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Incremental Stability Analysis of Lurie Systems

Lanlan Su, Ross Drummond, and Sei Zhen Khong

***Abstract*— The incremental stability analysis of Lurie systems consisting of the feedback interconnection between a linear time-invariant (LTI) system and a slope-bounded nonlinearity is considered. We first show that the incremental input-output mappings generated by the set of slope-bounded nonlinearities satisfy a set of biased integral quadratic constraints (IQCs) defined by Popov multipliers. Then, a frequency-domain inequality (FDI) condition on the LTI system is proposed for establishing incremental closed-loop stability via an incremental form of IQC theory. Application of the KYP lemma yields an equivalent linear matrix inequality (LMI) condition.**

I. Introduction

The incremental analysis of dynamical systems is concerned with determining the properties of trajectories towards each other, not just with respect to a fixed equilibrium point. In nonlinear settings, incremental notions such as incremental stability and incremental dissipativity are of great significance for applications such as regulation [1], observer design [2] and synchronisation [3], [4]. The crucial role of incremental analysis in these applications has driven forward theoretical developments in recent years, including in the closely-related topic of contraction theory which has attracted significant attention since the late 1990s [5]. However, in spite of these application-orientated motivations, results on incremental properties have arguably stagnanted. As pointed out in studies such as [6], the field of nonlinear control systems has been dominated by non-incremental analysis since the 1980s and advances on the incremental problem have been limited. For example, whilst many novel methods have been devised to compute Lyapunov functions, these advances have not transferred over to the incremental problem. As a consequence, results on the incremental analysis have been scattered. Even though notable pioneers in the field placed significant value on the role of incremental analysis, notably George Zames in his 1966 papers [7] and [8], that perspective has been somewhat lost in recent years.

Similar to the non-incremental approach on nonlinear systems stability analysis, the incremental analysis can be treated in both the state-space (internal) and input-output (external) perspectives. The former is more recent, with results focusing on the search for incremental forms of Lyapunov-type theories [5], [9], [10], with an incremental dissipativity theory developed in [11] and employed in [3]. The input-output perspective of incremental notions was pioneered by Zames [7], [8] and developed in the classical textbooks [12], [13]. Building upon these results, the recent work of [6] showed that the incremental input-output properties studied in the 1970s and the incremental form of Lyapunov stability could be connected to the incremental form of dissipativity in a straightforward way.

This work develops an input-output approach for the incremental stability analysis of Lurie systems consisting of a linear time-invariant (LTI) system in feedback connection with a slope-bounded nonlinearity. We restrict the class of Lurie systems considered with the following assumptions:

H1) The feed-through term in the open-loop LTI system is assumed to be zero, as in $G(\infty) = 0$.

H2) The external input at the input side of the nonlinear operator is differentiable.

The main result of the paper is a Popov-criterion-like condition for the closed-loop incremental stability of such Lurie systems. The key implication is that the open-loop LTI component of the Lurie system is *not required to be passive* — the Popov multiplier can shift the phase of the LTI system to reduce the conservatism in the incremental stability certificates. The idea behind the result is to establish the incremental closed-loop stability via an incremental form of integral quadratic constraint (IQC) theory [14], [15], which encapsulates notions such as small-gain and passivity [16]. This is done by first capturing the incremental input-output mappings of slope-bounded nonlinearities by Popov multipliers and then deriving the frequency-domain inequality (FDI) condition defined by Popov multipliers on the open-loop LTI system. In this way, this work builds upon early results on the well-posedness of Lurie systems consisting of LTI systems in feedback with set-valued maximally monotone operators, such as [17]–[20]. However, note that these works impose a passivity assumption on the open-loop LTI system, a condition which can be relaxed by the Popov criterion of this paper.

The first step of the theory outlined above may appear to conflict with [21] – a seminal result on the incremental analysis of Lurie systems. In that paper, it was shown that the dynamic multipliers, which have been highly successful in reducing the conservatism in the non-incremental analysis of Lurie systems, fail for the incremental analysis. In particular, it was proven in [21] via a counter-example that Popov multipliers preserve incremental positivity of a monotone nonlinearity if and only if the nonlinearity is linear. The implication of this result is that dynamic multipliers do not generalise to the incremental stability analysis, a feature which has led to a bottleneck in this area. We circumvent the

L. Su and R. Drummond are with the Department of Automatic Control and Systems Engineering, University of Sheffield, Sheffield, S3 1JD, UK. {lanlan.su, ross.drummond}@sheffield.ac.uk.
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S. Z. Khong is with the Department of Electrical Engineering, National Sun Yat-sen University, Kaohsiung 80424, Taiwan. szkhong@mail.nsysu.edu.tw

bottleneck of [21] by: *i)* restricting the class of Lurie systems with assumptions **H1** and **H2**, and *ii)* introducing a bias in the definition of incremental positivity. This allows us to adopt a weaker form of incremental positivity of the slope-bounded nonlinerities, but from this weaker definition we are still able to infer a form of incremental stability results. We also show that, by applying the KYP lemma, the proposed condition is equivalent to a linear matrix inequality (LMI) which can then be efficiently verified using convex programming. Crucially, these results allow the passivity of G to be relaxed and so can reduce the conservatism of incremental stability conditions.

The rest of the paper is organised as follows. Section II introduces the notation and formulates the problem. Section III presents the main results including: the characterisation of the incremental property for slope-bounded nonlinearities by Popov multipliers, incremental stability analysis, and its relation to the result of [22]. Section IV contains the conclusions.

II. Notation and Problem Formulation

We use $\mathbf{L}_2$ and $\mathbf{L}_{2e}$ to denote the following signal spaces

$$\mathbf{L}_2 := \left\{ x : [0,\infty) \to \mathbb{R}^n \;\middle|\; \|x\|^2 = \int_0^\infty |x(t)|^2 dt < \infty \right\}$$

and

$$\mathbf{L}_{2e} := \left\{ x : [0,\infty) \to \mathbb{R}^n \mid \; P_T x \in \mathbf{L}_2, \; \forall T \geq 0 \right\},$$

where P_T denotes the truncation operator defined as

$$(P_T f)(t) = \begin{cases} f(t) & \text{for } t \leq T \\ 0 & \text{otherwise.} \end{cases}$$

For a signal x in $\mathbf{L}_{2e}$, denote its initial value as x_0, i.e., $x_0 = x(0)$. For any signal $x \in \mathbf{L}_{2e}$ and $T > 0$, let $x_T := P_T x$ and $\|x\|_T := \|x_T\|$. For $x, y \in \mathbf{L}_2$, $\langle x, y \rangle := \int_0^\infty x(t)^\top y(t) dt$. For $T \geq 0$ and $x, y \in \mathbf{L}_{2e}$, $\langle x, y \rangle_T := \int_0^T x(t)^\top y(t) dt$. A negative definite matrix $M \in \mathbb{R}^{n \times n}$ is denoted $M \prec 0$ and a positive-definite matrix is denoted $M \succ 0$.

A system $H : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ is said to be causal if $P_T H P_T = P_T H$ for all $T > 0$. In this work, the main object of study is incremental systems properties, with the notation

$$\delta z = z - \bar{z}$$

denoting the difference between a pair of objects z and $\bar{z}$.

In this paper, a static (a.k.a. memoryless) system $\Delta : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ is one for which there exists $\phi : \mathbb{R} \to \mathbb{R}$ such that $(\Delta u)(t) = \phi(u(t))$. For convenience, we will subsequently abuse the notation by using Δ to denote both the operator Δ mapping from $\mathbf{L}_{2e}$ into $\mathbf{L}_{2e}$ and the function ϕ mapping $\mathbb{R}$ into $\mathbb{R}$ interchangeably.

Problem formulation

The Lurie system studied in this work is given by the feedback interconnection depicted in Figure 1 with $G, \Delta : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ where G is an LTI system and Δ belongs to a nonlinearity set. Throughout this work, we assume both G, Δ are causal having single-input single-output. For the sake of simplicity, we are not considering the multi-input multi-output case but a parallel generalisation may be possible.

It is also assumed that $G \sim (A, B, C)$ with A being Hurwitz. Without loss of generality, we assume zero initial conditions $x(0) = 0$ of G as the unforced response of G due to nonzero initial condition can be lumped with the external signal e_2. Note that the unforced response of G and its derivative belong to $\mathbf{L}_2$ because of A being Hurtwitz. It is assumed that Δ belongs to the class of (α, β)-slope-bounded static nonlinearities that map 0 to 0, denoted by $\boldsymbol{\Delta}_{\alpha,\beta}$. That is, for each $\Delta \in \boldsymbol{\Delta}_{\alpha,\beta}$, it holds that $\Delta(0) = 0$, and

$$\alpha \leq \frac{\Delta(x_1) - \Delta(x_2)}{x_1 - x_2} \leq \beta, \forall x_1 \neq x_2, \; x_1, x_2 \in \mathbb{R}. \quad (1)$$

With a restricted external signal space, we define the well-posedness and incremental stability of the Lurie system in Figure 1 as follows.

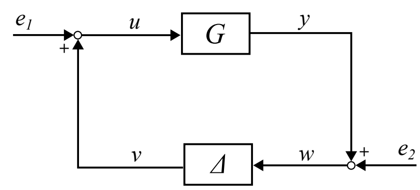

Fig. 1. Feedback configuration of the Lurie system.

Definition 1: The feedback system in Figure 1 is said to be well-posed if for any $e_1, e_2 \in \mathbf{L}_{2e}$ such that $\dot{e}_2 \in \mathbf{L}_{2e}$, there exists $u, y \in \mathbf{L}_{2e}$ that depend causally on e_1, e_2, and $\dot{y} \in \mathbf{L}_{2e}$.

Definition 2: The feedback system in Figure 1 is said to be incrementally stable if it is well-posed and there exist constants $\gamma_0 > 0$ and $\rho > 0$ such that

$$\|y - \bar{y}\|_T^2 + \|u - \bar{u}\|_T^2 \leq \gamma_0 \left| e_2(0) - \bar{e}_2(0) \right|^2 + \rho(\|e_1 - \bar{e}_1\|_T^2 + \|e_2 - \bar{e}_2\|_T^2 + \|\dot{e}_2 - \dot{\bar{e}}_2\|_T^2)$$

for all $T > 0$ and for arbitrary $e_2(0), \bar{e}_2(0) \in \mathbb{R}$ and $e_1, \bar{e}_1, e_2, \bar{e}_2 \in \mathbf{L}_{2e}$ such that $\dot{e}_2, \dot{\bar{e}}_2 \in \mathbf{L}_{2e}$.

For linear systems, incremental stability and its non-incremental counterpart exhibit equivalence, which does not hold for nonlinear systems. In contrast to the finite incremental $\mathbf{L}_2$-gain given in [23, Definition 2.1.5], Definition 2 imposes differentiability on the external input e_2 and allows a bias caused by the difference of the initial value of e_2. The requirement for smoothness applies solely to e_2 and not to e_1. This distinction arises from the fact that e_2 is injected into the input side of the nonlinear system Δ, while G is linear, stable, and satisfies **H1**. The aim of this work is to derive conditions on G such that the feedback system is incrementally stable.

III. Main results

A. *Popov multipliers*

The Popov multiplier is defined as

$$\Pi_P := \begin{bmatrix} 0 & -j\omega \\ j\omega & 0 \end{bmatrix}.$$

For notational convenience, we do not differentiate Π_P and its associated operator in time domain. Given signals $x, z \in \mathbf{L}_2$ such that $\dot{x} \in \mathbf{L}_2$, the inner product

$$\left\langle \begin{bmatrix} x \\ z \end{bmatrix}, \Pi_P \begin{bmatrix} x \\ z \end{bmatrix} \right\rangle$$

exists and is given by $2\langle \dot{x}, z\rangle$. Next, we show in the following that input-output pairs of $\Delta \in \mathbf{\Delta}_{\alpha,\beta}$ satisfy the integral quadratic constraints defined by the Popov multiplier.

Lemma 1: For any $u - \bar{u} \in \mathbf{L}_2$ such that $\dot{u} - \dot{\bar{u}} \in \mathbf{L}_2$, it holds that $y - \bar{y} \in \mathbf{L}_2$, and

$$\left\langle \begin{bmatrix} u-\bar{u} \\ y-\bar{y} \end{bmatrix}, \lambda\Pi_P \begin{bmatrix} u-\bar{u} \\ y-\bar{y} \end{bmatrix} \right\rangle \geq -\gamma \left| u_0 - \bar{u}_0 \right|^2, \quad \forall \lambda \in \mathbb{R}, \Delta \in \mathbf{\Delta}_{\alpha,\beta}$$

where $\gamma := |\lambda| \max(|\alpha|, |\beta|)$, $y = \Delta(u)$, and $\bar{y} = \Delta(\bar{u})$. This is referred to as a biased IQC.

Proof: First, note that $\Delta \in \mathbf{\Delta}_{\alpha,\beta}$ satisfies $(\Delta(x))(t) = \Delta(x(t))$ for all $t \geq 0$, $\Delta 0 = 0$, and (1). Therefore, $\|y - \bar{y}\| = \|\Delta u - \Delta \bar{u}\| \leq \max(|\alpha|, |\beta|)\|u - \bar{u}\|$, i.e. $y - \bar{y} \in \mathbf{L}_2$.

Recall that $u(T) - \bar{u}(T) \to 0$ as $T \to \infty$ since $u - \bar{u} \in \mathbf{L}_2$ and $\dot{u} - \dot{\bar{u}} \in \mathbf{L}_2$ [12, Section VI.6]. Let $\delta y := y - \bar{y}$ and $\delta u := u - \bar{u}$. Then, $\delta y(t) = y(t) - \bar{y}(t) = \Delta(u(t)) - \Delta(\bar{u}(t))$. By recalling (1), we have that $\alpha(\delta u(t))^2 \leq \delta y(t)\delta u(t) \leq \beta(\delta u(t))^2$. Next, we show that

$$\left\langle \begin{bmatrix} \delta u \\ \delta y \end{bmatrix}, \lambda\Pi_P \begin{bmatrix} \delta u \\ \delta y \end{bmatrix} \right\rangle \geq -\gamma \left| \delta u_0 \right|^2$$

with $\gamma = |\lambda| \max(|\alpha|, |\beta|)$. To see this, let $\delta\dot{u} = \dot{u} - \dot{\bar{u}}$ and observe that

$$\begin{aligned} 2\lambda \lim_{T\to\infty} \int_0^T \delta y \delta\dot{u} \; dt &= 2\lambda \lim_{T\to\infty} \int_{\delta u_0}^{\delta u(T)} \delta y \; d(\delta u) \\ &= -2\lambda \int_0^{\delta u_0} \delta y \; d(\delta u) \\ &\geq -2\gamma \int_0^{\delta u_0} \delta u \; d(\delta u) = -\gamma |\delta u_0|^2, \end{aligned}$$

where the second equality follows from $\delta u(T) \to 0$ as $T \to \infty$ and the last inequality holds because $\alpha(\delta u(t))^2 \leq \delta y(t)\delta u(t) \leq \beta(\delta u(t))^2$. As $\delta y, \delta\dot{u} \in \mathbf{L}_2$, it follows that $\lim_{T\to\infty} \int_0^T \delta y \delta\dot{u} \; dt = \langle \delta y, \delta\dot{u}\rangle$. The inequality in Lemma 1 holds as

$$2\lambda\langle \delta y, \delta\dot{u}\rangle = \left\langle \begin{bmatrix} \delta u \\ \delta y \end{bmatrix}, \lambda\Pi_P \begin{bmatrix} \delta u \\ \delta y \end{bmatrix} \right\rangle.$$

■

Next, a more general class of Popov multipliers given by $\Pi_B + \lambda\Pi_P$ are used to capture the incremental property of the class $\mathbf{\Delta}_{\alpha,\beta}$ with

$$\Pi_B := \begin{bmatrix} -2\alpha\beta & \alpha+\beta \\ \alpha+\beta & -2 \end{bmatrix}.$$

It is well known that a (α, β)-sector bounded nonlinearity satisfies the IQC defined by Π_B (see, [12], [24]). From the proof of Lemma 1, it follows straightforwardly that

$$\left\langle \begin{bmatrix} u-\bar{u} \\ y-\bar{y} \end{bmatrix}, \Pi_B \begin{bmatrix} u-\bar{u} \\ y-\bar{y} \end{bmatrix} \right\rangle \geq 0, \quad \forall u, \bar{u} \in \mathbf{L}_2, \; \Delta \in \mathbf{\Delta}_{\alpha,\beta}$$

where $y = \Delta(u), \bar{y} = \Delta(\bar{u})$. This in combination with Lemma 1 leads directly to the following lemma.

Lemma 2: For any $\Delta \in \mathbf{\Delta}_{\alpha,\beta}$, it holds that

$$\left\langle \begin{bmatrix} u-\bar{u} \\ y-\bar{y} \end{bmatrix}, (\Pi_B + \lambda\Pi_P) \begin{bmatrix} u-\bar{u} \\ y-\bar{y} \end{bmatrix} \right\rangle \geq -\gamma \left| u_0 - \bar{u}_0 \right|^2, \quad \forall u, \bar{u}, \dot{u}, \dot{\bar{u}} \in \mathbf{L}_2$$

where $\gamma := |\lambda| \max(|\alpha|, |\beta|)$, and $y = \Delta(u), \bar{y} = \Delta(\bar{u})$.

Note that when $0 \leq \alpha \leq \beta$, $\mathbf{\Delta}_{\alpha,\beta}$ is a set of monotone memoryless nonlinearities, the positivity of which is known to be preserved by several classes of multilpliers including Zames–Falb multipliers [25]–[27]. The work [21] presents a fundamental result on the monotone analysis of nonlinear systems, and so merits special mention. It was proven that "*stability multipliers such as Zames–Falb multipliers, Popov multipliers, and RL/RC multipliers, known to preserve positivity of monotone memoryless nonlinearities, do not, in general, preserve their incremental positivity*". In particular, [21] established by counterexample that for a static monotone function Δ (i.e., $\Delta \in \mathbf{\Delta}_{0,\infty}$), it holds that

$$\left\langle \begin{bmatrix} u-\bar{u} \\ \Delta u - \Delta\bar{u} \end{bmatrix}, \begin{bmatrix} 0 & 1-\lambda j\omega \\ 1+\lambda j\omega & 0 \end{bmatrix} \begin{bmatrix} u-\bar{u} \\ \Delta u - \Delta\bar{u} \end{bmatrix} \right\rangle \geq 0 \quad \forall \lambda \in \mathbb{R}, u, \bar{u} \in \mathbf{L}_2 \text{ such that } u, \dot{u} \in \mathbf{L}_2$$

if and only if Δ is linear. From this statement, it would appear that the counterexample of [21] gives a definitive statement on the irrelevance of dynamics multipliers for verifying incremental positivity of monotone nonlinearities. This seminal result has since acted as a roadblock to the development of novel methods for incremental stability analysis of nonlinear systems. In fact, the least conservative method has remained the Circle criterion, which was developed by Zames in 1966 [8]. Nevertheless, as will be shown in the latter sections of this paper, there is scope to develop incremental results around this apparent roadblock, and Lemma 1 represents one way to achieve this. As a comparison, Lemma 1 shows that Popov multipliers preserve incremental positivity of slope-bounded nonlinearities in a weak form in that it is dependent on the initial value of the input. With such a distinction, Lemma 1 then shows that incremental positivity results can be obtained via Popov multipliers, which can then be used for incremental stability analysis.

B. Incremental stability analysis

In this subsection, we present the main results of the paper that propose a condition on G to establish incremental stability of the Lurie system in Figure 1. Before proceeding to the main result, well-posedness of the Lurie system is first shown.

Lemma 3: Suppose $G \sim (A,B,C)$ with A being Hurwitz, and $\Delta \in \mathbf{\Delta}_{\alpha,\beta}$. The feedback system in Figure 1 is well-posed.

Proof: Based on Theorem 4.1 in [28], the feedback system $[G,\Delta]$ is well-posed in the conventional sense, i.e., for any $e_1, e_2 \in \mathbf{L}_{2e}$, there exists $u, y \in \mathbf{L}_{2e}$ that depend causally on e_1, e_2. The fact that G is stable and strictly proper implies that the mapping from u to $\dot{y}$ is stable, and thus if $u \in \mathbf{L}_{2e}$ then $\dot{y} \in \mathbf{L}_{2e}$. Hence, the feedback system $[G,\Delta]$ is well-posed as per Definition 2. ■

The next theorem provides a condition on $G \sim (A,B,C)$ for the feedback incremental stability defined in Definition 2 by exploiting the class of Popov multipliers $\Pi_B + \lambda\Pi_P$.

Theorem 1: Suppose $G \sim (A,B,C)$ with A being Hurwitz, $\alpha \le 0 \le \beta$, and $\Delta \in \mathbf{\Delta}_{\alpha,\beta}$. The feedback system in Figure 1 is incrementally stable if there exist $\lambda \in \mathbb{R}, \epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ 1 \end{bmatrix}^* (\Pi_B + \lambda\Pi_P) \begin{bmatrix} G(j\omega) \\ 1 \end{bmatrix} \le -\epsilon, \forall \omega \in \mathbb{R}. \quad (2)$$

Proof: Since $\alpha \le 0 \le \beta$, it is clear that if $\Delta \in \mathbf{\Delta}_{\alpha,\beta}$, then $\tau\Delta \in \mathbf{\Delta}_{\alpha,\beta}$ for all $\tau \in [0,1]$. By Lemma 3, we know that $[G,\tau\Delta]$ is well-posed for all $\tau \in [0,1]$ and $\Delta \in \mathbf{\Delta}_{\alpha,\beta}$. Since $\Delta \in \mathbf{\Delta}_{\alpha,\beta}$, we have from Lemma 2 that

$$\left\langle \begin{bmatrix} \delta w \\ \delta v \end{bmatrix}, \Pi_B + \lambda\Pi_P \begin{bmatrix} \delta w \\ \delta v \end{bmatrix} \right\rangle \ge -\gamma \left|\delta w_0\right|^2,$$

for all $w, \bar{w}, \dot{w}, \dot{\bar{w}} \in \mathbf{L}_2$ where $\gamma := |\lambda| \max(|\alpha|, |\beta|)$, and $v = \Delta(w), \bar{v} = \Delta(\bar{w})$, which can be rewritten equivalently into

$$\left\langle \begin{bmatrix} \delta \dot{w} \\ \delta w \\ \delta v \end{bmatrix}, \left[\begin{array}{c|c} 0 & 0 \quad \lambda \\ \hline 0 & \\ \lambda & \Pi_B \end{array}\right] \begin{bmatrix} \delta \dot{w} \\ \delta w \\ \delta v \end{bmatrix} \right\rangle \ge -\gamma \left|\delta w_0\right|^2, \quad (3)$$
$$\forall w, \bar{w}, \dot{w}, \dot{\bar{w}} \in \mathbf{L}_2.$$

Now rewrite (2) as

$$\begin{bmatrix} j\omega G(j\omega) \\ G(j\omega) \\ 1 \end{bmatrix}^* \left[\begin{array}{c|c} 0 & 0 \quad \lambda \\ \hline 0 & \\ \lambda & \Pi_B \end{array}\right] \begin{bmatrix} j\omega G(j\omega) \\ G(j\omega) \\ 1 \end{bmatrix} \le -\epsilon, \quad \forall \omega \le 0.$$

Note that $y, \dot{y} \in \mathbf{L}_2$ when $u \in \mathbf{L}_2$ as G is stable and strictly proper. By Proposition 4 in [24], the frequency domain inequality is equivalent to

$$\left\langle \begin{bmatrix} \delta \dot{y} \\ \delta y \\ \delta u \end{bmatrix}, \left[\begin{array}{c|c} 0 & 0 \quad \lambda \\ \hline 0 & \\ \lambda & \Pi_B \end{array}\right] \begin{bmatrix} \delta \dot{y} \\ \delta y \\ \delta u \end{bmatrix} \right\rangle \le -\epsilon \|\delta u\|^2,$$
$$\forall u, \bar{u} \in \mathbf{L}_2, y = Gu, \bar{y} = G\bar{u}.$$

As G is stable and the mapping from δu to $\delta \dot{y}$ is bounded, the above condition implies that there exists $\epsilon > 0$ such that

$$\left\langle \begin{bmatrix} \delta \dot{y} \\ \delta y \\ \delta u \end{bmatrix}, \left[\begin{array}{c|c} 0 & 0 \quad \lambda \\ \hline 0 & \\ \lambda & \Pi_B \end{array}\right] \begin{bmatrix} \delta \dot{y} \\ \delta y \\ \delta u \end{bmatrix} \right\rangle \le -\epsilon \left\| \begin{bmatrix} \delta \dot{y} \\ \delta y \\ \delta u \end{bmatrix} \right\|^2, \quad (4)$$
$$\forall u, \bar{u} \in \mathbf{L}_2, y = Gu, \bar{y} = G\bar{u}.$$

Next, a homotopy argument that mimics that in [14, Theorem 1] is used, which is completed in three steps.

Step 1. From (3)&(4), we show in what follows that there exist $\gamma_0, \rho > 0$ such that

$$\|\delta y\|^2 + \|\delta u\|^2 \le \gamma_0 \|\delta e_2(0)\|^2 + \rho \cdot (\|\delta e_1\|^2 + \|\delta e_2\|^2 + \|\delta \dot{e}_2\|^2)$$

for all $w, \bar{w}, \dot{w}, \dot{\bar{w}} \in \mathbf{L}_2$ and $u, \bar{u} \in \mathbf{L}_2, y = Gu, \bar{y} = G\bar{u}$. For notational simplicity, let

$$M := \left[\begin{array}{c|c} 0 & 0 \quad \lambda \\ \hline 0 & \\ \lambda & \Pi_B \end{array}\right], \; p := \begin{bmatrix} \delta \dot{y} \\ \delta y \\ \delta u \end{bmatrix}, \; q := \begin{bmatrix} \delta \dot{w} \\ \delta w \\ \delta v \end{bmatrix}. \quad (5)$$

Summing up (3)&(4) yields

$$\begin{aligned} &-\gamma|\delta w_0|^2 + \epsilon\|p\|^2 \\ \le& \langle q, Mq\rangle - \langle p, Mp\rangle \\ =& \langle (q-p), M(q-p)\rangle + \\ & \langle p, M(q-p)\rangle + \langle (q-p), Mp\rangle \\ \le& \|M\|\|q-p\|^2 + 2\|M\|\|p\|\|q-p\| \\ \le& \|M\|\|q-p\|^2 + \frac{2\|M\|^2\|q-p\|^2}{\epsilon} + \frac{\epsilon}{2}\|p\|^2. \end{aligned}$$

Therefore, we obtain

$$\frac{\epsilon}{2}\|p\|^2 \le \gamma|\delta w_0|^2 + \left(\|M\| + \frac{2\|M\|^2}{\epsilon}\right)\|q-p\|^2.$$

Since G is strictly proper and has zero initial condition, we have $y(0) = 0$. Hence, $\delta w_0 = \delta y(0) + \delta e_2(0) = \delta e_2(0)$. By noting $\|\delta e_1\|^2 + \|\delta e_2\|^2 + \|\delta \dot{e}_2\|^2 = \|q-p\|^2$, the claim is proved by letting $\gamma_0 = \frac{2\gamma}{\epsilon}$ and $\rho = \frac{2}{\epsilon}\left(\|M\| + \frac{2\|M\|^2}{\epsilon}\right)$.

Step 2. Given the homotopy $\tau\Delta, \tau \in [0,1]$, we show that if $[G, \tau_1\Delta]$ is incrementally stable, then there exists a constant $d > 0$ such that $[G, \tau_2\Delta]$ is also incrementally stable for all τ_2 such that $|\tau_2 - \tau_1| \le d$ where d depends only on α, β and ρ in Step 1. Given $\tau_2 \in [0,1]$, recall from Lemma 3 that $[G, \tau_2\Delta]$ is well-posed as $\tau_2\Delta \in \mathbf{\Delta}_{\alpha,\beta}$. Hence, it follows that for any $(e_1, e_2, \dot{e}_2) \in \mathbf{L}_{2e}$, there exist $(u, y, \dot{y}) \in \mathbf{L}_{2e}$ such that for all $T \ge 0$,

$$y_T = P_T G u_T, \; u_T = e_{1T} + v_T$$
$$v_T = \tau_2 \Delta w_T, \; w_T = y_T + e_{2T}.$$

One can rewrite $\tau_2\Delta = \tau_1\Delta + (\tau_2 - \tau_1)\Delta$, so $v = \tau_1\Delta w + (\tau_2 - \tau_1)\Delta w$. Let $v^* := \tau_1 \Delta w$ and $e^*_{1T} := e_{1T} + (\tau_2 - \tau_1)\Delta w_T$. Then, the above equations can be rewritten as

$$y_T = P_T G u_T, \; u_T = e^*_{1T} + v^*_T$$
$$v^*_T = \tau_1 \Delta w_T, \; w_T = y_T + e_{2T},$$

which can be taken as the feedback interconnection $[G, \tau_1\Delta]$ with external input (e^*_1, e_2). The assumed incremental stability of $[G, \tau_1\Delta]$ implies

$$\|\delta y\|_T^2 + \|\delta u\|_T^2 \le \gamma_0 \left|\delta e_2(0)\right|^2 + \rho \cdot (\|\delta e^*_1\|_T^2 + \|\delta e_2\|_T^2 + \|\delta \dot{e}_2\|_T^2)$$

for all $T > 0$ and for arbitrary $w_0, \bar{w}_0 \in \mathbb{R}$. Since $e^*_{1T} := e_{1T} + (\tau_2 - \tau_1)\Delta w_T$, we have $\|\delta e^*_1\|_T^2 \le 2\|\delta e_1\|_T^2 + 2|\tau_2 - \tau_1| \cdot \|(\Delta w - \Delta \bar{w})\|_T^2 \le 2\|\delta e_1\|_T^2 +$

$2|\tau_2 - \tau_1|\max\{|\alpha|,|\beta|\}(\|\delta w\|_T^2) \leq 2\|\delta e_1\|_T^2 + 4|\tau_2 - \tau_1|\max\{|\alpha|,|\beta|\}(\|\delta e_2\|_T^2 + \|\delta y\|_T^2)$ where the first and the third lines hold because $(a+b)^2 \leq 2a^2 + 2b^2$ and the second line follows from (1). Suppose $|\tau_2 - \tau_1| \leq d := \frac{1}{4\rho\max\{|\alpha|,|\beta|\}}$. Then, we have $\|\delta u\|_T^2 \leq \gamma_0 |\delta e_2(0)|^2 + 2\rho\|\delta e_1\|_T^2 + (\rho+1)(\|\delta e_2\|_T^2 + \|\delta \dot{e}_2\|_T^2)$, which, in combination with,

$$\|\delta y\|_T^2 + \|\delta u\|_T^2 \leq (\|G\|^2 + 1)\|\delta u\|_T^2$$

implies that $[G, \tau_2\Delta]$ is incrementally stable.

Step 3. Since the open-loop system $[G, 0]$ is stable and linear, $[G, 0]$ is incrementally stable, and by the iterative application of the conclusion in Steps 1&2, it can be shown that $[G, \tau\Delta]$ is incrementally stable for $\tau \in [0, 1]$. This completes the proof. ■

Remark 1: When λ in Theorem 1 is taken to be $\lambda = 0$, the condition (2) reduces to the circle criterion. The use of Popov multiplier allows the condition of G to be relaxed and thus reduce the conservatism of incremental stability conditions.

C. Linear matrix inequality conditions

In the following, the frequency domain inequality (FDI) of (2) from Theorem 1 is translated into a linear matrix inequality (LMI) condition by applying the KYP lemma, and so can be verified using semi-definite programming.

By augmenting the dimension of output of G, define the state-space system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ \tilde{y} &= \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix} x + \begin{bmatrix} 0 \\ CB \end{bmatrix} u. \end{aligned}$$

Denote the corresponding operator mapping from u to $\tilde{y}$ as $\tilde{G}$, which has a state-space realisation

$$\tilde{G} \sim \left(A, B, \begin{bmatrix} C \\ CA \end{bmatrix}, \begin{bmatrix} 0 \\ CB \end{bmatrix} \right).$$

It follows that $\tilde{G}$ is stable and its frequency response is

$$\tilde{G}(j\omega) = \begin{bmatrix} j\omega G(j\omega) \\ G(j\omega) \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix} (j\omega I - A)^{-1} B + \begin{bmatrix} 0 \\ CB \end{bmatrix}.$$

Hence, the FDI (2) can be rewritten as

$$\begin{bmatrix} \tilde{G}(j\omega) \\ 1 \end{bmatrix}^* \left[\begin{array}{c|c} 0 & 0 \quad \lambda \\ \hline 0 & \\ \lambda & \Pi_B \end{array}\right] \begin{bmatrix} \tilde{G}(j\omega) \\ 1 \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} (j\omega I - A))^{-1} B \\ 1 \end{bmatrix}^* M \begin{bmatrix} (j\omega I - A))^{-1} B \\ 1 \end{bmatrix} \leq -\epsilon, \forall \omega \in \mathbb{R} \tag{7}$$

where

$$M := \begin{bmatrix} C & 0 \\ CA & CB \\ 0 & 1 \end{bmatrix}^T \left[\begin{array}{c|c} 0 & 0 \quad \lambda \\ \hline 0 & \\ \lambda & \Pi_B \end{array}\right] \begin{bmatrix} C & 0 \\ CA & CB \\ 0 & 1 \end{bmatrix}.$$

Now we are ready to present the following corollary.

Corollary 1: Suppose $G \sim (A, B, C)$ with A being Hurwitz, $\alpha \leq 0 \leq \beta$, and $\Delta \in \mathbf{\Delta}_{\alpha,\beta}$. The feedback system in Figure 1 is incrementally stable if there exist $\lambda \in \mathbb{R}, P = P^T$ such that

$$\begin{bmatrix} PA + A^T P & PB \\ B^T P & 0 \end{bmatrix} + M \prec 0 \tag{8}$$

with M as in (5).

Proof: From Theorem 1 and (7), it follows that the feedback system in Figure 1 is incrementally stable if there exists $\lambda \in \mathbb{R}, \epsilon > 0$ such that

$$\begin{bmatrix} (j\omega I - A))^{-1} B \\ 1 \end{bmatrix}^* \left(M + \begin{bmatrix} 0 & 0 \\ 0 & \epsilon \end{bmatrix} \right) \begin{bmatrix} (j\omega I - A))^{-1} B \\ 1 \end{bmatrix} \prec 0, \quad \forall \omega \in \mathbb{R}$$

which is equivalent, by invoking the KYP lemma [24, Theorem 8], to showing that there exists $\lambda \in \mathbb{R}$, $\epsilon > 0$, $P = P^T$ such that

$$\begin{bmatrix} PA + A^T P & PB \\ B^T P & 0 \end{bmatrix} + M + \begin{bmatrix} 0 & 0 \\ 0 & \epsilon \end{bmatrix} \prec 0. \tag{9}$$

That inequality (8) holds then implies that there exists some $\epsilon > 0$ such that (9) is satisfied which completes the proof. ■

D. A special case

The condition (2) was shown to be a sufficient condition in [22, Theorem 1] for establishing the stability of the feedback system in Figure 1 with Δ being (α, β)-sector bounded, $e_2 = 0$, $G \sim (A, B, C)$ with possibly nonzero initial condition. In what follows, we consider the same system setting as in [22] and show by an interpolation argument that the feedback system the sector-bounded nonlinearity set replaced by $\mathbf{\Delta}_{\alpha,\beta}$ is incrementally stable.

Consider the feedback system in Figure 1 where $G \sim (A, B, C)$ with initial state $x(0) = x_0$, and $e_2 = 0$, satisfies

$$\begin{aligned} \dot{x} &= Ax + Bu, \; x(0) = x_0, \\ y &= Cx, \\ v &= \Delta(y), \quad u = v + e_1. \end{aligned} \tag{10}$$

Proposition 1: Assume A is Hurwitz, $\alpha \leq 0 \leq \beta$ and $\Delta \in \mathbf{\Delta}_{\alpha,\beta}$. Consider the feedback system (10). If there exists $\lambda \in \mathbb{R}, \epsilon > 0$ such that (2) holds, then, there exist constants $\gamma, \rho > 0$ such that

$$\|y - \bar{y}\|_T^2 + \|u - \bar{u}\|_T^2 \leq \gamma |x_0 - \bar{x}_0|^2 + \rho\|e_1 - \bar{e}_1\|_T^2$$

for all $T \geq 0$, $x_0, \bar{x}_0 \in \mathbb{R}$, and $e_1, \bar{e}_1 \in \mathbf{L}_{2e}$.

Proof: Since $\alpha \leq 0 \leq \beta$, $\tau\Delta$ is (α, β)-sector bounded for all $\tau \in [0, 1]$ if Δ is (α, β)-sector bounded. Similarly to Lemma 3, it can be shown that $[G, \tau\Delta]$ with any (α, β)-sector bounded Δ is well-posed by the definition of [22], i.e., for any $e_1 \in \mathbf{L}_{2e}$, there exists $u, x, \dot{x} \in \mathbf{L}_{2e}$ that depend causally on e_1. Then, using both Example 1 and Theorem 1 from [22], one can show that the feedback system $[G, \psi]$ is stable when ψ is a (α, β)-sector bounded nonlinearity, i.e, for all $e_1 \in \mathbf{L}_{2e}$ and $x_0 \in \mathbb{R}$, there exists $\gamma_0, \rho > 0$ such that $\|y\|_T^2 + \|u\|_T^2 \leq \gamma_0|x_0| + \rho\|e_1\|_T^2$ for all $T \geq 0$.

Next, we prove that the stability of $[G, \psi]$ with ψ being (α, β)-sector-bounded implies the incremental stability of

$[G, \Delta]$ with Δ being (α, β)-slope-bounded. Suppose by contradiction that $[G, \Delta^*]$ is not incrementally stable for some $\Delta^* \in \mathbf{\Delta}_{\alpha,\beta}$. That is, there exists $x_0^*, \bar{x}_0^* \in \mathbb{R}^n, e_1^*, \bar{e}_1^* \in \mathbf{L}_{2e}$ such that there exists no γ_0, ρ satisfying

$$\|\delta y^*\|_T + \|\delta u^*\|_T^2 \leq \gamma_0 |\delta x_0^*|^2 + \rho \|\delta e_1^*\|_T^2, \quad \forall T \geq 0.$$

Now consider the feedback system $[G, \psi]$, and let $e_1 = \delta e_1^*$ and $x_0 = \delta x_0^*$. Let ψ^* be the (α, β)-sector bounded system that maps δy^* to δv^*. It follows from the linearity of G that

$$\begin{aligned} \delta y^* &= G(\delta u^*) \\ \delta v^* &= \psi^*(\delta y^*) \\ \delta u^* &= \delta e_1^* + \delta v^*. \end{aligned}$$

Hence, the feedback system $[G, \psi^*]$ is not stable, which completes the proof. ■

Remark 2: In comparison with Proposition 1, Theorem 1 provides a more general result that allows nonzero external e_2. In fact, Proposition 1 can be recovered from Theorem 1. To see this, we note that by shifting effect of the nonzero initial condition of G to e_2 the feedback system (10) can be equivalently represented as

$$\begin{aligned} &\dot{x} = Ax + Bu_1, x(0) = 0, \\ &y = Cx, \; e_2 = Ce^{At}x_0, \; w = y + e_2, \\ &u = \Delta(w) + e_1, \end{aligned}$$

where G maps 0 to 0. For $x_0, \bar{x}_0$, let $\delta e_2 = Ce^{At}x_0 - Ce^{At}\bar{x}_0$. Since A is Hurwitz, there exist constants γ_1, γ_2 such that $\|\delta e_2\|^2 = \|Ce^{At}\delta x_0\|^2 \leq \gamma_1 |\delta x_0|^2$ and $\|\dot{e}_2\|^2 = \|CAe^{At}\delta x_0\|^2 \leq \gamma_2 \|\delta x_0\|^2$. At the initial time instant, $e_2(0) = Cx_0$. If condition (2) is satisfied, then Theorem 1 implies there exists constants $\gamma_0 > 0$ and $\rho > 0$ such that

$$\begin{aligned} &\|\delta y\|_T^2 + \|\delta u\|_T^2 \\ \leq &\gamma \left|\delta e_2(0)\right|^2 + \rho(\|\delta e_1\|_T^2 + \|\delta e_2\|_T^2 + \|\delta \dot{e}_2\|_T^2) \\ \leq &\gamma_0 \|C\|^2 |\delta x_0|^2 + \rho(\|\delta e_1\|_T^2 + \gamma_1 |\delta x_0|^2 + \gamma_2 |\delta x_0|^2) \\ = &(\gamma_0 \|C\|^2 + \rho\gamma_1 + \rho\gamma_2)|\delta x_0|^2 + \rho \|\delta e_1\|_T^2 \end{aligned}$$

for all $T > 0$ and for arbitrary $x_0, \bar{x}_0 \in \mathbb{R}$ and $e_1, \bar{e}_1 \in \mathbf{L}_{2e}$. Hence, Proposition 1 is recovered.

IV. Conclusion

This work considered Lurie systems consisting of a feedback interconnection of a stable and strictly proper open-loop LTI system and a slope-bounded nonlinearity. We imposed differentiablity on one of the two exogenous signal spaces which enabled incremental properties of the slope-bounded nonlinearities to be analysed using Popov multipliers. Using an incremental version of an IQC theorem, conditions for incremental closed-loop stability have been proposed in the form of FDIs and LMIs.

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