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State Estimation Using a Network of Observers for a Class of Nonlinear Systems With Communication Delay

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***Abstract*— Distributed observer design is critical for large-scale systems to collectively estimate the system state via networked sensors. In this paper, we propose a novel distributed observer scheme for estimating the states of a class of nonlinear systems. Unknown and time-varying communication delays are considered due to ubiquitous network latency when information is exchanged among observer nodes. Based on the Lyapunov stability criterion, a set of linear matrix inequalities (LMIs) are derived for the design of observer gains, which ensure asymptotic convergence of the state estimates to the true state trajectories in the presence of communication delays. Simulation results are given to validate the effectiveness of the proposed method and its advantage over a recent approach without considering communication delays.**

I. INTRODUCTION

In recent years, there have been many modern systems that are large-scale cyber-physical systems (LSCPS), which involve a large number of networked sensors that interact with each other, such as smart power grids, traffic networks, water systems, etc. Real-time monitoring of the states of LSCPSs is taken by a set of spatially distributed sensors that are unable to individually observe all states of the system. Classical centralized state estimation requires all nodes to transmit their measurements to a central computational unit, which leads to high communication costs and risks for LSCPSs. In this situation, the distributed state estimation scheme can circumvent such a limitation, where each local observer estimates the entire state of the system by exploiting the local measurements and the information shared from its neighbors over a communication network.

In the literature, the general idea of distributed state estimation is to extend the centralized observer design method for linear systems by considering data exchange over the communication network. In [1]–[3], the classical Kalman filtering approach was modified to the distributed estimation network, where each node reaches consensus with their neighboring estimators. Alternatively, [4]–[6] dealt with a Luenberger-type observer, which was extended from a traditional Luenberger observer by introducing a consensus term. In particular, [6] achieved a decentralized design for continuous-time linear time-invariant (LTI) systems. By transforming the systems to the real Jordan canonical form, another decentralized design for distributed observers was proposed in [7]. In [8], by introducing the augmented states, the necessary and sufficient conditions were developed for the existence of parameter choice for the distributed observers. In [9], by exploiting the method of multi-sensor observable canonical decomposition, a kind of Luenberger-based distributed observer was designed for discrete-time LTI systems. By introducing the notion of ρ-hop output matrix, a distributed state estimation approach was developed based on an iterative decomposition in [10]. In [11] and [12], the authors proposed a kind of distributed observers that can achieve asymptotic convergence of state estimation error at a pre-assigned convergence rate. More recently, by utilizing the Volterra operator and non-asymptotic kernels, [13], [14] proposed the design of kernel-based distributed finite-time observers, which enable each node to reconstruct the states of the system within a fixed time interval.

In addition to the aforementioned studies concerning linear autonomous systems, more complex scenarios have been addressed recently, including nonlinearities, unknown input, dynamic network graph, etc. More specifically, in [15]–[17], the time-varying network topology was investigated. To address unknown external disturbance and measurement noise, a robust distributed observer design was proposed in [18]. In [19], the authors addressed the challenge imposed on distributed state estimation, where each observer is requested to recover the full state vector in the presence of locally unknown input signals. In [20] and [21], the resilient state estimation problem was studied to improve the reliability of observer schemes in adversarial environments. In terms of ubiquitous network and sensing delays imposed on the distributed estimation problem, [22]–[24] provided solutions for constant delays, while time-varying delays are addressed in [25] and [26]. Moreover, distributed state estimation for nonlinear systems was investigated in [27]–[29].

In contrast to existing work, where communication delays and nonlinearities are separately studied, this paper aims to solve the distributed estimation problem with consideration of both aspects. As illustrated in Fig. 1, a Luenberger-type distributed state is designed, where each local observer has access to its local measurement and delayed information

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from neighboring sensors. The design criteria of the observer gains are reduced to linear matrix inequalities (LMIs) according to the Lyapunov stability criterion. It has been shown that the design can sufficiently guarantee the asymptotic convergence of each local observer. To show the effectiveness and advantages of the proposed design over existing methods, we compare our method with a recently proposed approach [30] for nonlinear systems, but ignoring communication delays.

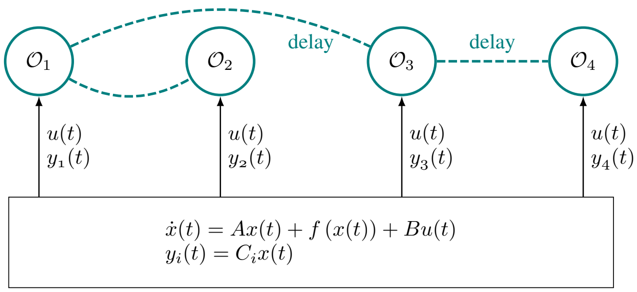

Fig. 1. An example of distributed observer consisting of 4 nodes: each local observer $\mathcal{O}_i$ has access to the input and local measurement $u(t)$ and $y_i(t)$. Furthermore, neighboring estimates are exchanged over an undirected communication network (dashed line), which may suffer from information transmission delays.

The organization of this paper is as follows. In Section II, we present the key notation. In Section III, the problem of distributed estimation for a class of nonlinear systems with communication delays is formulated. In Section IV, we present the design of the distributed observer. The simulation results are given in Section V. Concluding remarks and future work are presented in Section VI.

II. Preliminaries

A. Notation

Throughout the paper, the following notation is considered. $\mathbb{R}$ is the set of real numbers. $\mathbb{R}_{>0}$ denotes the set of positive real numbers. I_n denotes the $n \times n$ identity matrix. $\mathbf{0}_{n\times m}$ is an $n \times m$ all-zeros matrix, and for simplicity, we assume that $\mathbf{0}$ is appropriately sized according to the context. $\|\cdot\|$ is the standard Euclidean norm. $\otimes$ denotes the Kronecker product.
For a square matrix M, let $M \succ 0$ or $M \succeq 0$ if it is symmetric positive definite or symmetric positive semi-definite. Given matrices $M_1, M_2, \ldots, M_n$, $\mathrm{diag}(M_1, M_2, \ldots, M_n)$ denotes the block diagonal matrix composed of M's, and $\mathrm{col}(M_1, M_2, \ldots, M_n)$ denotes the stacked matrix $[M_1^\top, M_2^\top, \cdots, M_n^\top]^\top$.

B. Graph Theory

Communication among network observers is described by an undirected graph denoted by $\mathcal{G} = (\mathbf{N}, \mathcal{E}, \mathcal{A})$ where $\mathbf{N} = \{1, 2, \ldots, N\}$ is a finite nonempty set of nodes of the graph (describing the N sets of observers with local sensors), $\mathcal{E} \subseteq \mathbf{N} \times \mathbf{N}$ represents the edges of the graph (describing communication links among the nodes) and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix where a_{ij} is positive if there exists an edge between Node i and Node j, and it is zero otherwise. Moreover, we define an undirected graph as connected if there is a path of edges between each two nodes of the graph.

Define the Laplacian matrix associated with the graph $\mathcal{G}$ as $\mathcal{L} := \mathcal{D} - \mathcal{A}$ where the i-th entry of the diagonal matrix $\mathcal{D}$ is given by $d_i = \sum_{j=1}^N a_{ij}$.

III. Problem Statement

Consider a class of nonlinear systems in the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(x(t)) + Bu(t), \\ y_i(t) &= C_i x(t), \quad i \in \mathbf{N}, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ represents the state vector, $u \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n\times n}$ is the state matrix, $B \in \mathbb{R}^{n\times m}$ denotes the input matrix, $f(x) \in \mathbb{R}^n$ is a nonlinear function of the states, and $y_i \in \mathbb{R}^{p_i}$ is the measurement output in the i-th node. Accordingly, $C_i \in \mathbb{R}^{p_i\times n}$ is the output matrix associated with the i-th node. In this condition, the collection of all the outputs can be represented as

$$y(t) = \mathrm{col}\,(y_1(t), y_2(t), \cdots, y_N(t))$$

with $\sum_{i=1}^N p_i = p$. Next, we introduce some essential assumptions about the described system with distributed measurements.

Assumption 1: The communication graph associated with the observer network is connected.

Assumption 2: The matrix A is constructed such that the system is jointly observable, i.e., the pair (C, A) is observable where $C = \mathrm{col}(C_1, C_2, \cdots, C_N)$, but (C_i, A) is not necessarily observable for all $i \in \mathbf{N}$.

Assumption 3: We assume that x belongs to a domain $\mathscr{D}$, such that $f(x)$ is Lipschitz on $\mathscr{D}$ as follows [31]–[33]:

$$\|f(x_1) - f(x_2)\| \le \gamma \|x_1 - x_2\|, \forall x_1, x_2 \in \mathscr{D},$$

where $\gamma \in \mathbb{R}_{>0}$.

Assumption 4: We assume unknown time-varying delays τ_{ij}, $\forall (i,j) \in \mathcal{E}$, exist between communication links in the sensor network. The universal bound of τ_{ij}, $\forall (i,j) \in \mathcal{E}$, is known as a finite number $\bar{\tau}$.

Considering nonlinear system (1), the objective is to exploit the joint observability property of the system to design a network of distributed observers such that, in spite of the existence of communication delays subject to Assumption 4, the estimated states of each observer converge to the states of the system as

$$\lim_{t\to\infty} (\hat{x}_i(t) - x(t)) = \mathbf{0}_{n\times 1}, \quad \forall i \in \mathbf{N}.$$

The observer scheme and the analytical results are stated in the next section.

IV. Observer Design and Analysis

Suppose that all nodes have synchronized clocks and that all data transmissions have time stamps [26]. By utilizing the buffers, the distributed observer at node i, $i \in \mathbf{N}$, is designed as

$$\begin{aligned} \dot{\hat{x}}_i(t) =& A\hat{x}_i(t) + L_i\big(C_i\hat{x}_i(t) - y_i(t)\big) + f(\hat{x}_i(t)) + Bu(t) \\ &+ \chi P_i^{-1}\Bigg(\sum_{j=1}^N a_{ij}\big(\hat{x}_j(t-\bar{\tau}) - \hat{x}_i(t-\bar{\tau})\big)\Bigg) \end{aligned} \tag{2}$$

where $L_i \in \mathbb{R}^{n\times p_i}$ and $P_i \in \mathbb{R}^{n\times n}$ are the observer gains that will be designed in the following section. $\chi \in \mathbb{R}_{>0}$ is a positive scalar number, which assigns the weight of the consensus part $\sum_{j=1}^{N} a_{ij}\big(\hat{x}_j(t-\bar{\tau}) - \hat{x}_i(t-\bar{\tau})\big)$ in (2).

To illustrate the main result, we introduce the following lemma [34]:

Lemma 1: For a vector-valued function $\delta(t) \in \mathbb{R}^{n_\delta}$, if its first order derivative exists and is continuous, then the following inequality

$$
\begin{aligned}
&-\int_{t-\tau}^{t} \dot{\delta}(s)^\top R \dot{\delta}(s)\mathrm{d}s \\
&\le \begin{bmatrix}\delta(t)\\ \delta(t-\tau)\end{bmatrix}^\top \begin{bmatrix} M_1^\top + M_1 & -M_1^\top + M_2 \\ -M_1 + M_2^\top & -M_2^\top - M_2\end{bmatrix} \begin{bmatrix}\delta(t)\\ \delta(t-\tau)\end{bmatrix} \\
&\quad + \tau \begin{bmatrix}\delta(t)\\ \delta(t-\tau)\end{bmatrix}^\top \begin{bmatrix} M_1^\top \\ M_2^\top\end{bmatrix} R^{-1} \begin{bmatrix} M_1 & M_2\end{bmatrix} \begin{bmatrix}\delta(t)\\ \delta(t-\tau)\end{bmatrix}
\end{aligned} \tag{3}
$$

holds for any matrices $M_1, M_2, R \in \mathbb{R}^{n_\delta \times n_\delta}$ and $R = R^\top \succ 0$, and a scalar $\tau \ge 0$.

In order to introduce the LMI condition for our design, let us introduce the matrix $Y_i \in \mathbb{R}^{n\times p_i}$, $i \in \mathbf{N}$, which is defined as $Y_i = P_i L_i$. As a consequence, the observer gains L_i can be obtained by $L_i = P_i^{-1} Y_i$.

Theorem 1: Consider the nonlinear system described in (1) under Assumptions 1-4 and the distributed observers given in (2). The estimation error $e_i(t) = x(t) - \hat{x}_i(t)$, $i \in \mathbf{N}$, converges to zero if there exist matrices M_1 and M_2, positive definite symmetric matrices P_i and Q_i, $i \in \mathbf{N}$, positive numbers α_i, $i \in \mathbf{N}$, introduced matrices Y_i, $i \in \mathbf{N}$, positive tuning variables χ and μ, such that the following LMIs are satisfied

$$\Phi \prec 0 \tag{4a}$$
$$R - \mu P \prec 0 \tag{4b}$$
$$P - I_{nN} \succ 0 \tag{4c}$$

where

$$
\Phi = \begin{bmatrix}
\eta_{11} & \eta_{12} & \bar{A}^\top & M_1^\top & \sqrt{\gamma}P & \bar{A}^\top & \gamma R & \mathbf{0} \\
\eta_{12}^\top & \eta_{22} & \eta_{23} & M_2^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} & \eta_{28} \\
\bar{A} & \eta_{23}^\top & \eta_{33} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
M_1 & M_2 & \mathbf{0} & -\frac{1}{\bar{\tau}}R & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\sqrt{\gamma}P & \mathbf{0} & \mathbf{0} & \mathbf{0} & -I_{nN} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\bar{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \eta_{66} & \mathbf{0} & \mathbf{0} \\
\gamma R & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \eta_{77} & \mathbf{0} \\
\mathbf{0} & \eta_{28}^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \eta_{88}
\end{bmatrix} \tag{5}
$$

with

$$
\begin{aligned}
&\eta_{11} = \Lambda + Q + \gamma I_{nN} + \bar{\tau}\gamma^2 R + M_1^\top + M_1 \\
&\eta_{12} = -\chi(\mathcal{L}\otimes I_n) - M_1^\top + M_2 \\
&\eta_{22} = -Q - (M_2^\top + M_2) \\
&\eta_{23} = -\chi(\mathcal{L}\otimes I_n),\ \eta_{33} = -\frac{1}{\bar{\tau}\mu}P \\
&\eta_{28} = (\mathcal{L}\otimes I_n),\ \eta_{66} = -\frac{1}{\bar{\tau}}P \\
&\eta_{77} = -\frac{1}{(\chi+1)\bar{\tau}} I_{nN},\ \eta_{88} = -\frac{1}{\bar{\tau}\chi}P
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&P = \mathrm{diag}(P_1, P_2, \cdots, P_N) \\
&Q = \mathrm{diag}(Q_1, Q_2, \cdots, Q_N) \\
&R = \mathrm{diag}(\alpha_1 I_n, \alpha_2 I_n, \cdots, \alpha_N I_n) \\
&\Lambda_i = A^\top P_i + P_i A + C_i^\top Y_i^\top + Y_i C_i \\
&\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2, \cdots, \Lambda_N) \\
&\bar{A} = \mathrm{diag}(P_1 A + Y_1 C_1, \cdots, P_N A + Y_N C_N)
\end{aligned} \tag{7}
$$

Proof: The time derivative of estimation error $e_i(t)$, $i \in \mathbf{N}$, follows

$$
\begin{aligned}
\dot{e}_i(t) =& (A + L_i C_i)e_i(t) + f(x(t)) - f(\hat{x}_i(t)) \\
&- \chi P_i^{-1}\Big(\sum_{j=1}^{N} a_{ij}\big(e_i(t-\bar{\tau}) - e_j(t-\bar{\tau})\big)\Big)
\end{aligned} \tag{8}
$$

Consider the following Lyapunov-Krasovskii functional

$$
\begin{aligned}
V(t) =& \sum_{i=1}^{N} e_i(t)^\top P_i e_i(t) + \sum_{i=1}^{N}\int_{t-\bar{\tau}}^{t} e_i^\top(s) Q_i e_i(s)\mathrm{d}s \\
&+ \sum_{i=1}^{N}\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} \alpha_i \dot{e}_i(s)^\top \dot{e}_i(s)\mathrm{d}s\mathrm{d}\theta
\end{aligned} \tag{9}
$$

The derivative of $V(t)$ along the error's trajectory satisfies

$$
\begin{aligned}
\dot{V}(t) =& \sum_{i=1}^{N} e_i(t)^\top\big((A+L_iC_i)^\top P_i + P_i(A+L_iC_i)\big)e_i(t) \\
&- 2\chi\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} e_i(t)^\top\big(e_i(t-\bar{\tau}) - e_j(t-\bar{\tau})\big) \\
&+ 2\sum_{i=1}^{N} e_i(t)^\top P_i\big(f(x(t)) - f(\hat{x}_i(t))\big) \\
&+ \sum_{i=1}^{N} e_i(t)^\top Q_i e_i(t) - \sum_{i=1}^{N} e_i(t-\bar{\tau})^\top Q_i e_i(t-\bar{\tau}) \\
&+ \bar{\tau}\sum_{i=1}^{N}\alpha_i \dot{e}_i(t)^\top \dot{e}_i(t) - \sum_{i=1}^{N}\int_{t-\bar{\tau}}^{t}\alpha_i \dot{e}_i^\top(s)\dot{e}_i(s)\mathrm{d}s
\end{aligned} \tag{10}
$$

According to Assumption 3, one gets

$$
2\sum_{i=1}^{N} e_i(t)^\top P_i\big(f(x(t)) - f(\hat{x}_i(t))\big) \le 2\gamma\sum_{i=1}^{N}\|e_i(t)\|\|P_i e_i(t)\| \tag{11}
$$

Let $e = \mathrm{col}(e_1, e_2, \cdots e_N)$. Since $2\|e_i\|\|P_i e_i\| \le e_i^\top e_i + e_i^\top P_i^2 e_i$, one can obtain

$$
2\sum_{i=1}^{N} e_i(t)^\top P_i\big(f(x(t)) - f(\hat{x}_i(t))\big) \le \gamma e(t)^\top(I_{Nn} + P^2)e(t) \tag{12}
$$

In view of (8), it holds that

$$
\begin{aligned}
&\sum_{i=1}^{N} \alpha_i \dot{e}_i(t)^\top \dot{e}_i(t) \\
&= e(t)^\top A_L^\top R A_L e(t) + F(t)^\top R F(t) + 2e(t)^\top A_L^\top R F(t) \\
&\quad - 2\chi e(t)^\top A_L^\top R P^{-1}(\mathcal{L} \otimes I_n) e(t-\bar{\tau}) \\
&\quad + \chi^2 e(t-\bar{\tau})^\top (\mathcal{L} \otimes I_n) P^{-1} R P^{-1} (\mathcal{L} \otimes I_n) e(t-\bar{\tau}) \\
&\quad - 2\chi e(t-\bar{\tau})^\top (\mathcal{L} \otimes I_n) P^{-1} R F(t)
\end{aligned} \tag{13}
$$

where

$$
\begin{aligned}
A_L &= \mathrm{diag}(A + L_1 C_1, A + L_2 C_2, \cdots, A + L_N C_N) \\
F(t) &= \mathrm{col}\big(f(x(t)) - f(\hat{x}_1(t)), \cdots, f(x(t)) - f(\hat{x}_N(t))\big)
\end{aligned}
$$

From Assumption 3, we have

$$
\begin{aligned}
F(t)^\top R F(t) &\le \gamma^2 e(t)^\top R e(t) \\
2e(t)^\top A_L^\top R F(t) &\le e(t)^\top (A_L^\top A_L + \gamma^2 R^2) e(t)
\end{aligned} \tag{14}
$$

and

$$
\begin{aligned}
&- 2e(t-\bar{\tau})^\top (\mathcal{L} \otimes I_n) P^{-1} R F(t) \\
&\le e(t-\bar{\tau})^\top (\mathcal{L} \otimes I_n) P^{-2} (\mathcal{L} \otimes I_n) e(t-\bar{\tau}) \\
&\quad + \gamma^2 e(t) R^2 e(t)
\end{aligned} \tag{15}
$$

By applying Lemma 1, we can obtain

$$
\begin{aligned}
&- \sum_{i=1}^{N} \int_{t-\bar{\tau}}^{t} \dot{e}_i(s)^\top R_i \dot{e}_i(s) \mathrm{d}s \le \begin{bmatrix} e(t) \\ e(t-\bar{\tau}) \end{bmatrix}^\top \\
&\times \begin{bmatrix} M_1^\top + M_1 & -M_1^\top + M_2 \\ -M_1 + M_2^\top & -M_2^\top - M_2 \end{bmatrix} \begin{bmatrix} e(t) \\ e(t-\bar{\tau}) \end{bmatrix} \\
&+ \bar{\tau} \begin{bmatrix} e(t) \\ e(t-\bar{\tau}) \end{bmatrix}^\top \begin{bmatrix} M_1^\top \\ M_2^\top \end{bmatrix} R^{-1} \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} e(t) \\ e(t-\bar{\tau}) \end{bmatrix} \\
&= \begin{bmatrix} e(t) \\ e(t-\bar{\tau}) \end{bmatrix}^\top \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \begin{bmatrix} e(t) \\ e(t-\bar{\tau}) \end{bmatrix}
\end{aligned} \tag{16}
$$

where

$$
\begin{aligned}
\omega_{11} &= M_1^\top + M_1 + \bar{\tau} M_1^\top R^{-1} M_1 \\
\omega_{12} &= \omega_{21}^\top = -M_1^\top + M_2 + \bar{\tau} M_1^\top R^{-1} M_2 \\
\omega_{22} &= -M_2^\top - M_2 + \bar{\tau} M_2^\top R^{-1} M_2
\end{aligned} \tag{17}
$$

By applying (12)-(16) to (10), it can be obtained that

$$
\dot{V}(t) \le \begin{bmatrix} e(t) \\ e(t-\bar{\tau}) \end{bmatrix}^\top \Sigma \begin{bmatrix} e(t) \\ e(t-\bar{\tau}) \end{bmatrix} \tag{18}
$$

where

$$
\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12}^\top & \sigma_{22} \end{bmatrix} \tag{19}
$$

with

$$
\begin{aligned}
\sigma_{11} &= \Lambda + Q + \gamma(I_{Nn} + P^2) + \bar{\tau}(A_L^\top R A_L + \gamma^2 R \\
&\quad + A_L^\top A_L + \gamma^2 R^2 + \chi \gamma^2 R^2) + \omega_{11} \\
\sigma_{12} &= -\chi(\mathcal{L} \otimes I_n) - \bar{\tau}\chi A_L^\top R P^{-1} (\mathcal{L} \otimes I_n) + \omega_{12} \\
\sigma_{22} &= -Q + \bar{\tau}\chi^2 (\mathcal{L} \otimes I_n) P^{-1} R P^{-1} (\mathcal{L} \otimes I_n) \\
&\quad + \bar{\tau}\chi (\mathcal{L} \otimes I_n) P^{-2} (\mathcal{L} \otimes I_n) + \omega_{22}
\end{aligned} \tag{20}
$$

Owing to (4b) and (4c), it is immediate to show

$$
\begin{aligned}
-PR^{-1}P &\prec -\frac{1}{\mu} P \\
-P^2 &\prec -P
\end{aligned} \tag{21}
$$

Next, in view of the equation of Φ in (5), combining the LMI (4a), we construct a matrix Ψ, such that

$$
\Psi \prec \Phi \prec 0 \tag{22}
$$

where

$$
\Psi = \begin{bmatrix}
\eta_{11} & \eta_{12} & \bar{A}^\top & M_1^\top & \sqrt{\gamma}P & \bar{A}^\top & \gamma R & \mathbf{0} \\
\eta_{12}^\top & \eta_{22} & \eta_{23} & M_2^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} & \eta_{28} \\
\bar{A} & \eta_{23}^\top & \tilde{\eta}_{33} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
M_1 & M_2 & \mathbf{0} & -\frac{1}{\bar{\tau}}R & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\sqrt{\gamma}P & \mathbf{0} & \mathbf{0} & \mathbf{0} & -I_{nN} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\bar{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{\eta}_{66} & \mathbf{0} & \mathbf{0} \\
\gamma R & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \eta_{77} & \mathbf{0} \\
\mathbf{0} & \eta_{28}^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{\eta}_{88}
\end{bmatrix} \tag{23}
$$

with

$$
\tilde{\eta}_{33} = -\frac{1}{\bar{\tau}} P R^{-1} P,\ \tilde{\eta}_{66} = -\frac{1}{\bar{\tau}} P^2,\ \tilde{\eta}_{88} = -\frac{1}{\bar{\tau}\chi} P^2 \tag{24}
$$

Then, pre- and post-multiplying diagonal matrices

$$
\mathrm{diag}(I_{nN}, I_{nN}, RP^{-1}, I_{nN}, I_{nN}, P^{-1}, I_{nN}, P^{-1})
$$

and

$$
\mathrm{diag}(I_{nN}, I_{nN}, P^{-1}R, I_{nN}, I_{nN}, P^{-1}, I_{nN}, P^{-1})
$$

on the matrix inequality $\Psi \prec 0$ respectively, one can obtain a matrix inequality

$$
\Theta \prec 0 \tag{25}
$$

where

$$
\Theta = \left[\begin{array}{cc:cccccc}
\eta_{11} & \eta_{12} & A_L^\top R & M_1^\top & \sqrt{\gamma}P & A_L^\top & \gamma R & \mathbf{0} \\
\eta_{12}^\top & \eta_{22} & \hat{\eta}_{23} & M_2^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\eta}_{28} \\
\hdashline
RA_L & \hat{\eta}_{23}^\top & -\frac{1}{\bar{\tau}}R & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
M_1 & M_2 & \mathbf{0} & -\frac{1}{\bar{\tau}}R & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\sqrt{\gamma}P & \mathbf{0} & \mathbf{0} & \mathbf{0} & -I_{nN} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
A_L & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\eta}_{66} & \mathbf{0} & \mathbf{0} \\
\gamma R & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \eta_{77} & \mathbf{0} \\
\mathbf{0} & \hat{\eta}_{28}^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\eta}_{88}
\end{array}\right] \tag{26}
$$

with

$$
\begin{aligned}
\hat{\eta}_{23} &= -\chi(\mathcal{L} \otimes I_n) P^{-1} R,\ \hat{\eta}_{66} = -\frac{1}{\bar{\tau}} I_{nN} \\
\hat{\eta}_{28} &= (\mathcal{L} \otimes I_n) P^{-1},\ \hat{\eta}_{88} = -\frac{1}{\bar{\tau}\chi} I_{nN}
\end{aligned} \tag{27}
$$

According to the Schur Complement [35], combining the matrix-blocking of Θ in (26), (25) further implies

$$
\Sigma \prec 0 \tag{28}
$$

which guarantees $\dot{V}(t) < 0$. Therefore, $V(t)$ asymptotically converges to zero and the proof is completed. ■

V. SIMULATION EXAMPLE

In this section, a numerical example is performed to show the effectiveness of the proposed observer design and the results are benchmarked against a method [30], where the communication delay is ignored. Consider a system in the form of (1) with $x = \begin{bmatrix} x^{(1)} & x^{(2)} & x^{(3)} \end{bmatrix}^\top \in \mathbb{R}^3$ and $x(0) = \begin{bmatrix} 1 & -1.5 & -1 \end{bmatrix}^\top$,

$$A = \begin{bmatrix} -0.7 & 0 & -0.3 \\ 0 & -0.6 & 0.0 \\ 0.5 & 0 & 0.4 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \ C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \ C_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$
$$C_4 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \ C_5 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

and the nonlinear function

$$f(x) = \begin{bmatrix} 0.5\sin(x^{(1)}) \\ 0.05x^{(2)}\cos(x^{(2)}) \\ 0.3\sin(x^{(3)})\cos(x^{(3)}) \end{bmatrix}$$

with the Lipschitz constant $\gamma = 0.5$.

The communication among agents is modelled by an undirected graph shown in Fig. 2 whose Laplacian is given by

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

And the time-varying delays existing in the communication network between sensor nodes are bounded by $\bar{\tau} = 0.198$.

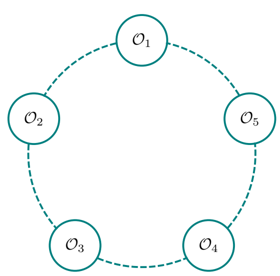

Fig. 2. Network communication topology in simulation example. The communication between sensor nodes is subject to time-varying delays bounded by $\bar{\tau} = 0.198$.

Following the distributed observer design in (2), the observer gains are obtained by solving the LMIs defined in (4), shown as follows:

$$L_1 = \begin{bmatrix} 0.2255 \\ -0.0000 \\ -2.9461 \end{bmatrix}, \ L_2 = \begin{bmatrix} -1.9619 \\ 0.0000 \\ -0.5024 \end{bmatrix}, \ L_3 = \begin{bmatrix} 0.2956 \\ -0.0000 \\ -3.0698 \end{bmatrix}$$
$$L_4 = \begin{bmatrix} 0.1449 \\ 0.0000 \\ -3.0702 \end{bmatrix}, \ L_5 = \begin{bmatrix} -0.0000 \\ -1.7630 \\ 0.0000 \end{bmatrix}$$

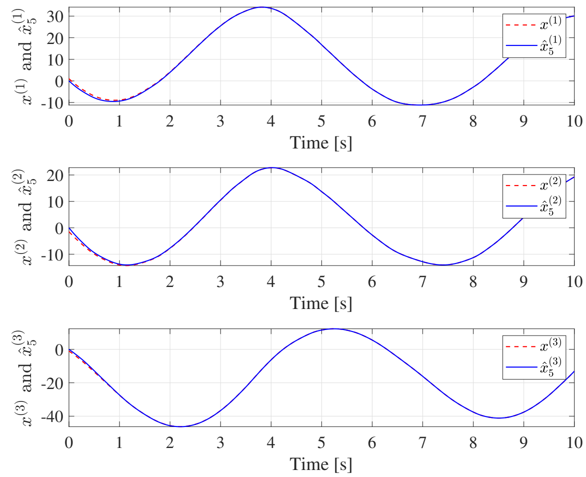

Fig. 3. The estimated states of observer 5 following (2) and the states of the system.

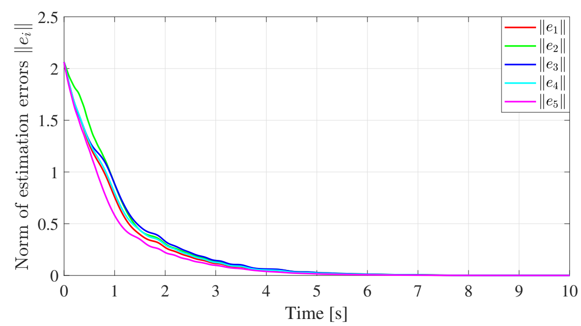

Fig. 4. Norm of estimation errors of the observers designed in (2).

$$P_1 = \begin{bmatrix} 1.5143 & 0.0000 & -0.0465 \\ 0.0000 & 1.5909 & 0.0000 \\ -0.0465 & 0.0000 & 2.6166 \end{bmatrix}$$
$$P_2 = \begin{bmatrix} 2.3281 & -0.0000 & 0.1836 \\ -0.0000 & 1.6240 & -0.0000 \\ 0.1836 & -0.0000 & 1.0261 \end{bmatrix}$$
$$P_3 = \begin{bmatrix} 1.6315 & 0.0000 & -0.1007 \\ 0.0000 & 1.6382 & 0.0000 \\ -0.1007 & 0.0000 & 2.5763 \end{bmatrix}$$
$$P_4 = \begin{bmatrix} 1.6491 & -0.0000 & -0.0439 \\ -0.0000 & 1.6267 & 0.0000 \\ -0.0439 & 0.0000 & 2.5635 \end{bmatrix}$$
$$P_5 = \begin{bmatrix} 2.0594 & -0.0000 & 0.0775 \\ -0.0000 & 2.1713 & 0.0000 \\ 0.0775 & 0.0000 & 1.0062 \end{bmatrix}$$

and the weight of consensus term is $\chi = 1.25$.

The simulation results are shown in Fig. 3-Fig. 5. Fig. 3 shows that, in the presence of bounded time-varying communication delays, the entire states of the system are locally reconstructed at node 5. In Fig. 4, the estimation errors of each observer designed according to (2) asymptotically converge to zero. Compared with a recent study [30] without considering communication delays, Fig. 5 shows that its estimation errors of the estimated states do not converge,

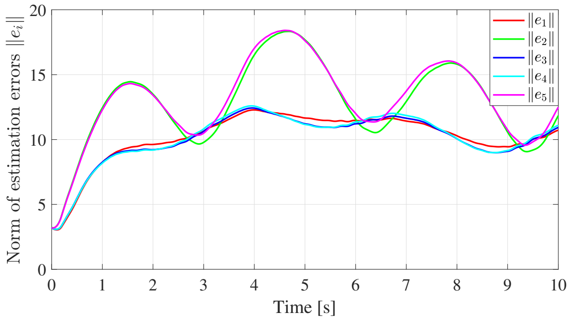

Fig. 5. Norm of estimation errors of the observers designed in [30].

which verifies the benefits of the proposed design.

VI. Conclusions

In this study, we introduced a distributed observer design tailored for a class of nonlinear systems, when information transmission among local observers is affected by unknown and time-varying delays. The analysis shows that the design of the observer gains can be reduced to a set of LMIs, which can be easily solved and leads to an asymptotic stable distributed observer scheme. Numerical results demonstrate the advantages of the proposed method over traditional approaches. This work sets the foundation for robust distributed observer designs against communication faults for large-scale systems. In our future research, we plan to modify the proposed model to factor in process and measurement noise and to expand the spectrum of nonlinear systems under observation.

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