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Global Pose and Velocity-Bias Observer Design on SE(3) Using Synergistic-Based Hybrid Method

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Abstract— **This paper presents an approach for the design of globally asymptotically stable pose and velocity-bias observers. Based on the generic framework for pose and velocity-bias observers, we construct synergistic potential functions on SE(3) and SO(3) via angular warping and use the gradients of the formers to construct innovation terms. Under the synergistic-based hybrid method, different potential functions are selected along with the innovation terms to avoid the undesired critical points on the manifold. The proposed observer can be expressed in terms of measurements and modified measurements in some cases. Simulations are also conducted to show the advantages of the synergistic-based hybrid observer over the reset-based hybrid observer.**

I. INTRODUCTION

These years have seen an increasing demand for the development of robust pose (i.e., position and attitude) estimation algorithms for various applications such as unmanned aerial vehicles, robots and autonomous underwater vehicles [1]. Based on whether the dynamics of the considered system are utilized, the estimation algorithms can be categorized as the static determination algorithms and the dynamic estimation algorithms, with the formers directly constructing the state information utilizing measurements [2] while the latters combining the system dynamics with measurements to recover the attitude or pose information, in the sense that the noise disturbance is suppressed. The filters and observers are the two main dynamic estimation methods. The filters, such as Kalman filters [3], unscented Kalman filters [4] and particle filters [5], are constructed with stability property that is difficult to prove rigorously. In contrast, the observers are derived in the deterministic systems framework whose stability can be proven via Lyapunov theory.

In recent years, a class of nonlinear attitude and pose estimation algorithms (observers), mostly derived on SO(3) and SE(3), respectively, have appeared and attracted the interest of many researchers. This approach, also called nonlinear observers, extracts the attitude or pose information from measurements by introducing an innovation term to the kinematic prediction term [6], and the innovation term is mostly determined by incorporating latest estimates and measurements. Nonlinear observers date back to the work of Salcudean [7], and are greatly extended by Mahony et al [6]. Early works about nonlinear observers are constructed on SO(3) for attitude estimation problems [6], [8]. Motivated by these works, nonlinear pose observers are then developed on SE(3) [9]-[13], among which, different pose observers are designed to pursue different properties, while almost the same gradient-based observer frame is considered, where a right invariant non-degenerate Morse-Bott cost function is constructed and its gradient is used to form the innovation term to correct the estimates. Because of the topological obstruction on SE(3), these classical smooth gradient-based observers can only achieve almost global stability, i.e., the pose converges to the actual one from almost any initial condition except from a set of measure zero. In fact, when the pose on the manifold is near the unstable critical points, it shows to converge slowly [2]. To achieve global stability, the generic gradient-based approach for hybrid estimation on SE(3) was utilized in [1], [14]. This approach, motivated by [15], avoids the undesired critical points by directly changing the point (i.e., the pose) close to the undesired critical points on the manifold, to another point in the decreasing direction of cost function. In [2], [16], utilizing the angular warping method, a central family of synergistic potential functions on SO(3) was constructed and then used in the innovation term. The resulting synergistic potential functions have different undesired critical points but share the same desired critical point. By switching the synergistic potential functions utilized to form the innovation term, the undesired critical points can be avoided successfully. This approach leads to globally stable observers on SO(3).

In this paper, we are devoted to deriving a hybrid pose and velocity-bias observers on SE(3) with global asymptotic stability. Inspired by [2], [15], [16], we utilize the angular warping method to construct synergistic potential functions on SE(3) and SO(3) and then use the gradients of the formers in the innovation term of the observers. We show that under necessary but practically reasonable assumptions, global asymptotic stability is guaranteed. The proposed hybrid pose observer is also re-expressed in terms of measurements and modified measurements in some cases.

The rest of the paper is organized as follows: Section II introduces preliminaries. The problem is formulated in Section III. In Section IV, the hybrid observer is designed and expressed in explicit form in some cases. Simulations are conducted in Section V. Conclusions are drawn in Section VI.

II. BACKGROUND

A. Notations

Throughout this paper, we denote the sets of real, nonnegative real and natural numbers by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$, respectively.

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Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and let $\mathbb{S}^n$ denote the unit n-sphere embedded in $\mathbb{R}^{n+1}$. For two matrices $A, B \in \mathbb{R}^{m\times n}$, their Euclidean inner product is defined as $\langle\langle A, B\rangle\rangle = \mathrm{tr}(A^T B)$. Let the Euclidean norm of a vector $x \in \mathbb{R}^n$ be defined as $||x|| = \sqrt{x^T x}$, and let the Frobenius norm of a matrix $X \in \mathbb{R}^{n\times m}$ be defined as $||X||_F = \sqrt{\langle\langle X, X\rangle\rangle}$. For a square matrix $A \in \mathbb{R}^{n\times n}$, we use $\lambda_i^A, \lambda_{\min}^A, \lambda_{\max}^A$ to denote, respectively, the i-th, the minimum and maximum eigenvalue of A. We define $\mathcal{E}(A)$ as the set of all eigenvectors of A.

B. Pose Representation and Preliminaries

We denote the inertial frame by $\mathcal{I}$ and body-fixed frame of a rigid body by $\mathcal{B}$. Let $p \in \mathbb{R}^3$ denote the rigid body position expressed in $\mathcal{I}$, and $R \in SO(3)$ denote the attitude of $\mathcal{B}$ relative to $\mathcal{I}$, expressed in $\mathcal{I}$. The angular velocity and translational velocity of $\mathcal{B}$ with respect to $\mathcal{I}$ expressed in $\mathcal{B}$ are denoted by $\omega \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$, respectively.

We define the map $(\cdot)^\times : \mathbb{R}^3 \to \mathfrak{so}(3)$ such that $x^\times y = x \times y$ for any $x, y \in \mathbb{R}^3$. Define $\mathrm{vex}() : \mathfrak{so}(3) \to \mathbb{R}^3$ as the inverse operation of $(\cdot)^\times$ satisfying $\mathrm{vex}(\omega^\times) = \omega$ for $\omega \in \mathbb{R}^3$ and $(\mathrm{vex}(\Omega))^\times = \Omega$ for $\Omega \in \mathfrak{so}(3)$. We use the elements on 3-dimensional *Special Euclidean group* $SE(3) = SO(3) \times \mathbb{R}^3$ to represent the pose of a rigid body, which is given by

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}.$$

Denoting by $\mathfrak{se}(3)$ the Lie algebra of $SE(3)$, we have

$$\mathfrak{se}(3) = \{X \in \mathbb{R}^{4\times 4} | X = \begin{bmatrix} \omega^\times & v \\ 0 & 0 \end{bmatrix}, \omega, v \in \mathbb{R}^3\}.$$

Define a wedge map $(\cdot)^\wedge : \mathbb{R}^6 \to \mathfrak{se}(3)$ as

$$\xi^\wedge = \begin{bmatrix} \omega^\times & v \\ 0 & 0 \end{bmatrix}, \xi = \begin{bmatrix} \omega \\ v \end{bmatrix}, \omega, v \in \mathbb{R}^3.$$

The tangent spaces of $SO(3)$ and $SE(3)$ are identified respectively by

$$\begin{aligned} T_R SO(3) =& \{R\Omega | R \in SO(3), \Omega \in \mathfrak{so}(3)\}, \\ T_g SE(3) =& \{gX | g \in SE(3), X \in \mathfrak{se}(3)\}. \end{aligned}$$

Defining $\mathbb{P}_a : \mathbb{R}^{3\times 3} \to \mathfrak{so}(3)$ as the projection map on $\mathfrak{so}(3)$ such that $\mathbb{P}_a(A) = (A - A^T)/2$ for any $A \in \mathbb{R}^{3\times 3}$. Defining $\mathbb{P} : \mathbb{R}^{4\times 4} \to \mathfrak{se}(3)$ as the projection map on $\mathfrak{se}(3)$ such that for all $A \in \mathbb{R}^{3\times 3}, b, c^T \in \mathbb{R}^3$ and $d \in \mathbb{R}$, we have

$$\mathbb{P}\left(\begin{bmatrix} A & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \mathbb{P}_a(A) & b \\ 0 & 0 \end{bmatrix}.$$

For any matrix $A = \{a_{ij}\} \in \mathbb{R}^{3\times 3}$, we define the composition map ψ_a as

$$\psi_a(A) = \mathrm{vex}(\mathbb{P}_a(A)) = [a_{32} - a_{23}, a_{13} - a_{31}, a_{21} - a_{12}]^T/2.$$

Besides, for any matrix $\mathbb{A} = \{a_{ij}\} \in \mathbb{R}^{4\times 4}, y \in \mathbb{R}^6$, we define ψ as

$$\psi(\mathbb{A}) = [a_{32} - a_{23}, a_{13} - a_{31}, a_{21} - a_{12}, a_{14}, a_{24}, a_{34}]^T/2,$$

which satisfies $\langle\langle \mathbb{A}, y^\wedge\rangle\rangle = 2\psi(\mathbb{A})^T y$.

Let $\langle\cdot,\cdot\rangle_g : T_g SE(3) \times T_g SE(3) \to \mathbb{R}$ be a left invariant Riemannian metric on $SE(3)$, such that

$$\langle gU_1, gU_2\rangle_g = \langle\langle U_1, U_2\rangle\rangle, g \in SE(3), U_1, U_2 \in \mathfrak{se}(3).$$

For a smooth function $f : SE(3) \to \mathbb{R}^+$, we denote its gradient by $\nabla_g f$ for any $g \in SE(3)$, which is often determined by Riemannian metric. Let $\mathcal{R}_a : \mathbb{R} \times \mathbb{S}^2 \to SO(3)$ be the angle-axis parametrization of $SO(3)$, which is defined as $\mathcal{R}_a(\theta, u) = I_3 + \sin\theta u^\times + (1 - \cos\theta)(u^\times)^2$, with $\theta \in \mathbb{R}, u \in \mathbb{S}^2$. For any $R \in SO(3)$, we define $|R|_I \in [0, 1]$ satisfying $|R|_I^2 = ||I_3 - R||_F^2/8 = \mathrm{tr}(I_3 - R)/4$ as the normalized Euclidean distance on $SO(3)$.

For any $g \in SE(3), X \in \mathfrak{se}(3)$, we define a map $\mathrm{Ad}_g(\cdot) : SE(3)\times\mathfrak{se}(3) \to \mathfrak{se}(3)$ as $\mathrm{Ad}_g(X) = gXg^{-1}$ with its matrix representation on $\mathfrak{se}(3)$ given by

$$\mathrm{Ad}_g = \begin{bmatrix} R & 0 \\ p^\times R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}.$$

We also denote its transpose map by $\mathrm{Ad}_g^T(\cdot)$. For any $r = (r_v, r_s)^T, b = (b_v, b_s)^T \in \mathbb{R}^4$ with r_v, b_v the first three elements of r, b, respectively, we define the wedge product $\wedge$ as

$$b \wedge r = \begin{bmatrix} b_v \times r_v \\ b_s r_v - r_s b_v \end{bmatrix} \in \mathbb{R}^6.$$

C. Hybrid Systems Framework

The hybrid framework considered here can be described as follows

$$\begin{aligned} \dot{x} =& F(x, q), \quad (x, q) \in \mathcal{F}, \\ q^+ \in& J(x, q), \quad (x, q) \in \mathcal{J}. \end{aligned}$$

The flow map $F : \mathcal{M}\times\mathcal{Q} \to \mathcal{TM}$ drives the continuous flow of x on $\mathcal{M}$, which means that the state of the system evolves continuously according to the given kinematic relations. The flow set $\mathcal{F} \subset \mathcal{M} \times \mathcal{Q}$ sets the occurring domain for continuous flow. $\mathcal{Q} \subset \mathbb{N}$ is a finite index set, in which different q means different evolution paths. The jump map $J : \mathcal{M} \times \mathcal{Q} \to \mathcal{Q}$ governs discrete jumps of q, and the jump set $\mathcal{J} \subset \mathcal{M} \times \mathcal{Q}$ dictates where the jumps could happen. In the flow set $\mathcal{F}$, q remains unchanged and during the jump set $\mathcal{J}$, the state x of the system remains unchanged.

III. PROBLEM FORMULATION

Let $g \in SE(3)$ denote the pose of a rigid body, $\xi = [\omega^T, v^T]^T \in \mathbb{R}^6$ denote the generalized velocity. Then g evolves according to the following kinematics equation

$$\dot{g} = g\xi^\wedge. \tag{1}$$

We suppose that ξ is bounded and continuous. The measured velocity, denoted by $\xi_y = [\omega_y^T, v_y^T]^T \in \mathbb{R}^6$ is contaminated by a constant and slowly varying bias $b_a = [b_\omega^T, b_v^T]^T \in \mathbb{R}^6$ such that

$$\xi_y = \xi + b_a. \tag{2}$$

Suppose that a set of n constant reference vectors $r_i \in \mathbb{R}^4, i = 1, \ldots, n$ in $\mathcal{I}$ are known and their measurements in $\mathcal{B}$ are expressed as

$$b_i = g^{-1} r_i, i = 1, \ldots, n. \tag{3}$$

The n reference vectors include n_1 landmarks and $n-n_1$ inertial vectors, with the form of

$$r_i = [(p_i^{\mathcal{I}})^T, 1]^T, \quad p_i^{\mathcal{I}} \in \mathbb{R}^3, \quad i = 1, \dots, n_1,$$
$$r_i = [(v_i^{\mathcal{I}})^T, 0]^T, \quad v_i^{\mathcal{I}} \in \mathbb{R}^3, \quad i = n_1+1, \dots, n,$$

respectively. Then we can denote the measurements of the landmarks and inertial vectors respectively by

$$b_i = [(p_i^{\mathcal{B}})^T, 1]^T, \ p_i^{\mathcal{B}} = R^T(p_i^{\mathcal{I}} - p) \in \mathbb{R}^3, \ i = 1, \dots, n_1,$$
$$b_i = [(v_i^{\mathcal{B}})^T, 0]^T, \ v_i^{\mathcal{B}} = R^T v_i^{\mathcal{I}} \in \mathbb{R}^3, \ i = n_1+1, \dots, n.$$

We define the weighted geometric center of all the landmarks and their measurements as follows

$$p_c^{\mathcal{I}} = \sum_{i=1}^{n_1} \alpha_i p_i^{\mathcal{I}}/l, \quad p_c^{\mathcal{B}} = \sum_{i=1}^{n_1} \alpha_i p_i^{\mathcal{B}}/l = R^T(p_c^{\mathcal{I}} - p), \quad (4)$$

where $\alpha_i > 0, i = 1, \dots, n_1$ and $l = \sum_{i=1}^{n_1} \alpha_i$. Here we give the following modified inertial vectors

$$v_i^{\mathcal{I}} = p_i^{\mathcal{I}} - p_c^{\mathcal{I}}, \quad v_i^{\mathcal{B}} = p_i^{\mathcal{B}} - p_c^{\mathcal{B}} = R^T v_i^{\mathcal{I}}, \quad (5)$$

for all $i = 1, \dots, n_1$. We define the set of all the inertial vectors (including the directly measured ones and the modified ones) as $V^{\mathcal{I}} = \{v_i^{\mathcal{I}}, i = 1, \dots, n\}$. To guarantee the observability of the system, at least one landmark point should be measured, and at least two vectors from $V^{\mathcal{I}}$ are non-collinear, among the n measurements. This is a standard measurement case in estimation problems on $SE(3)$, but not the only one.

Now we introduce the following matrix

$$\mathbb{A} = \sum_{i=1}^{n} k_i r_i r_i^T = \begin{bmatrix} A & b \\ b^T & d \end{bmatrix} \in \mathbb{R}^{4\times 4}, \quad (6)$$

where $k_i > 0, i = 1, \dots, n$, $d = \sum_{i=1}^{n_1} k_i \in \mathbb{R}$, $b = \sum_{i=1}^{n_1} k_i p_i^{\mathcal{I}} \in \mathbb{R}^3$ and $A = \sum_{i=1}^{n_1} k_i p_i^{\mathcal{I}} (p_i^{\mathcal{I}})^T + \sum_{i=n_1+1}^{n} k_i v_i^{\mathcal{I}} (v_i^{\mathcal{I}})^T \in \mathbb{R}^{3\times 3}$. We set $\alpha_i = k_i, i = 1, \dots, n_1$. It can be verified that

$$A - bb^T d^{-1} = \sum_{i=1}^{n} k_i v_i^{\mathcal{I}} (v_i^{\mathcal{I}})^T.$$

Lemma 1: Consider the matrix $\mathbb{A}$ defined in (6). Define the matrix $Q = A - bb^T d^{-1}$, which is positive definite. Then the matrix $W_Q = \mathrm{tr}(Q) I_3 - Q$ is also positive definite.

Lemma 2: Consider the matrix $\mathbb{A}$ defined in (6) and the matrix Q defined above. Then for any $g \in SE(3)$, the following relations can be obtained

$$\begin{aligned} \mathrm{tr}((I_4 - g)\mathbb{A}(I_4 - g)^T) &= \sum_{i=1}^{n} k_i ||r_i - g^{-1} r_i||^2, \\ \psi(\mathbb{P}((I_4 - g^{-1})\mathbb{A})) &= \frac{1}{2} \sum_{i=1}^{n} k_i (g^{-1} r_i) \wedge r_i, \\ \mathrm{tr}(Q(I_3 - R)) &= \frac{1}{2} \sum_{i=1}^{n} k_i ||v_i^{\mathcal{I}} - R^T v_i^{\mathcal{I}}||^2, \\ \mathrm{tr}((I_4 - g)\mathbb{A}(I_4 - g)^T) &= 2\mathrm{tr}(Q(I_3 - R)) \\ &\quad + d||p - (I_3 - R) b d^{-1}||^2. \end{aligned}$$

We aim at designing a globally stable pose and velocity-bias estimation algorithm with the synergistic-based hybrid method, using available inertial vectors and landmarks measurements in some cases.

IV. HYBRID OBSERVER DESIGN

Consider a positive-valued smooth function $\mathcal{U} : SE(3) \to \mathbb{R}^+$. It can be used as a potential function on $SE(3)$ if $\mathcal{U}(g) \geq 0$ for any $g \in SE(3)$ and $\mathcal{U}(g) = 0$ if and only if $g = I_4$. We use $\nabla_g \mathcal{U}(g)$ to denote its gradient with respect to g. Because of the special manifold structure, there are at least four critical points of $\mathcal{U}$ on $SE(3)$, among which only one is stable. Let $\Psi_{\mathcal{U}}$ denote the set of all the critical points.

A. Hybrid Pose and Velocity-Bias Observer Design

Let $\hat{g}$ and $\hat{b}_a$ denote the estimates of g and b_a, respectively. We define their estimation errors respectively, as $\tilde{g} = g\hat{g}^{-1}, \tilde{b}_a = \hat{b}_a - b_a$. Let $\mathcal{Q} \subset \mathbb{N}$ be a finite index set. We adopt the generic gradient-based pose observer framework proposed in [1] and get the potential function $\mathcal{U}$ indexed by $q \in \mathcal{Q}$, such that

$$\begin{aligned} \dot{\hat{g}} &= \hat{g}(\xi_y - \hat{b}_a + k_\beta \beta)^\wedge, \quad (7) \\ \dot{\hat{b}}_a &= -\Gamma \sigma_b, \quad (8) \\ \beta &= \mathrm{Ad}_{\hat{g}^{-1}} \psi(\tilde{g}^{-1} \nabla_{\tilde{g}} \mathcal{U}(\tilde{g}, q)), \\ \sigma_b &= \mathrm{Ad}_{\hat{g}}^T \psi(\tilde{g}^{-1} \nabla_{\tilde{g}} \mathcal{U}(\tilde{g}, q)), \end{aligned}$$

where $\hat{g}(0) \in SE(3), \hat{b}_a(0) \in \mathbb{R}^6, \Gamma = \mathrm{diag}(k_\omega I_3, k_v I_3) \in \mathbb{R}^{6\times 6}$, and $k_\omega, k_v, k_\beta > 0$. The discrete jump variable q is govern by the following hybrid mechanism

$$\begin{cases} \dot{q} = 0, & (\tilde{g}, q) \in \mathcal{F}, \\ q^+ \in \arg\min_{p \in \mathcal{Q}} \mathcal{U}(\tilde{g}, p), & (\tilde{g}, q) \in \mathcal{J}, \end{cases} \quad (9)$$

in which the flow set $\mathcal{F}$ and jump set $\mathcal{J}$ are defined such that

$$\mathcal{F} = \{(\tilde{g}, q) : \mathcal{U}(\tilde{g}, q) - \min_{p \in \mathcal{Q}} \mathcal{U}(\tilde{g}, p) \leq \delta\}, \quad (10)$$
$$\mathcal{J} = \{(\tilde{g}, q) : \mathcal{U}(\tilde{g}, q) - \min_{p \in \mathcal{Q}} \mathcal{U}(\tilde{g}, p) \geq \delta\}, \quad (11)$$

for some $\delta > 0$ to be determined later.

Theorem 1: Consider the pose kinematics (1) coupled with the observer (7)-(11). Suppose that the potential function $\mathcal{U}(\tilde{g}, q)$ are smooth on $SE(3)$ and the index set $\mathcal{Q}$ along with δ is chosen such that all the undesired critical points of $\mathcal{U}$ lie in $\mathcal{J}$. Then the number of discrete jumps is finite and the observer is uniformly globally asymptotically stable.

Proof: Let us consider the following Lyapunov function candidate on $SE(3)$

$$V(\tilde{g}, \tilde{b}_a, q) = \mathcal{U}(\tilde{g}, q) + \tilde{b}_a^T \Gamma^{-1} \tilde{b}_a. \quad (12)$$

For any $(\tilde{g}, q) \in \mathcal{F}$, the closed loop dynamics are given by

$$\begin{aligned} \dot{\tilde{g}} &= \tilde{g}(\mathrm{Ad}_{\hat{g}} \tilde{b}_a - k_\beta \psi(\tilde{g}^{-1} \nabla_{\tilde{g}} \mathcal{U}(\tilde{g}, q)))^\wedge, \\ \dot{\tilde{b}}_a &= -\Gamma \mathrm{Ad}_{\hat{g}}^T \psi(\tilde{g}^{-1} \nabla_{\tilde{g}} \mathcal{U}(\tilde{g}, q)). \end{aligned} \quad (13)$$

Taking the time derivative of V along the trajectories of (13), we have

$$
\begin{aligned}
&\dot{V}(\tilde{g},\tilde{b}_a,q)\\
&=\langle\nabla_{\tilde{g}}\mathcal{U}(\tilde{g},q),\tilde{g}(\mathrm{Ad}_{\hat{g}}\tilde{b}_a-k_\beta\psi(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}(\tilde{g},q)))^\wedge\rangle_{\tilde{g}}\\
&\quad-2\tilde{b}_a^T\mathrm{Ad}_{\hat{g}}^T\psi(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}(\tilde{g},q))\\
&=2\psi(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}(\tilde{g},q))^T(\mathrm{Ad}_{\hat{g}}\tilde{b}_a-k_\beta\psi(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}(\tilde{g},q)))\\
&\quad-2\tilde{b}_a^T\mathrm{Ad}_{\hat{g}}^T\psi(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}(\tilde{g},q))\\
&=-2k_\beta||\psi(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}(\tilde{g},q))||^2<0,
\end{aligned}
$$

for any $(\tilde{g},q)\in\mathcal{F}$. As for $(\tilde{g},q)\in\mathcal{J}$, we have

$$V(\tilde{g},\tilde{b}_a,q^+)-V(\tilde{g},\tilde{b}_a,q)\leq-\delta.$$

Therefore, it is guaranteed that $V(\tilde{g},\tilde{b}_a,q)$ strictly decreases in the whole process. According to Barbalat's lemma, $\lim_{t\to\infty}\tilde{g}=I_4$. In view of (13), $\lim_{t\to\infty}\tilde{b}_a=0$. The observer is uniformly globally asymptotically stable.

For all $(t,j)\in\mathrm{dom}(\tilde{g},\tilde{b}_a,q)$, we have $0<V(t,j)\leq V(0,0)-\delta j$, which leads to $j\leq(V(0,0)-V(t,j))/\delta$. Hence, we can conclude that the number of discrete jumps is finite. The proof is complete. ∎

B. Construction of the Potential Function and Explicit Hybrid Observer Expression Using Measurements

In this part, we extend the synergistic-based hybrid method in [16] from 3-dimentional to 6-dimension and apply it to the observer. Let us consider the following potential function

$$\mathcal{U}_A(\tilde{R})=\mathrm{tr}(Q(I_3-\tilde{R}))/2\lambda_{\max}^{W_Q},\tag{14}$$

with Q and W_Q defined in lemma 1. It has been widely used in the attitude estimation problems. Besides, we have

$$\nabla_{\tilde{R}}\mathcal{U}_A(\tilde{R})=\tilde{R}\mathbb{P}_a(Q\tilde{R})/2\lambda_{\max}^{W_Q},\tag{15}$$

$$\Psi_{\mathcal{U}_A}=\{I_3\}\cup\{\tilde{R}\in SO(3)|\tilde{R}=\mathcal{R}_a(\pi,v),v\in\mathcal{E}(Q)\}.\tag{16}$$

We then introduce the following angular warping transformation

$$\Gamma_A(\tilde{R},q)=\tilde{R}\mathfrak{R}_A(\tilde{R},q),\tag{17}$$

$$\mathfrak{R}_A(\tilde{R},q)=\mathcal{R}_a(2\sin^{-1}(k\mathcal{U}_A(\tilde{R})),\nu(q)),\tag{18}$$

where $\mathcal{U}_A(\tilde{R})$ is defined in (14), $q\in\mathcal{Q}\subset\mathbb{N}$, the map $\nu(q):\mathcal{Q}\to\mathbb{S}^2$ is to be determined and the scalar k satisfies

$$0<k<\bar{k}=(2\lambda_{\max}^{W_Q}\sqrt{1+4\xi})^{-1},\tag{19}$$

with $\xi=\lambda_{\max}^{W_Q}/\lambda_{\min}^{W_Q}$. Let

$$\Phi(\tilde{R},q)=\mathcal{U}_A\circ\Gamma_A(\tilde{R},q).\tag{20}$$

By introducing the transformation, we stretch and compress the manifold $SO(3)$ by moving all the undesired critical points to different locations.

Consider the following parameters design: $\Phi(\tilde{R},q)$ are defined in (20), Q is defined in lemma 1 with distinct eigenvalues $0<\lambda_1^Q<\lambda_2^Q<\lambda_3^Q,\mathcal{Q}\in\{1,2\},\nu(1)=-\nu(2)=u,u\in\mathbb{S}^2$ and satisfies that if $\lambda_2^Q\lambda_3^Q-\lambda_1^Q\lambda_2^Q-\lambda_1^Q\lambda_3^Q\geq0$, then

$$u^Tv_1=0,\quad(u^Tv_i)^2=\lambda_i^Q/(\lambda_2^Q+\lambda_3^Q),\quad i=2,3,$$

otherwise,

$$(u^Tv_i)^2=1-4\Pi_{j\neq i}\lambda_j^Q/(\Sigma_l\Sigma_{k\neq l}\lambda_l^Q\lambda_k^Q),$$

for $i\in\{1,2,3\}$. v_i is the corresponding eigenvector of Q to the eigenvalue λ_i^Q. Define $\delta_R\in\mathbb{R}$ such that $0<\delta_R<\Delta=(8k^2\bar{V}^2-8\lambda_{\min}^{W_Q}k^2\bar{V}+\varpi-1)(-1+\varpi)^2/16k^4\bar{V}^3$ with k selected as in (19), $\varpi=\sqrt{1+16\lambda_{\min}^{W_Q}k^2\bar{V}}$, and

$$\bar{V}=\begin{cases}\lambda_1^Q & \text{if}\,\lambda_2^Q\lambda_3^Q-\lambda_1^Q\lambda_2^Q-\lambda_1^Q\lambda_3^Q\geq0,\\ \frac{4\Pi_j\lambda_j^Q}{\Sigma_l\Sigma_{k\neq l}\lambda_l^Q\lambda_k^Q} & \text{otherwise.}\end{cases}$$

Define the hybrid mechanism

$$\begin{cases}\dot{q}=0, & (\tilde{R},q)\in\mathcal{F}_R,\\ q^+\in\arg\min_{p\in\mathcal{Q}}\Phi(\tilde{R},p), & (\tilde{R},q)\in\mathcal{J}_R,\end{cases}$$

with

$$\mathcal{F}_R=\{(\tilde{R},q):\Phi(\tilde{R},q)-\min_{p\in\mathcal{Q}}\Phi(\tilde{R},p)\leq\delta_R\},$$

$$\mathcal{J}_R=\{(\tilde{R},q):\Phi(\tilde{R},q)-\min_{p\in\mathcal{Q}}\Phi(\tilde{R},p)\geq\delta_R\}.$$

In view of [16], for all $(\tilde{R},q)\in\mathcal{F}_R$, we have

$$\Gamma_A(\tilde{R},q)\notin\{\tilde{R}\in SO(3)|\tilde{R}=\mathcal{R}_a(\pi,v),v\in\mathcal{E}(Q)\},$$

and for any

$$\Gamma_A(\tilde{R},q)\in\{\tilde{R}\in SO(3)|\tilde{R}=\mathcal{R}_a(\pi,v),v\in\mathcal{E}(Q)\},$$

we have $(\tilde{R},q)\in\mathcal{J}_R$. With the parameters defined above, the 3-dimensional synergistic-based hybrid method is presented. Now we go ahead to extend it from 3-dimention to 6-dimension.

Consider the potential function on $SE(3)$ as

$$\mathcal{U}_s(\tilde{g})=\mathrm{tr}((I_4-\tilde{g})\mathbb{A}(I_4-\tilde{g})^T)/2,\tag{21}$$

with $\mathbb{A}$ defined in (6). $\mathcal{U}_s(\tilde{g})$ can be used as a potential function for pose estimation problems. Besides,

$$\nabla_{\tilde{g}}\mathcal{U}_s(\tilde{g})=\tilde{g}\mathbb{P}((I_4-\tilde{g}^{-1})\mathbb{A}),\tag{22}$$

$$\begin{aligned}\Psi_{\mathcal{U}_s}=\{I_4\}\cup\{\tilde{g}\in &SE(3)|\tilde{R}=\mathcal{R}_a(\pi,v),\\ &\tilde{p}=(I_3-\mathcal{R}_a(\pi,v))bd^{-1},v\in\mathcal{E}(Q)\},\end{aligned}\tag{23}$$

In view of (16) and (23), both $\mathcal{U}_A(\tilde{R})$ and $\mathcal{U}_s(\tilde{g})$ have three undesired critical points and a desired critical point which have the one-to-one correspondence. Any pair of the points share the same expression for $\tilde{R}$. Therefore, we can use the angular wrapping method to design the synergistic potential functions $\mathcal{U}(\tilde{g},q)$ on $SE(3)$. Assume that all the undesired critical points of $\mathcal{U}(\tilde{g},q)$ and $\Phi(\tilde{R},q)$ have the one-to-one correspondence for any $q\in\mathcal{Q}$. Then all the undesired critical points of $\mathcal{U}(\tilde{g},q)$ on the manifold $SE(3)$ are removed to the jump set J, so long as $\mathcal{F}=\mathcal{F}_R\times\mathbb{R}^3,\mathcal{J}=\mathcal{J}_R\times\mathbb{R}^3$.

Lemma 3: Consider the transformation Γ_A in (17) and (18). For any $X \in SO(3)$ and $\omega \in \mathbb{R}^3$ such that $\dot{X} = X\omega^\times$, we have

$$\frac{d}{dt}\Gamma_A(X,q) = \Gamma_A(X,q)(\Theta_A(X,q)\omega)^\times, \qquad (24)$$

where $\Theta_A(X,q)$ is given by

$$\Theta_A(X,q) = \mathfrak{R}_A(X,q)^T + \frac{2k\nu(q)\psi_a(QX)^T}{\lambda_{\max}^{W_Q}\sqrt{1-k^2\mathcal{U}_A^2(X)}}. \qquad (25)$$

To ensure the one-to-one correspondence of the three undesired critical points and a desired critical point between $\Phi(\tilde{R},q)$ and $\mathcal{U}(\tilde{g},q)$, and that any pair of the points share the same expression for $\tilde{R}$, we define

$$\mathcal{U}(\tilde{g},q) = \mathcal{U}_s \circ \Gamma_A(\tilde{g},q), \qquad (26)$$

with

$$\Gamma_A(\tilde{g},q) = \begin{bmatrix} \tilde{R}\mathfrak{R}_A(\tilde{R},q) & \tilde{p} + (I_3 - \tilde{R}\mathfrak{R}_A(\tilde{R},q))bd^{-1} \\ 0 & 1 \end{bmatrix}, \qquad (27)$$

where $\mathfrak{R}_A(\tilde{R},q)$ is defined in (18). With this design, $\mathcal{F} = \mathcal{F}_R \times \mathbb{R}^3, \mathcal{J} = \mathcal{J}_R \times \mathbb{R}^3$ are guaranteed when $\delta = 2\lambda_{\max}^{W_Q}\delta_R$, in view of lemma 2. Taking the time derivative of $\Gamma_A(\tilde{g},q)$, we have

$$\begin{aligned} \frac{d}{dt}\Gamma_A(\tilde{g},q) &= \begin{bmatrix} \Gamma_A(\tilde{R},q)(\Theta_A(\tilde{R},q)\tilde{\omega})^\times & \Upsilon_1 \\ 0 & 0 \end{bmatrix} \\ &= \Gamma_A(\tilde{g},q)(\Theta_A(\tilde{g},q)\xi)^\wedge, \end{aligned} \qquad (28)$$

with

$$\Upsilon_1 = \tilde{R}\tilde{v} - \Gamma_A(\tilde{R},q)(\Theta_A(\tilde{R},q)\tilde{\omega})^\times bd^{-1},$$

$$\Theta_A(\tilde{g},q) = \begin{bmatrix} \Theta_A(\tilde{R},q) & 0 \\ (bd^{-1})^\times\Theta_A(\tilde{R},q) & (\mathfrak{R}_A(\tilde{R},q))^{-1} \end{bmatrix}, \qquad (29)$$

where $\Theta_A(\tilde{R},q)$ is defined in (25), $\xi = [\tilde{\omega}^T, \tilde{v}^T]^T \in \mathbb{R}^6$ such that $\dot{\tilde{g}} = \tilde{g}\xi^\wedge$. Calculating the gradient of $\mathcal{U}(\tilde{g},q)$ with respect to $\tilde{g}$ and using the chain rule, we have

$$\begin{aligned} \nabla_{\tilde{g}}\mathcal{U}(\tilde{g},q) =& \tilde{g}[\Theta_A(\tilde{g},q)^T\psi(\mathbb{P}((I_4 - \Gamma_A(\tilde{g},q)^{-1})\mathbb{A}))]^\wedge \\ & \times \mathrm{diag}(I_3, 2). \end{aligned} \qquad (30)$$

In view of lemma 2, it is easy to get

$$\begin{aligned} \beta =& \frac{1}{2}\mathrm{Ad}_{\hat{g}^{-1}}\left(\Theta_A(\tilde{g},q)^T\sum_{i=1}^n k_i((g^{-1}\Gamma_A(\tilde{g},q))^{-1}b_i)\wedge r_i\right), \\ \sigma_b =& \frac{1}{2}\mathrm{Ad}_{\hat{g}}^T\left(\Theta_A(\tilde{g},q)^T\sum_{i=1}^n k_i((g^{-1}\Gamma_A(\tilde{g},q))^{-1}b_i)\wedge r_i\right), \end{aligned} \qquad (31)$$

with

$$(g^{-1}\Gamma_A(\tilde{g},q))^{-1} = \begin{bmatrix} \mathfrak{R}_A(\tilde{R},q)^T\hat{R} & \Upsilon_2 \\ 0 & 1 \end{bmatrix}, \qquad (32)$$

$$\Upsilon_2 = \mathfrak{R}_A(\tilde{R},q)^T\hat{p} + (I_3 - \tilde{R}\mathfrak{R}_A(\tilde{R},q))^T bd^{-1},$$

and $\Theta_A(\tilde{g},q)$ defined in (29).

The term $\tilde{R}$ in (32) may be unavailable in some practical applications because no onboard sensors can directly obtain the measurement of R. We can deal with this problem by appropriately choosing the landmarks and their corresponding gain parameters such that $bd^{-1} = 0$, or directly reconstructing R from the measurements through static determination algorithms, which is a common practice and not detailed here.

By combining (14) (18) and (25), the following quantities are expressed in an explicit form

$$\mathfrak{R}_A(\tilde{R},q) = \mathcal{R}_a(2\sin^{-1}(k\vartheta), \nu(q)), \qquad (33)$$

$$\vartheta = \frac{1}{4\lambda_{\max}^{W_Q}}\sum_{i=1}^n k_i||v_i^{\mathcal{B}} - \hat{R}^T v_i^{\mathcal{I}}||^2, \qquad (34)$$

$$\Theta_A(\tilde{R},q)^T = \mathfrak{R}_A(\tilde{R},q) + \frac{k\hat{R}\sum_{i=1}^n k_i(v_i^{\mathcal{B}} \times \hat{R}^T v_i^{\mathcal{I}})\nu(q)^T}{\lambda_{\max}^{W_Q}\sqrt{1-k^2\vartheta^2}}. \qquad (35)$$

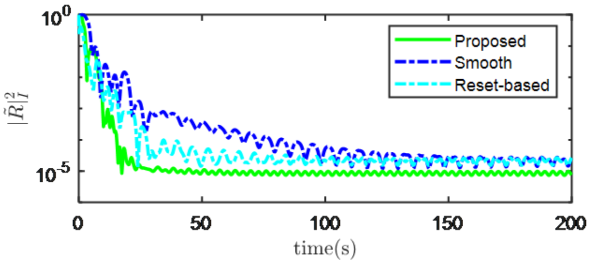
V. SIMULATIONS

Suppose that the angular velocity and linear velocity of the system are given by $\omega(t) = 0.01[-\sin(t), \cos(t), \sin(t)]^T \mathrm{rad/s}$ and $v(t) = 0.2[\cos(t), \sin(t), 0]^T \mathrm{m/s}$, respectively. The measured group velocity is corrupted by the slowly varying bias $b_\omega = 0.001\cos(t)[-2,2,1]^T \mathrm{rad/s}$ and $b_v = 0.01\cos(t)[2,-1,1]^T \mathrm{m/s}$. Assume that one landmark and three inertial vectors are measured with known vector elements in the inertial frame $p_1 = [\sqrt{2}/2, \sqrt{2}/2, 2]^T, v_1 = [1,-1,1]^T/\sqrt{3}, v_2 = [0,0,1]^T, v_3 = [1,0,0]^T$ and the corresponding gain parameters $k_1 = 1, k_2 = 1, k_3 = 3, k_4 = 1$. The other parameters are selected as $\delta_R = 0.8\Delta, k = 0.95\bar{k}, k_\beta = 0.8, k_\omega = 0.4, k_v = 0.4, q(0) = 2$. The pose of the system is initialized at $R(0) = I_3, p(0) = [0,1,4]^T \mathrm{m}$. The initial estimates are given by $\hat{R}(0) = \mathcal{R}_a(\pi, v_1)^T R(0), \hat{p}(0) = (I_3 - \hat{R}(0))bd^{-1}, \hat{b}_\omega(0) = 0, \hat{b}_v(0) = 0$. For comparison purpose, we also implement the reset-based non-decoupled hybrid pose observer proposed in [1] with $\theta = 2\pi/3, \delta = 0.2, q(0) = 0$ and the smooth pose observer.

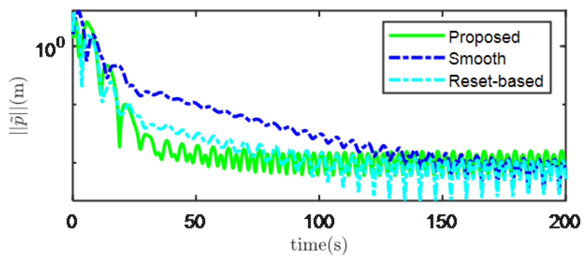
The simulation results are given in Fig.1. It can be seen that the proposed observer ensures faster convergence of the estimation errors compared to the reset-based hybrid observer and the smooth observer in both rotational and translational channel. In the steady-state phase, our observer achieves higher estimation accuracy than the others especially in the rotational channel. Also note that both the proposed observer and the reset-based observer trigger the jump maps. When the jump occurs, the proposed observer selects the other synergistic potential function and thus a different innovation term is utilized, leading to new estimation errors convergence rates, while the reset-based observer changes the state estimates directly along the decreasing direction of the potential function, with the innovation term remaining unchanged. This partly explains the faster convergence rates of the estimation errors acquired by the proposed observer.

VI. CONCLUSION

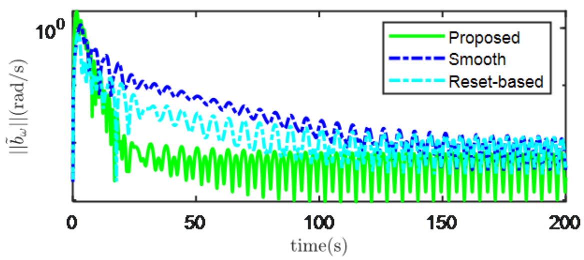
A globally asymptotically stable hybrid pose and velocity-bias observer has been proposed. We construct synergistic potential functions on SE(3) and SO(3) via angular warping



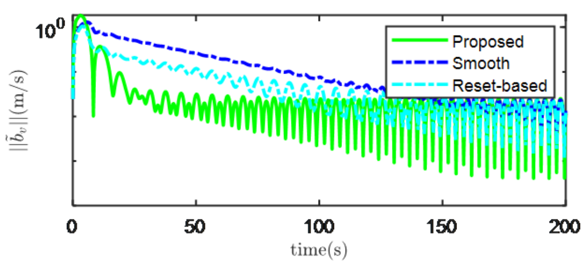
(a) Attitude estimation error



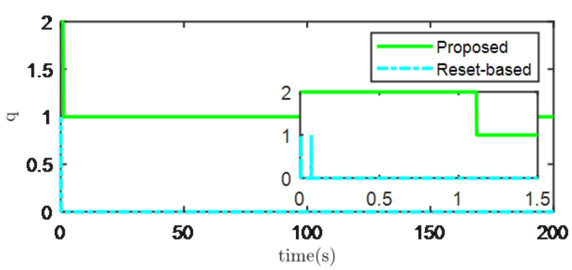
(b) Position estimation error



(c) Angular velocity-bias estimation error



(d) Linear velocity-bias estimation error



(e) Results for q

Fig. 1. Behavior of the proposed observer

and use the gradients of the formers in the innovation term. We show that under the developed switching mechanism, the undesired critical points can be avoided successfully. The proposed hybrid observer can be explicitly expressed in terms of inertial vector measurements, modified inertial vectors measurements and landmark measurements in some cases. Simulations are conducted to show the performance of the hybrid observer.

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