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Interval Consensus of Hybrid Multi-agent Systems over Cooperative-antagonistic Network*

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Abstract— **This paper considers the interval consensus problem for a class of hybrid linear multi-agent systems with periodic jumps over a signed digraph, without assuming specific connectivity properties. By utilizing the concept of independent strongly connected components, we provide a clear characterization of network clusters and explicitly depict the consensus behavior of a trajectory. A hybrid distributed state feedback approach is developed to achieve the convergence of agents under stabilizable condition in the hybrid time domain. Importantly, our results establish the solvability of the interval consensus problem for both discrete-time and sampled-data continuous-time multi-agent systems under signed graphs. To validate the theoretical findings, a practical interval consensus of multiple bouncing disks is studied and simulated.**

I. INTRODUCTION

In recent decades, there has been a growing interest in multi-agent systems, which aim to achieve distributed coordination using local information. These systems hold significant potential across various applications, such as social networks [2], smart grids [4], formation control [28], to name just a few. Control problems in multi-agent systems often revolve around developing a distributed control law to steer all agents toward a consensus behavior. This consensus can take two main forms: a global consensus, which is applicable to all agents when the graph satisfies certain connectivity conditions like strong connectivity and the spanning tree condition [16], [22], or specific consensus behaviors that emerge within distinct clusters where graph connectivity may not be guaranteed [7], [13], [21]. Typically, these studies assume cooperative interactions among agents.

The rise of applications like social networks, network adversarial attacks, and the study of inhibitory effects of neurons have promoted the necessity for a general consensus that handles interactions with cooperative and antagonistic relationships. For this purpose, a signed graph was intruded in [26], where positive edges and negative edges represent cooperation and antagonism, respectively. Extensive research efforts have been devoted to the exploration of signed graphs. For instance, in structurally balanced signed graphs, the final convergence split into two values with opposite signs but equal magnitudes, a phenomenon known as bipartite consensus [1], [15]. Here, under structurally balanced condition, agents can be divided into two opposing groups with cooperative interactions within each group. However, this division does not hold for structurally balanced graphs. In such a case, under strong connectivity, all agents asymptotically converge to zero [1]. When a graph includes only a spanning tree or even lacks connectivity, the convergence behaviors become more diverse. For example, in a signed graph containing a spanning tree, agents exhibit interval bipartite consensus [15], [20], but they do not converge to a common trajectory. Meanwhile, the containment tracking problem is explored in [14], assuming each follower can be reached by at least one leader in signed graphs without requiring connectivity across the entire network. Various extensions of interval bipartite consensus have been studied for more general models, including coupled harmonic oscillators [25] and higher-order linear models [11].

All the aforementioned studies have primarily focused on either continuous-time or discrete-time multi-agent systems. However, practical scenarios such as bouncing balls and control domains like sampled-data control and reset linear control motivate a more reasonable consideration of systems that exhibits characteristics of both continuous-time (flow) dynamics and discrete-time (jump) dynamics. Such systems are referred to as hybrid systems, with a more detailed discussion available in [8]. Recently, hybrid multi-agent systems with cooperative networks have been addressed recently. For instance, single consensus problems were tackled under spanning trees condition [27], and multi-consensus problems were solved without requiring connectivity conditions [5], [6]. Notably, the consensus problem of hybrid multi-agent systems with cooperative-antagonistic networks is not investigated yet.

This paper aims to develop a hybrid distributed state feedback approach to solve the interval consensus problem for a class of hybrid linear multi-agent systems over a signed graph without assuming specific connectivity properties. Our contributions can be summarized in two aspects. Firstly, we employ the decomposition of signed digraphs based on independent strongly connected components (iSCCs) to explicitly characterize the asymptotic behaviors of all agents. Specifically, we demonstrate that agents within the same iSCCs achieve either bipartite consensus or stability, depending on whether the iSCCs exhibit structurally balanced and structurally unbalanced, respectively, while the agents outside iSCCs converge to convex hull spanned by the final consensus values of those agents in iSCCs. Secondly, our hybrid distributed state feedback provides the solvability of the interval consensus problem for both discrete-time and sampled-data continuous-time multi-agent systems under signed graphs, extending the results in [12], [23], [29], [25]

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to general signed graphs without assuming any connectivity.

The rest of this paper is structured as follows: Section II introduces signed graphs without connectivity and formulates the associated problem. Section III explores the asymptotic behavior and coordination analysis of hybrid multi-agent systems. An illustrative example demonstrating the effectiveness of the theoretical results is provided in Section IV. Finally, conclusions are summarized in Section V.

Notations: $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^n$ denotes the n dimensional Euclidean space, and $\mathbb{R}^{n\times n}$ denotes the set of all $n\times n$ matrices with real entries. $\mathbb{C}^{n\times n}$ denotes the set of all $n\times n$ matrices with complex entries. I_n denotes the $n\times n$ identity matrix, and $\mathbf{1}_n$ denotes the n dimensional column vector with every element being 1. Let $\mathbb{N}=\{0,\mathbb{Z}^+\}$ where $\mathbb{Z}^+$ denotes the set of all positive integers. $\sigma_0(A)$ denotes the set composed by all nonzero eigenvalues of a square matrix A. For any positive integer p, $\mathbb{Z}_p=\{1,\dots,p\}$. For a vector $u\in\mathbb{R}^p$, if $R=\{i_1<\cdots<i_r\}\subseteq\mathbb{Z}_p$, $u_R=[u_{i_1},\dots,u_{i_r}]^\mathsf{T}\in\mathbb{R}^r$. Given a finite set R, $\#R$ denotes the number of the elements.

II. Preliminaries and Problem Formulation

A. Signed Graphs

The communication network of multi-agent systems can be modeled by a directed graph (digraph) $\mathcal{G}=(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ represents the set of nodes and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ denotes the set of edges. A directed path from node i to node j in $\mathcal{G}$ is a sequence of pairs $(i,i_1),(i_1,i_2),\dots,(i_k,j)$ in $\mathcal{E}$. A digraph $\mathcal{G}$ is said to be strongly connected if for any $i,j\in\mathcal{V}$ with $i\neq j$, there exists a directed path from i to j. Let $\mathcal{A}=[a_{ij}]\in\mathbb{R}^N$ be a signed adjacency matrix of $\mathcal{G}$, where $a_{ij}\neq 0$ if $(j,i)\in\mathcal{E}$ and $a_{ij}=0$ if $(j,i)\notin\mathcal{E}$. $\mathcal{G}$ is termed an unsigned digraph if $a_{ij}\geq 0$ for all $i,j\in\mathcal{V}$; otherwise, it is considered a signed digraph. Following the terminology in [1], a signed digraph $\mathcal{G}$ is structurally balanced if there is a partition $\{\mathcal{V}_1,\mathcal{V}_2\}$ of $\mathcal{V}$, where $\mathcal{V}=\mathcal{V}_1\cup\mathcal{V}_2$ and $\mathcal{V}_1\cap\mathcal{V}_2=\emptyset$, such that $a_{ij}\geq 0$ when nodes i and j are in the same subset and $a_{ij}\leq 0$ otherwise. If this condition is not satisfied, the signed digraph is considered structurally unbalanced. Let $\mathcal{L}=[l_{ij}]\in\mathbb{R}^{N\times N}$ be signed Laplacian matrix for $\mathcal{G}$ satisfying that $l_{ii}=\sum_{k\neq i}|a_{ik}|$ and $l_{ij}=-a_{ij}$ for any $i\neq j$.

For any subset $\mathcal{V}'$ of $\mathcal{V}$, $\mathcal{G}_{\mathcal{V}'}=(\mathcal{V}',\mathcal{E}_{\mathcal{V}'})$ is a subgraph of $\mathcal{G}$ with its node set $\mathcal{V}'$ and edge set $\mathcal{E}_{\mathcal{V}'}=\mathcal{E}\cap(\mathcal{V}'\times\mathcal{V}')$. A subgraph is called a *strongly connected component* (SCC) of $\mathcal{G}$ if it is a maximal subgraph that is strongly connected. It is termed an *independent strongly connected component (iSCC)* if it is a SCC and has no edge (i,j) with $i\notin\mathcal{V}'$ and $j\in\mathcal{V}'$. Let us denote the family of node sets for all iSCCs as $\{\mathcal{X}_1,\dots,\mathcal{X}_m\}$. Define $\mathcal{X}_{m+1}=\mathcal{V}\backslash\bigcup_{s=1}^m\mathcal{X}_s$. Set $N_s=\#\mathcal{X}_s$ for any $s\in\mathbb{Z}_{m+1}$. After suitably reordering the nodes in $\mathcal{V}$, the signed Laplacian matrix $\mathcal{L}$ can be rewritten into the following form [14]:

$$\mathcal{L}=\begin{bmatrix}\mathcal{L}_1 & 0 & \cdots & 0 & 0\\ 0 & \mathcal{L}_2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \mathcal{L}_m & 0\\ \mathcal{L}_1^* & \mathcal{L}_2^* & \cdots & \mathcal{L}_m^* & \mathcal{M}\end{bmatrix} \tag{1}$$

where for each $s\in\mathbb{Z}_m$, $\mathcal{L}_s\in\mathbb{R}^{N_s\times N_s}$ is a signed Laplacian matrix associated to the iSCC $G_{\mathcal{X}_s}$ and $\mathcal{L}_s^*\in\mathbb{R}^{N_{m+1}\times N_s}$ denotes the connections from $\mathcal{X}_s$ to $\mathcal{X}_{m+1}$; $\mathcal{M}\in\mathbb{R}^{N_{m+1}\times N_{m+1}}$ define the internal connections in the $\mathcal{X}_{m+1}$. For each $s\in\mathbb{Z}_m$, $G_{\mathcal{X}_s}$ is strongly connected, and hence, according to [1], [24], the associated signed Laplacian matrix $\mathcal{L}_s$ has the following properties.

Lemma 1: Consider a signed digraph $\mathcal{G}$ with its signed Laplacian matrix being given by (1), then

1) When $\mathcal{G}_{\mathcal{X}_s}$ is structurally balanced, there exists a diagonal matrix $D_s=\mathrm{diag}\{d_{s1},\dots,d_{sN_s}\}$ with $d_{sj}\in\{\pm 1\}$ for $1\leq j\leq N_s$ such that $D_sL_sD_s$ has property that all the off-diagonal entries are nonpositive and all row sums are zero. Moreover, zero is a simple eigenvalue of $D_sL_sD_s$.
2) When $\mathcal{G}_{\mathcal{X}_s}$ is structurally unbalanced, all eigenvalues of $\mathcal{L}_s$ have positive real parts.

Through the decomposition (1), for any $j\in\mathcal{X}_{m+1}$, there always exists $i\in\bigcup_{s=1}^m\mathcal{X}_s$ such that a directed path from node i to node j exists. Consequently, the subsequent result can be directly deduced from [14, Lemma 4], clarifying the characteristics of the submatrix $\mathcal{M}$.

Lemma 2: Consider a signed digraph $\mathcal{G}$ with its signed Laplacian matrix being given by (1). All eigenvalues of $\mathcal{M}$ have positive real parts.

B. Problem Statement

Consider a class of hybrid linear multi-agent systems governed by the flow dynamics

$$\dot{\tau}=1,\ \dot{x}_i=Ax_i+Bu_{Fi},\ i=1,\dots,N \tag{2a}$$

whether $(\tau,x_i)\in[0,\tau_d]\times\mathbb{R}^n$ and the jump dynamics

$$\tau^+=0,\ x_i^+=Ex_i+Fu_{Ji},\ i=1,\dots,N \tag{2b}$$

whether $(\tau,x_i)\in\{\tau_d\}\times\mathbb{R}^n$, where $\tau\in\mathbb{R}$, $x_i\in\mathbb{R}^n$, $u_{Fi}\in\mathbb{R}^{m_1}$, and $u_{Ji}\in\mathbb{R}^{m_2}$ are the clock variable, state, flow input, and jump input of the i-th subsystem, respectively. $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m_1}$, $E\in\mathbb{R}^{n\times n}$, and $F\in\mathbb{R}^{n\times m_2}$ are all constant matrices. $\tau_d>0$ is a known constant that represents the dwell-time between two consecutive jumps. All solutions to system (2a)-(2b) are defined on the common hybrid time domain $\mathcal{T}:=\{(t,k):t\in[t_k,t_{k+1}],k\in\mathbb{N},t_k:=k\tau_d\}$. In this paper, the topology of network (2) is described by a signed graph $\mathcal{G}$ with $\mathcal{V}=\mathbb{Z}_N$ being the node set, $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ being a signed adjacency matrix, and $\mathcal{L}=[l_{ij}]\in\mathbb{R}^{N\times N}$ being a signed Laplacian matrix.

Remark 1: System (2) characterizes a class of hybrid dynamics involving periodic time jumps, exhibiting characteristics of both continuous-time (flow) dynamics and discrete-time (jump) dynamics. In this scenario, for any $t \in [k\tau_d, (k+1)\tau_d)$, the dynamics are driven by continuous-time (flow) dynamics (2a), while a jump occurs at $t = k\tau_d$, transitioning to discrete-time (jump) dynamics (2b).

Remark 2: As highlighted in [27], system (2) is able to characterize various practical engineering systems, say spinning and bouncing disk and RC circuits, and practical control systems like sampled-data control systems and periodic linear impulsive systems. It is noteworthy that each subsystem of (2) operates within a common hybrid time domain denoted as $\mathcal{T}$ where all the subsystems jump simultaneously. This condition is not just reasonable but necessary since achieving consensus requires synchronization of all subsystem states at steady-state, mandating simultaneous jumps. A similar discussion is available in [3].

As is usually considered for hybrid linear systems [17], [27], each subsystem is assumed to be stabilizable throughout this paper.

Assumption 1: Each subsystem of the hybrid linear multi-agent system (2) is stabilizable.

In what follows, a technical lemma related to Assumption 1 are recalled. To begin with, we define

$$\tilde{A} = e^{A\tau_d}E,\ \tilde{B} = \begin{bmatrix} e^{A\tau_d}F & R_{A,B} \end{bmatrix} \tag{3}$$

where $R_{A,B} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$. According to Lemma 1 of [27], there exists a nonsingular matrix $V_c \in \mathbb{R}^{(nm_1+m_2)\times(nm_1+m_2)}$ such that $\tilde{B}V_c = [\tilde{B}_v, 0]$, where $\tilde{B}_v$ has full column rank. Moreover, Assumption 1 is equivalent to the stabilizability of $(\tilde{A}, \tilde{B}_v)$ (in the discrete-time sense). The following lemma delineates the methodology for selecting K such that $\tilde{A} + \lambda\tilde{B}_v K$ is Schur.

Lemma 3 (Lemma 3 of [27]): Consider the modified H_∞ type Riccati inequality

$$\tilde{A}^\mathsf{T} P\tilde{A} - P - (1-\delta^2)\tilde{A}^\mathsf{T} P\tilde{B}_v(\tilde{B}_v^\mathsf{T} P\tilde{B}_v)^{-1}\tilde{B}_v^\mathsf{T} P\tilde{A} < 0. \tag{4}$$

Let

$$\delta_c = \sup_{\delta>0}\{\delta | \exists P \text{ s.t. the inequality (4) holds}\}. \tag{5}$$

Given a complex number $\lambda \neq 0$. If there exists $\alpha \in \mathbb{R}$ such that $|1-\alpha\lambda| < \delta_c$, then, under Assumption 1, there exists a positive definite matrix P solving (4) with $\delta = |1-\alpha\lambda|$. Moreover, $K = -\alpha(\tilde{B}_v^\mathsf{T} P\tilde{B}_v)^{-1}\tilde{B}_v P\tilde{A}$ is such that $\tilde{A} + \lambda\tilde{B}_v K$ is Schur.

Remark 3: As indicated in [9], under Assumption 1, a lower bound for δ_c always exists since $(\tilde{A}, \tilde{B}_v)$ is stabilizable (in the discrete-time sense). This lower bound is associated with the unstable eigenvalues of $\tilde{A}$. Specifically, when $\tilde{A}$ contains no unstable eigenvalues, it is always possible to select δ_c as 1.

To be specific, we consider the following dynamic feedback controller, with flow dynamics

$$\dot{\tau} = 1,\quad \dot{v}_i = -A^\mathsf{T} v_i \tag{6a}$$

whether $(\tau, x_i, v_i) \in [0, \tau_d] \times \mathbb{R}^n \times \mathbb{R}^n$, jump dynamics

$$\tau^+ = 0,\quad v_i^+ = e^{A^\mathsf{T}\tau_d}\bar{K}_F \sum_{j\neq i} a_{ij}(\mathrm{sgn}(a_{ij})x_i - x_j) \tag{6b}$$

whether $(\tau, x_i, v_i) \in \{\tau_d\} \times \mathbb{R}^n \times \mathbb{R}^n$, and controller output

$$u_{Fi} = B^\mathsf{T} v_i,\quad u_{Ji} = K_J \sum_{j\neq i} a_{ij}(\mathrm{sgn}(a_{ij})x_i - x_j) \tag{6c}$$

where $\bar{K}_F$ and K_J are determined by Algorithm 1.

Algorithm 1 The design for matrices $\bar{K}_F$ and K_J

Require: Assumption 1 is satisfied.
1: **Input** τ_d, A, B, E, F
2: Define $\tilde{A}$ and $\tilde{B}$ by (3)
3: Find the matrix V_c such that $\tilde{B}V_c = [\tilde{B}_v, 0]$, where $\tilde{B}_v$ has full column rank (see Lemma 1 of [27])
4: Given δ_c satisfying (5), find $\alpha_c \in \mathbb{R}$ such that

$$|1-\alpha_c\lambda| < \delta_c,\ \forall\ \lambda \in \sigma_0(\mathcal{L}). \tag{7}$$

5: Find a positive definite matrix P satisfying (4) with $\delta = \delta(\alpha_c) \triangleq \max_{\lambda\in\sigma_0(\mathcal{L})} |1-\alpha_c\lambda|$
6: Define $[K_J^\mathsf{T}, K_F^\mathsf{T}]^\mathsf{T} = [K_v^\mathsf{T}, 0]^\mathsf{T} V_c^\mathsf{T}$, where $K_v = -\alpha_c(\tilde{B}_v^\mathsf{T} P\tilde{B}_v)^{-1}\tilde{B}_v^\mathsf{T} P\tilde{A}$
7: Obtain $\bar{K}_F$ by solving $R_{A,B}K_F = G(\tau_d)\bar{K}_F$ with $G(\tau_d) = \int_0^{\tau_d} e^{A(\tau_d-\tau)}BB^\mathsf{T} e^{A^\mathsf{T}(\tau_d-\tau)}\mathrm{d}\tau$
8: **Output** $\bar{K}_F$, K_J

This paper aims to *explore the asymptotic behaviors for a class of hybrid linear multi-agent systems* (2) *under dynamic state controller* (6)*, interacting through signed digraphs without connectivity.* Additionally, a comprehensive convergence analysis is presented.

III. MAIN RESULTS

In this section, we discuss the asymptotic behaviors of hybrid closed-loop system composed by (2) and (6) based on the decomposition of Laplacian matrix as in (1).

For the sake of compactness and without loss of generality, suppose that the signed Laplacian matrix is of the form (1), and the node sets $\mathcal{X}_s$ with $s \in \mathbb{Z}_p$ $(p \leq m)$ and $\mathcal{X}_s$ with $s = \mathbb{Z}_m \backslash \mathbb{Z}_p$ correspond to structurally balanced and structurally unbalanced iSCCs, respectively. Denote $x = [x_1^\mathsf{T}, \ldots, x_N^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{nN}$. Then, by a reordering of states, one has $\mathbf{x} = [\mathbf{x}_1^\mathsf{T}, \ldots, \mathbf{x}_m^\mathsf{T}, \mathbf{x}_{m+1}^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{nN}$ where $\mathbf{x}_s = x_{\mathcal{X}_s}$ for any $s \in \mathbb{Z}_{m+1}$. Let D_s with $s \in \mathbb{Z}_p$ be given by Lemma 1. We are now in a position to present our main results.

Theorem 1: Consider a class of hybrid linear multi-agent system (2) with a signed digraph $\mathcal{G}$. Let δ_c satisfying (5). If there exists a $\alpha_c \in \mathbb{R}$ satisfying the inequality (7), then, under Assumption 1, for any initial conditions, the solution of the closed-loop system composed by (2) and (6) satisfies:

1) For any $s \in \mathbb{Z}_p$, $\lim_{t+k\to\infty} \mathbf{x}_s(t,k) = (D_s\mathbf{1}_{N_s}r_s^\mathsf{T} D_s \otimes Ee^{A(t-t_k)})\mathbf{x}_s(t_k,k)$ with $r_s^\mathsf{T} \in \mathbb{R}^{N_s}$ satisfying $r_s^\mathsf{T} D_s\mathcal{L}_s D_s = 0$ and $r_s^\mathsf{T}\mathbf{1}_{N_s} = 1$;
2) For any $s \in \mathbb{Z}_m \backslash \mathbb{Z}_p$, $\lim_{t+k\to\infty} \mathbf{x}_s(t,k) = 0$;

3) $\lim_{t+k\to\infty} \mathbf{x}_{m+1}(t,k) = -\sum_{s=1}^{m}(M^{-1}\mathcal{L}_s^* \otimes I_n)\mathbf{x}_s(t,k)$ with $\mathcal{M}$ and $\mathcal{L}_s^*$ being given by (1).

Proof: The proof is divided into the following four steps:

Step 1: The decomposition of closed-loop system. Let $x = [x_1^\mathsf{T}, \dots, x_N^\mathsf{T}]^\mathsf{T}$ and $v = [v_1^\mathsf{T}, \dots, v_N^\mathsf{T}]^\mathsf{T}$. Partition the state x and v as $\mathbf{x} = [\mathbf{x}_1^\mathsf{T}, \dots, \mathbf{x}_m^\mathsf{T}, \mathbf{x}_{m+1}^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{nN}$ and $\mathbf{v} = [\mathbf{v}_1^\mathsf{T}, \dots, \mathbf{v}_m^\mathsf{T}, \mathbf{v}_{m+1}^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{nN}$, respectively, where $\mathbf{x}_s = x_{\mathcal{X}_s}$ and $\mathbf{v}_s = v_{\mathcal{X}_s}$ for any $s \in \mathbb{Z}_{m+1}$. Correspondingly, the closed-loop system composed by (2) and (6) is equivalent to the following form:

$$\dot{\tau} = 1, \tag{8a}$$
$$\dot{\mathbf{x}}_s = (I_{N_s} \otimes A)\mathbf{x}_s + (I_{N_s} \otimes BB^\mathsf{T})\mathbf{v}_s, \ \forall\ s \in \mathbb{Z}_m, \tag{8b}$$
$$\dot{\mathbf{v}}_s = -(I_{N_s} \otimes A^\mathsf{T})\mathbf{v}_s, \ \forall\ s \in \mathbb{Z}_m, \tag{8c}$$
$$\dot{\mathbf{x}}_{m+1} = (I_{N_{m+1}} \otimes A)\mathbf{x}_{m+1} + (I_{N_{m+1}} \otimes BB^\mathsf{T})\mathbf{v}_{m+1}, \tag{8d}$$
$$\dot{\mathbf{v}}_{m+1} = -(I_{N_{m+1}} \otimes A^\mathsf{T})\mathbf{v}_{m+1} \tag{8e}$$

whether $(\tau, \mathbf{x}, \mathbf{v}) \in [0, \tau_d] \times \mathbb{R}^{nN} \times \mathbb{R}^{nN}$, and

$$\tau^+ = 0, \tag{8f}$$
$$\mathbf{x}_s^+ = (I_{N_s} \otimes E)\mathbf{x}_s + (\mathcal{L}_s \otimes FK_J)\mathbf{x}_s, \ \forall\ s \in \mathbb{Z}_m, \tag{8g}$$
$$\mathbf{v}_s^+ = (\mathcal{L}_s \otimes e^{A^\mathsf{T}\tau_d}\bar{K}_F)\mathbf{x}_s, \ \forall\ s \in \mathbb{Z}_m, \tag{8h}$$
$$\mathbf{x}_{m+1}^+ = (I_{N_{m+1}} \otimes E)\mathbf{x}_{m+1} + (\mathcal{M} \otimes FK_J)\mathbf{x}_{m+1} + \sum_{s=1}^{m}(\mathcal{L}_s^* \otimes FK_J)\mathbf{x}_s, \tag{8i}$$
$$\mathbf{v}_{m+1}^+ = (\mathcal{M} \otimes e^{A^\mathsf{T}\tau_d}\bar{K}_F)\mathbf{x}_{m+1} + \sum_{s=1}^{m}(\mathcal{L}_s^* \otimes e^{A^\mathsf{T}\tau_d}\bar{K}_F)\mathbf{x}_s \tag{8j}$$

whether $(\tau, \mathbf{x}, \mathbf{v}) \in \{\tau_d\} \times \mathbb{R}^{nN} \times \mathbb{R}^{nN}$. In what follows, we use the notation $\eta_s = [\mathbf{x}_s^\mathsf{T}, \mathbf{v}_s^\mathsf{T}]^\mathsf{T}$ and yield that

$$\dot{\tau} = 1, \ \dot{\eta}_s = A_{\eta_s}\eta_s \tag{9a}$$

whether $(\tau, \eta_s) \in [0, \tau_d] \times \mathbb{R}^{2nN_s}$, and

$$\tau^+ = 0, \ \eta_s^+ = E_{\eta_s}\eta_s \tag{9b}$$

whether $(\tau, \eta_s) \in \{\tau_d\} \times \mathbb{R}^{2nN_s}$, where the matrices A_{η_s} and E_{η_s} are of the following form

$$A_{\eta_s} = \begin{bmatrix} I_{N_s} \otimes A & I_{N_s} \otimes BB^\mathsf{T} \\ 0 & -(I_{N_s} \otimes A^\mathsf{T}) \end{bmatrix},$$
$$E_{\eta_s} = \begin{bmatrix} I_{N_s} \otimes E + \mathcal{L}_s \otimes FK_J & 0 \\ \mathcal{L}_s \otimes e^{A^\mathsf{T}\tau_d}\bar{K}_F & 0 \end{bmatrix}.$$

Step 2: Bipartite consensuses for the subsystems associated with structurally balanced iSCCs, where $s \in \mathbb{Z}_p$. By the assumption, for each $s \in \mathbb{Z}_p$, $\mathcal{G}_{\mathcal{X}_s}$ is structurally balanced. According to Lemma 1, there exists a diagonal matrix $D_s = \operatorname{diag}\{d_{s1}, \dots, d_{sN_s}\}$ with $d_{sj} \in \{\pm 1\}$ for $1 \le j \le N_s$ such that $\bar{\mathcal{L}}_s = D_s\mathcal{L}_sD_s$ can be viewed as a Laplacian matrix of a strongly connected unsigned digraph. Notice that $D_s^{-1} = D_s$. Define $\bar{\mathbf{x}}_s = (D_s \otimes I_n)\mathbf{x}_s$ and $\bar{\mathbf{v}}_s = (D_s \otimes I_n)\mathbf{v}_s$. Then, system (8a) – (8c) with $s \in \mathbb{Z}_p$ is equivalent to the following form:

$$\dot{\tau} = 1$$
$$\dot{\bar{\mathbf{x}}}_s = (I_{N_s} \otimes A)\bar{\mathbf{x}}_s + (I_{N_s} \otimes BB^\mathsf{T})\bar{\mathbf{v}}_s,$$
$$\dot{\bar{\mathbf{v}}}_s = -(I_{N_s} \otimes A^\mathsf{T})\bar{\mathbf{v}}_s$$

whether $(\tau, \bar{\mathbf{x}}_s, \bar{\mathbf{v}}_s) \in [0, \tau_d] \times \mathbb{R}^{nN_s} \times \mathbb{R}^{nN_s}$, and system (8g) – (8h) with $s \in \mathbb{Z}_p$ is equivalent to the following form:

$$\tau^+ = 0,$$
$$\bar{\mathbf{x}}_s^+ = (I_{N_s} \otimes E)\bar{\mathbf{x}}_s + (\bar{\mathcal{L}}_s \otimes FK_J)\bar{\mathbf{x}}_s,$$
$$\bar{\mathbf{v}}_s^+ = (\bar{\mathcal{L}}_s \otimes e^{A^\mathsf{T}\tau_d}\bar{K}_F)\bar{\mathbf{x}}_s$$

whether $(\tau, \bar{\mathbf{x}}_s, \bar{\mathbf{v}}_s) \in \{\tau_d\} \times \mathbb{R}^{nN_s} \times \mathbb{R}^{nN_s}$. Notice that the unsigned graph associated with $\bar{\mathcal{L}}_s$ is strongly connected. The result in [27] can be directly used to obtain that if given δ_c satisfying (5), there exists a $\alpha_c \in \mathbb{R}$ such that$|1 - \alpha_c\lambda| < \delta_c, \ \forall\lambda \in \sigma_0(\mathcal{L}_s)$, then, under Assumption 1, by taking $K_v = -\alpha_c(\tilde{B}_v^\mathsf{T}P\tilde{B}_v)^{-1}\tilde{B}_v^\mathsf{T}P\tilde{A}$, $\lim_{t+k\to\infty}\bar{\mathbf{x}}_s(t,k) = (\mathbf{1}_{N_s}r_s^\mathsf{T} \otimes Ee^{A(t-t_k)})\bar{\mathbf{x}}_s(t_k,k)$. Since D_s is nonsingular, we have $\lim_{t+k\to\infty}\mathbf{x}_s(t,k) = (D_s\mathbf{1}_{N_s}r_s^\mathsf{T} \otimes Ee^{A(t-t_k)}D_s)\mathbf{x}_s(t_k,k)$. This shows part 1).

Step 3: Stability for the subsystem associated with structurally unbalanced iSCCs, where $s \in \mathbb{Z}_m \backslash \mathbb{Z}_p$. According to (9), for all $k \in \mathbb{N}$, $\eta_s(t_{k+1}) = E_{\eta_s}e^{A_{\eta_s}\tau_d}\eta_s(t_k)$, where $t_0 = 0$ and $\eta_s(t_k) \triangleq \eta_s(t_k,k)$ for simplicity. One can choose a nonsingular matrix $V_s \in \mathbb{C}^{N_s \times N_s}$ such that $\hat{\mathcal{L}}_s = V_s^{-1}\mathcal{L}_sV_s$ be an upper triangular matrix where the diagonal entries are all eigenvalues of $\mathcal{L}_s$. Let

$$\tilde{\eta}_s(t_k) = \begin{bmatrix} V_s^{-1} \otimes I_n & 0 \\ 0 & V_s^{-1} \otimes I_n \end{bmatrix}\bar{\eta}_s(t_k).$$

where $\bar{\eta}_s(t_k) = e^{A_{\eta_s}\tau_d}\eta_s(t_k)$. Then, it holds that

$$\tilde{\eta}_s(t_{k+1}) = \begin{bmatrix} I_{N_s} \otimes \tilde{A} + \hat{\mathcal{L}}_s \otimes \tilde{B}K & 0 \\ \hat{\mathcal{L}}_s \otimes \bar{K}_F & 0 \end{bmatrix}\tilde{\eta}_s(t_k). \tag{10}$$

By part 2) of Lemma 1, $I_{N_s} \otimes \tilde{A} + \hat{\mathcal{L}}_s \otimes \tilde{B}K$ is a block upper triangular matrix whose diagonal matrices are $\tilde{A} + \lambda\tilde{B}K = \tilde{A} + \lambda\tilde{B}_vK_v, \ \lambda \in \sigma_0(\mathcal{L}_s)$ with $s \in \mathbb{Z}_m \backslash \mathbb{Z}_p$. From Lemma 3, under Assumption 1, if given δ_c satisfying (5), there exists a $\alpha_c \in \mathbb{R}$ such that $|1 - \alpha_c\lambda| < \delta_c, \ \forall\ \lambda \in \sigma_0(\mathcal{L}_s)$ with $s \in \mathbb{Z}_m \backslash \mathbb{Z}_p$, then, by taking $K_v = -\alpha_c(\tilde{B}_v^\mathsf{T}P\tilde{B}_v)^{-1}\tilde{B}_v^\mathsf{T}P\tilde{A}$, $\tilde{A} + \lambda\tilde{B}_vK_v$ and hence $\tilde{A} + \lambda\tilde{B}K$ is Schur. Thus, system (10) is asymptotically stable. Then we have $\lim_{k\to\infty}||\tilde{\eta}_s(t_k)|| = 0$ and hence $\lim_{k\to\infty}||\eta_s(t_k)|| = 0$. Consequently, from (9a),

$$\lim_{t+k\to\infty}||\eta_s(t,k)|| \le e^{||A_{\eta_s}||\tau_d}\lim_{k\to\infty}||\eta_s(t_k)|| = 0,$$

which, in turn, implies that $\lim_{t+k\to\infty}\mathbf{x}_s(t,k) = 0$ with $s \in \mathbb{Z}_m \backslash \mathbb{Z}_p$. This shows part 2).

Step 4: The asymptotic behaviors of subsystem corresponding to subset $\mathcal{X}_{m+1}$. Define $\bar{\mathbf{x}}_{m+1} = \mathbf{x}_{m+1} + \sum_{s=1}^{m}(\mathcal{M}^{-1}\mathcal{L}_s^* \otimes I_n)\mathbf{x}_s$ and $\bar{\mathbf{v}}_{m+1} = \mathbf{v}_{m+1} + \sum_{s=1}^{m}(\mathcal{M}^{-1}\mathcal{L}_s^* \otimes I_n)\mathbf{v}_s$. Letting $\xi_{m+1} = [\bar{\mathbf{x}}_{m+1}^\mathsf{T}, \bar{\mathbf{v}}_{m+1}^\mathsf{T}]^\mathsf{T}$ yields that

$$\dot{\tau} = 1, \ \dot{\xi}_{m+1} = A_\xi\xi_{m+1} \tag{11a}$$

whether $(\tau, \xi_{m+1}) \in [0, \tau_d] \times \mathbb{R}^{2nN_{m+1}}$, and

$$\tau^+ = 0,\ \xi_{m+1}^+ = E_\xi \xi_{m+1} + \sum_{s=1}^{m} F_{\xi_s} \eta_s \tag{11b}$$

whether $(\tau, \xi_{m+1}) \in \{\tau_d\} \times \mathbb{R}^{2nN_{m+1}}$, where the matrices A_ξ, E_ξ, and F_{ξ_s} are of the following form

$$A_\xi = \begin{bmatrix} I_{N_{m+1}} \otimes A & I_{N_{m+1}} \otimes BB^\mathsf{T} \\ 0 & -(I_{N_{m+1}} \otimes A^\mathsf{T}) \end{bmatrix},$$

$$E_\xi = \begin{bmatrix} I_{N_{m+1}} \otimes E + \mathcal{M} \otimes FK_J & 0 \\ \mathcal{M} \otimes e^{A^\mathsf{T}\tau_d} \bar{K}_F & 0 \end{bmatrix},$$

$$F_{\xi_s} = \begin{bmatrix} \mathcal{M}^{-1}\mathcal{L}_s^* \mathcal{L}_s \otimes FK_J & 0 \\ \mathcal{M}^{-1}\mathcal{L}_s^* \mathcal{L}_s \otimes e^{A^\mathsf{T}\tau_d} \bar{K}_F & 0 \end{bmatrix}.$$

Thus, $\xi_{m+1}(t_{k+1}) = E_\xi e^{A_\xi \tau_d} \xi(t_k) + \sum_{s=1}^{m} F_{\xi_s} e^{A_{\eta_s}\tau_d} \eta_s(t_k)$ where $t_0 = 0$ and $\xi_{m+1}(t_k) \triangleq \xi_{m+1}(t_k, k)$ for simplicity. We can choose a nonsingular matrix $V \in \mathbb{C}^{N_{m+1} \times N_{m+1}}$ such that $\hat{\mathcal{M}} = V^{-1}\mathcal{M}V$ be an upper triangular matrix where the diagonal entries are all eigenvalues of $\mathcal{M}$. Let

$$\tilde{\xi}_{m+1}(t_k) = \begin{bmatrix} V^{-1} \otimes I_n & 0 \\ 0 & V^{-1} \otimes I_n \end{bmatrix} \bar{\xi}_{m+1}(t_k).$$

where $\bar{\xi}_{m+1}(t_k) = e^{A_\xi \tau_d} \xi_{m+1}(t_k)$. Then, it holds that

$$\tilde{\xi}_{m+1}(t_{k+1}) = \begin{bmatrix} I_{N_{m+1}} \otimes \tilde{A} + \hat{\mathcal{M}} \otimes \tilde{B}K & 0 \\ \hat{\mathcal{M}} \otimes \bar{K}_F & 0 \end{bmatrix} \tilde{\xi}_{m+1}(t_k)$$
$$+ \begin{bmatrix} V^{-1} \otimes I_n & 0 \\ 0 & V^{-1} \otimes I_n \end{bmatrix} e^{A_\xi \tau_d} \sum_{s=1}^{m} F_{\xi_s} e^{A_{\eta_s}\tau_d} \eta_s(t_k).$$

By Lemma 2, $I_{N_{m+1}} \otimes \tilde{A} + V^{-1}\mathcal{M}V \otimes \tilde{B}K$ is a block upper triangular matrix whose diagonal matrices are

$$\tilde{A} + \lambda \tilde{B}K = \tilde{A} + \lambda \tilde{B}_v K_v, \lambda \in \sigma_0(\mathcal{M}).$$

From Lemma 3, under Assumption 1, if given δ_c satisfying (5), there exists $\alpha_c \in \mathbb{R}$ such that $|1 - \alpha_c \lambda| < \delta_c$, $\forall\ \lambda \in \sigma_0(\mathcal{M})$, then, by taking $K_v = -\alpha_c(\tilde{B}_v^\mathsf{T} P \tilde{B}_v)^{-1} \tilde{B}_v^\mathsf{T} P \tilde{A}$, $\tilde{A} + \lambda \tilde{B}_v K_v$ and hence $\tilde{A} + \lambda \tilde{B}K$ is Schur.

From Step 2, for any $s \in \mathbb{Z}_p$, $\lim_{k\to\infty} \mathbf{x}_s(t_k) = \lim_{k\to\infty} (D_s \mathbf{1}_{N_s} r_s^\mathsf{T} D_s \otimes Ee^{A\tau_d}) \mathbf{x}_s(t_{k-1})$ and $\lim_{k\to\infty} \mathbf{v}_s(t_k) = 0$. From Step 3, for any $s \in \mathbb{Z}_m \backslash \mathbb{Z}_p$, $\lim_{k\to\infty} \mathbf{x}_s(t_k) = 0$ and $\lim_{k\to\infty} \mathbf{v}_s(t_k) = 0$. These both give that $\lim_{k\to\infty} (\mathcal{L}_s \otimes I_n) \mathbf{x}_s(t_k) = 0$ and $\lim_{k\to\infty} (\mathcal{L}_s \otimes I_n) \mathbf{v}_s(t_k) = 0$. Then, by a simple calculation, we have $\lim_{k\to\infty} F_{\xi_s} e^{A_{\eta_s}\tau_d} \eta_s(t_k) = 0$. Thus, we can deduce with the application of the input-to-state stability theory [10, Lemma 3.8] that $\lim_{k\to\infty} \|\tilde{\xi}_{m+1}(t_k)\| = 0$, and hence, $\lim_{k\to\infty} \|\xi_{m+1}(t_k)\| = 0$. Consequently, from (11a), $\lim_{t+k\to\infty} \|\xi_{m+1}(t,k)\| \le e^{\|A_\xi\|\tau_d} \lim_{k\to\infty} \|\xi_{m+1}(t_k)\| = 0$, which, in turn, implies that $\lim_{t+k\to\infty} \bar{\mathbf{x}}_{m+1}(t,k) = 0$. This shows part 3). □

Remark 4: In signed networks, akin to the analysis in unsigned networks [5], [6], the asymptotic behaviors are intricately linked to the underlying digraph's structure, as detailed in Theorem 1. When dealing with signed networks lacking connectivity, their Laplacian matrix can be rewritten as the form of (1). This decomposition facilitates a corresponding classification of dynamics. In particular, as shown in Theorem 1, dynamics associated with structurally balanced iSCCs reach bipartite consensus, while those associated with structurally unbalanced iSCCs asymptotically converge to zero. Additionally, the dynamics related to follower group (i.e., $\mathcal{X}_{m+1}$) converge towards the convex hull spanned by agents within iSCCs.

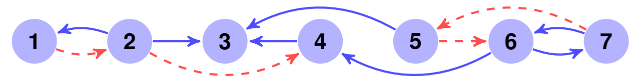

Fig. 1. The signed digraph $\mathcal{G}$ (Solid blue and dashed red lines are associated with positive and negative edges, respectively).

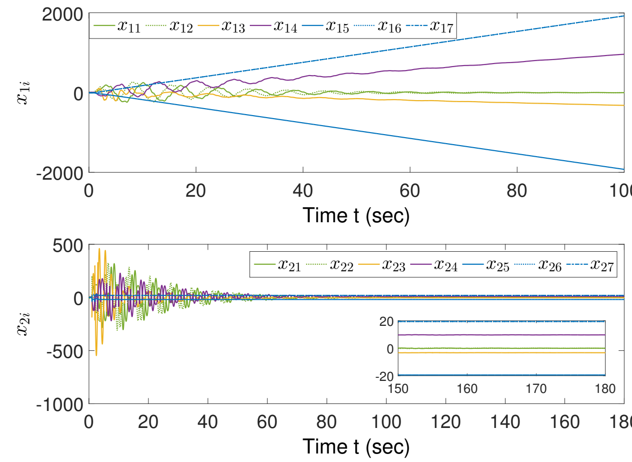

Fig. 2. Trajectories of x_{1i} and x_{2i} for control law (6)

Remark 5: When $E = I_n$ and $F = 0$, system (2) reduces to the continuous-time linear multi-agent system

$$\dot{x}_i = Ax_i + Bu_{Fi}, \quad i = 1, \ldots, N. \tag{12}$$

In this case, controller (6) reduces to a distributed sampled state feedback controller $u_{Fi}(t) = B^\mathsf{T} e^{-A^\mathsf{T}(t-k\tau_d)} e^{A^\mathsf{T}\tau_d} \bar{K}_F \sum_{j=1}^{N} a_{ij}(\mathrm{sgn}(a_{ij}) x_i(k\tau_d) - x_j(k\tau_d))$. Therefore, Theorem 1 provides a sampled-data control for interval consensus problem of system (12), which extends the previous results [25] beyond the requirement of a spanning tree (which only contains unique iSCC) to signed digraphs with lacking connectivity.

Remark 6: When $A = 0$ and $B = 0$, system (2) reduces to the discrete-time linear multi-agent system

$$x_i^+ = Ex_i + Fu_{Ji}, \quad i = 1, \ldots, N \tag{13}$$

In this case, controller (6) reduces to a distributed discrete-time state feedback controller $u_{Ji}(k) = K_J \sum_{j=1}^{N} a_{ij}(\mathrm{sgn}(a_{ij}) x_i(k) - x_j(k))$ that is consistent with that in some existing results, see [12], [29] for first-order system and [23] for general linear system. Thus, Theorem 1 also extends the results of system (13) in [12], [23], [29] to signed digraphs with lacking connectivity.

IV. Simulation

We consider seven infinitely massive disks in motion on a horizontal plane placed between parallel walls, with

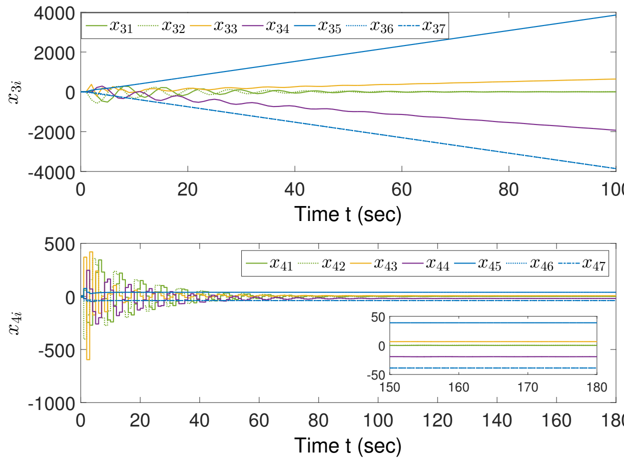

Fig. 3. Trajectories of x_{3i} and x_{4i} for control law (6)

identical state-space equations and parameters as given in [27]. The signed digraph $\mathcal{G}$ is described in Fig. 1, which does not satisfies any connectivity. It is evident that there exist two iSCCs: $\mathcal{X}_1 = \{1,2\}$ and $\mathcal{X}_2 = \{5,6,7\}$. In this scenario, $\mathcal{G}_{\mathcal{X}_1}$ is structurally unbalanced, while $\mathcal{G}_{\mathcal{X}_2}$ is structurally balanced. Moreover, with edge weight 0.5, $\sigma_0(\mathcal{L}) = \{1.5, 1, 0.5+0.5\iota, 0.5-0.5\iota, 1, 1, 1\}$ with ι being the imaginary unit. According to Algorithm 1, for given $\delta_c = 1$, we can take $\alpha_c = 1$ so that condition (7) holds with $\delta(\alpha_c) = 1/2$. The simulation results under control law (6) are shown in Figs. 2–3. It shows that the states of agents within $\mathcal{X}_1$ converge to zero, and those within $\mathcal{X}_2$ reach a bipartite consensus. Additionally, agents 3 and 4 asymptotically converge to the convex hull spanned by the agents belonging to the iSCCs.

V. CONCLUSIONS

The interval consensus problem of hybrid linear multi-agent systems with cooperative-antagonistic networks was investigated. A distributed dynamic state feedback has been presented to solve this problem, extending the consensus studies under signed digraph from purely continuous-time and discrete-time multi-agent systems to hybrid multi-agent systems. The theoretical results was demonstrated through the interval consensus of seven bouncing disks moving on a horizontal plane between parallel walls. Future work will aim to extend these results to systems with switching communication graphs.

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