conditions apply.

a) Final time t_f and final state x_f are fixed: $x(t_f) = x_f$
b) Final time t_f is fixed and final state x_f is free:

$$\begin{aligned}
\lambda^\top(t_f) &= \nabla_y L\big(t_f, y(t_f)\big) \cdot \nabla_x h\big(t_f, x(t_f)\big) && \text{(10)} \\
&\quad + \sum_{i=1}^{2} \beta_i^\top \cdot \nabla_y G_i\big(t_f, y(t_f)\big) \cdot \nabla_x h\big(t_f, x(t_f)\big)
\end{aligned}$$

c) Final time t_f is free and final state x_f is fixed:
$x(t_f) = x_f$ and

$$\begin{aligned}
0 &= \ell\big(t_f, y(t_f), u(t_f)\big) + \lambda^\top(t_f) \cdot f\big(t_f, x(t_f), u(t_f)\big) \\
&\quad + \frac{\partial}{\partial t} L\big(t_f, y(t_f)\big) \\
&\quad + \nabla_y L\big(t_f, y(t_f)\big) \cdot \frac{\partial}{\partial t} h\big(t_f, x(t_f)\big) && \text{(11)} \\
&\quad + \sum_{i=1}^{2} \beta_i^\top \cdot \nabla_y G_i\big(t_f, y(t_f)\big) \cdot \frac{\partial}{\partial t} h\big(t_f, x(t_f)\big) \\
&\quad + \sum_{i=1}^{2} \beta_i^\top \cdot \frac{\partial}{\partial t} G_i\big(t_f, y(t_f)\big)
\end{aligned}$$

d) Final time t_f and final state x_f are free: (10) and (11)

The following two final remarks of this section indicate alternative versions of Corollary 4.

i) If one of the functions ℓ, f, g_1, or g_2 is not differentiable with respect to the input u, Equation (9b) can be replaced by the more general formulation

$$u = \underset{\tilde{u}(\cdot)}{\operatorname{argmin}} \begin{cases} \ell\big(t, y(t), \tilde{u}(t)\big) + \lambda^\top(t) \cdot f\big(t, x(t), \tilde{u}(t)\big) \\ \text{s.t. (8a) and (8b).} \end{cases}$$

ii) As in Section II again $h\big(t, x(t)\big)$ can be replaced with $h\big(t, x(t), u(t)\big)$ in case the final time and final state is fixed.

IV. Example

To illustrate how to use the obtained conditions (9) an example representing the optimization of the speed profile $v(\cdot)$ and slope $\gamma(\cdot)$ to minimize the fuel consumption and time during a rocket launch is utilized. The states are the altitude $h(\cdot)$ and total mass $m(\cdot)$, while it is assumed that only the consumed fuel $y(\cdot) = m(0) - m(\cdot)$ is measured. The final time and mass are free, while the final altitude should be $9144\,\mathrm{m}$. Given some constraints for the speed, i.e. the velocity shall never be below the initial value of $128.6\,\mathrm{m\,s^{-1}}$ and should not exceed $300\,\mathrm{m\,s^{-1}}$. The slope should stay between $0\,\mathrm{deg}$ and $15\,\mathrm{deg}$. The constraint optimization problem now states as follows.

$$\begin{aligned}
&\min_{v(\cdot),\gamma(\cdot)} \alpha \cdot t_f + (1-\alpha) \cdot y(t_f) \\
&\qquad \dot{h} = v \cdot \sin(\gamma) \\
&\qquad \dot{m} = -C_{s1,T1} \cdot \left(1 + \frac{v}{C_{s2}}\right) \cdot \left(1 - \frac{h}{C_{T2}} + C_{T3} \cdot h^2\right) \\
&\qquad y = m_0 - m
\end{aligned}$$

$$\begin{aligned}
128.6 - v &\leq 0, & v - 300 &\leq 0, \\
-\gamma &\leq 0, & \gamma - 0.262 &\leq 0, \\
h_0 &= 3480, & m_0 &= 69\,000, \\
h_f &= 9144, & m_f &\ \text{free}
\end{aligned}$$

Here the coefficient $\alpha \in [0, 1]$ can be used to weight the two different objectives of minimizing, i.e. the time and the total fuel consumption. The values of the coefficients $C_{s1,T1} := C_{s1} \cdot C_{T1}$, C_{s2}, C_{T2}, and C_{T3} can be found in the following Table I and are like the entire example taken from [21]. The running cost $\ell(\cdot)$ is zero, while four constraints regarding the input variables must be fulfilled. Furthermore, one final state is fixed while the other one is free. Using the abbreviations

TABLE I
HAUL PARAMETERS

$C_{s1,T1}$	$1.487\,697\,7\,\mathrm{kg\,s^{-1}}$
C_{s2}	$441.54\,\mathrm{m\,s^{-1}}$
C_{T2}	$14\,909.9\,\mathrm{m}$
C_{T3}	$6.997 \times 10^{-10}\,\mathrm{m^2}$

$$\bar{v} := 1 + \frac{v}{C_{s2}},$$
$$\bar{h} := 1 - \frac{h}{C_{T2}} + C_{T3} \cdot h^2,$$
$$\text{and } \bar{h}' := -\frac{1}{C_{T2}} + 2C_{T3} \cdot h,$$

additionally to the equalities and inequalities in the optimization problem (9)-(11) lead to

$$\begin{aligned}
\begin{pmatrix}\dot{\lambda}_1 & \dot{\lambda}_2\end{pmatrix} &\overset{(9a)}{=} -\begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ -C_{s1,T1} \cdot \bar{v} \cdot \bar{h}' & 0 \end{pmatrix}, \\
0 &\overset{(9b)}{=} \begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix} \cdot \begin{pmatrix} \sin(\gamma) & v \cdot \cos(\gamma) \\ -\frac{C_{s1,T1}}{C_{s2}} \cdot \bar{h} & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix}-\alpha_1 + \alpha_2 & -\alpha_3 + \alpha_4\end{pmatrix}, \\
0 &\overset{(9e)}{=} \alpha_1 \cdot (128.6 - v), \\
0 &\overset{(9e)}{=} \alpha_2 \cdot (v - 300), \\
0 &\overset{(9e)}{=} \alpha_3 \cdot \gamma, \\
0 &\overset{(9e)}{=} \alpha_4 \cdot (\gamma - 0.262), \\
0 &\overset{(9g)}{\leq} \alpha_1, \alpha_2, \alpha_3, \alpha_4, \\
\lambda_2(t_f) &\overset{(10)}{=} \alpha - 1, \\
-\alpha &\overset{(11)}{=} \begin{pmatrix}\lambda_1(t_f) & \lambda_2(t_f)\end{pmatrix} \cdot \begin{pmatrix} v(t_f) \cdot \sin\big(\gamma(t_f)\big) \\ -C_{s1,T1} \cdot \bar{v}(t_f) \cdot \bar{h}(t_f) \end{pmatrix},
\end{aligned}$$

which can only be solved numerically. For the numerical solution we choose $\alpha = 0.01$, thus, preferring to limit the fuel consumption over the time to reach the target altitude. The optimizer Gurobi 10 [22] was used to obtain the solution. The time step for the discretization was set to $0.1\,\mathrm{s}$. The slope γ is constant and equals its upper boundary value of $0.262\,\mathrm{rad}$. For the second co-state it holds $\lambda_2(\cdot) \equiv \alpha - 1$ as can be seen directly from the equations. The evolution of the altitude h, output $y = m_0 - m$, velocity v, and co-state λ_1 is shown in Fig. 4. The desired altitude is reached in $90\,\mathrm{s}$ burning $\approx 182\,\mathrm{kg}$ of fuel.

V. Relation to the HJBE and Sufficiency

In this section it will be outlined how the results of Corollary 2 can be used to easily derive the HJBE ([3], [2]) for output-feedback systems. In [18] the other direction, i.e. how the HJBE can be transformed into PMP in case of state-feedback, has been shown.

The main idea is to start from equation (5a) and substitute $\lambda^\top(t)$ with

$$\nabla_y V\big(t, h(t, x(t))\big) \cdot \nabla_x h\big(t, x(t)\big) = \nabla_x V\big(t, h(t, x(t))\big),$$

where $V \in C^1\big(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n_y}; \mathbb{R}\big)$ is the so-called value function of the OCP (4).

$$V\big(t, y(t)\big) := \min_{u(\cdot)} \int_t^{t_f} \ell\big(\tau, y(\tau), u(\tau)\big)\, \mathrm{d}\tau + L\big(t_f, y(t_f)\big) \tag{12}$$

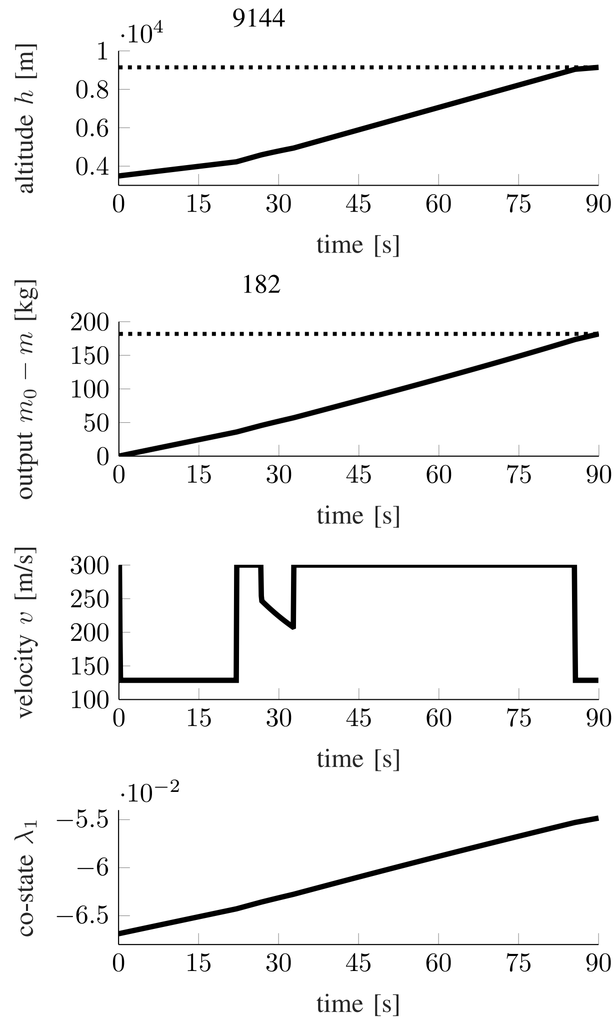

Fig. 4. Numerical solution of the rocket launch optimization problem

Keeping (5c) and (5d) in mind condition (5a) transforms into a gradient.

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\nabla_x V\big(t, h(t, x(t))\big)\Big) \\
&\quad + \nabla_x V\big(t, h(t, x(t))\big) \cdot \nabla_x f\big(t, x(t), u(t)\big) \\
&\quad + \nabla_y \ell\big(t, h(t, x(t)), u(t)\big) \cdot \nabla_x h\big(t, x(t)\big) \\
&= \nabla_x\Big[\frac{\partial}{\partial t} V\big(t, h(t, x(t))\big)\Big] \\
&\quad + \nabla_x\Big[\nabla_y V\big(t, h(t, x(t))\big) \cdot \frac{\partial}{\partial t} h\big(t, x(t)\big)\Big] \\
&\quad + \nabla_x\Big[\nabla_y V\big(t, h(t, x(t))\big) \cdot \nabla_x h\big(t, x(t)\big) \cdot f\big(t, x(t), u(t)\big)\Big] \\
&\quad + \nabla_x \ell\big(t, h(t, x(t)), u(t)\big)
\end{aligned} \tag{13}$$

The order of derivatives with respect to x and t can not simply be exchanged. The substitution of the derivative $\dot{x}(t)$ with the dynamics function $f\big(t, x(t), u(t)\big)$ leads to an extra dependency on the states. This issue is solved via incorporating the term $\nabla_x V\big(t, h(t, x(t))\big) \cdot \nabla_x f\big(t, x(t), u(t)\big)$ into the gradient. Integration with respect to x together with the terminal condition (7) yields

$$0 = \frac{\partial}{\partial t}V\big(t, y(t)\big) + \nabla_y V\big(t, y(t)\big) \cdot \frac{\partial}{\partial t}h\big(t, x(t)\big) + \nabla_y V\big(t, y(t)\big) \cdot \nabla_x h\big(t, x(t)\big) \cdot f\big(t, x(t), u(t)\big) + \ell\big(t, y(t), u(t)\big), \quad (14)$$

which is the HJBE for the output-feedback and time-dependent case. The second terminal condition (6) is included in the definition of the value function (12) since $V\big(t_f, y(t_f)\big) = L\big(t_f, y(t_f)\big)$. Finally, the derivative of (14) with respect to u leads to

$$0 = \nabla_y V\big(t, y(t)\big) \cdot \nabla_x h\big(t, x(t)\big) \cdot \nabla_u f\big(t, x(t), u(t)\big) + \nabla_u \ell\big(t, y(t), u(t)\big) \quad (15)$$

the first order optimality condition which is identical with (5b).

Corollary 4: (HJBE for output-feedback systems)
Consider an OCP (4) with ℓ, L, f, and h continuously differentiable. Then the Hamilton-Jacobi-Bellman equation is given by (14) and a first order optimality criteria by (15).

Having the HJBE which is a necessary and sufficient condition it is the our last goal to obtain the same for PMP. The reason why PMP and HJBE are not fully equivalent can be seen in the derivation. We integrated (13) to obtain (14) but had to chose a constant which in this case determines a specific trajectory $x(\cdot)$. However, sufficient conditions can be found in the literature, compare [17], [16], [11]. Following the strategies therein, we let $\mathcal{H}^*(t, x, \lambda)$ be the minimized Hamiltonian function, i.e.

$$\mathcal{H}^*(t, x, \lambda) = \min_{u(\cdot)} \big\{ \ell\big(t, h\big(t, x(t)\big), u(t)\big) + \lambda(t) \cdot f\big(t, x(t), u(t)\big) \big\}$$

and easily adapt the known theory to include the output-feedback case.

Corollary 5: (Sufficient conditions)
Suppose an OCP (4) and the constraints (8). If $\mathcal{H}^*(t, x, \lambda)$ is convex in x for all $\lambda \geq 0$ and there exists a solution $(t_f, x, y, u, \lambda, \alpha_1, \alpha_2, \beta_1, \beta_2)$ for (9)-(11) with

$$\lambda(\cdot) \geq 0,$$

then $(t_f, x, y, u, \lambda, \alpha_1, \alpha_2, \beta_1, \beta_2)$ is a global minimizer.

VI. Conclusions

In this work, Pontryagin's Minimum Principle has been generalized to output-feedback systems. Optimality criteria for four cases arising from whether or not the final time and the final output respectively state are fixed or free have been derived. It has also been investigated under which conditions the output function may depend on the input. Furthermore, the output-feedback case has been combined with the known results time-dependent equality and inequality constraints for the input and output variables as well as time-dependent terminal equality and inequality constraints. An example representing a rocket launch is provided to indicate how the theory could be applied. In the end, the generalized version of Pontryagin's Minimum Principle has been used to derive the Hamilton-Jacobi-Bellman equation for the output-feedback case and sufficient conditions were given.
Future work may focus on a discrete-time output-feedback version of Pontryagin's Minimum Principle.

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