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Reduced Order State Observer Via Centre Manifold and Sliding-Mode Theories

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***Abstract*— This study presents a new scheme for the synthesis of reduced order nonlinear state observer. It is based on the centre manifold and the sliding mode theories. The first one is well known as being very useful for the generating of reduced order models at the neighborhood of the Hopf bifurcation point while the sliding-mode theory is widely used for the synthesis of nonlinear state observers. Hence , this study investigates the opportunity to derive reduced order state observers from both theories. This way of proceeding obeys to a different spirit regarding that of the Luenberger observer which is based on the estimating of the only states that are not available from measurements while the proposed scheme gives a method to synthesize a reduced order observer from a reduced order model the output of which is forced to be convergent to the measured system output according to the sliding-modes principle. This permits the obtaining of an estimation of the centre variables based on which the estimation of the full order system state is determined.**

I. Introduction

The importance of sate observers for controlling and monitoring of dynamical systems is well established. They are more specifically required when system states are not accessible to measurement for technological and/or cost considerations, in order to estimate the system internal states. The literature about this topic refers to numerous techniques for the synthesis of linear and nonlinear state observers [15]–[17]. However, as a state observer is a copy of the dynamical system with an output gap term injected at its entrance, it generally exhibits the same complexity features as the system to observe more specifically its dimension. Thus for efficient practical use and for high dimension systems, one searches for the synthesize of reduced order state observers. The most common used way in this perspective is the one based on the considering of the number of state variables that are available from measurements. In this case, the resulting state observer dimension is decreased by the number of measurable outputs. This is the spirit of the Luenberger state observer which was defined for linear systems [3] and then subsequently extended to nonlinear systems [4]–[6] by defining suitable linear and nonlinear coordinate transformations, respectively. In this same framework, recent studies have proposed the synthesis of convergent reduced order state observers for nonlinear dynamical systems by using the contraction analysis and convex optimization [18], [19]. The concept of invariant manifolds was also recently proposed for the same objective [7], [20]. In these studies, asymptotic estimates of reduced order unknown state vectors were obtained by rendering attractive prior selected invariant manifolds. In [21], a different use of invariant manifolds was proposed for the synthesis of reduced order state observers. It consisted in the considering of the centre manifold to determine the nonlinear transformation mapping the system to be observed to a canonical form according to the Kazantzis and Kravaris condition [22], and based on which a reduced Luenberger-like observer is determined.

The present paper deals with the synthesis of reduced order state observers by using another paradigm. It proposes to build a reduced order state observer for a given nonlinear system by using its reduced order model instead of the full order version. More accurately, this study proposes the estimation of a full order system state by using a reduced state estimator which is obtained not by considering the availability of some states from the output measure as defined within Luenberger spirit, but by using a reduced order model determined by the centre manifold theorem [1], [2]. A copy of this reduced model is exploited together with the sliding mode theory in order to synthesize a sliding-mode observer [8]. The particular point is then to define a sliding surface as the gap between the measured output and the output of the reduced observer. The dual issue which consists in the defining of reduced order controllers from centre manifold based reduced models was recently considered and dealt with in [9].

According to the centre manifold theorem, a system motion takes place on an invariant manifold that is tangent to the centre linear sub-spaces at the Hopf bifurcation point, the centre sub-spaces being spanned by the eigenvectors corresponding to the pure imaginary eigenvalues. In the other hand, the sliding-mode theory is widely used to build convergent state observers as shown in numerous previous works [8], [10]–[13]. Then, this study investigates the opportunity to use the centre manifold based reduced order models in order to efficiently design convergent reduced order sliding mode state observers. In this framework, a convergence condition is derived for the proposed reduced observers. The associated performances are assessed by considering the problem of estimating friction-induced vibrations.

This paper is organized as follows. First, the considered problem is formulated in Section II. Then, the main principles of the centre manifold method for model order reduction are recalled in Section III. After that, the centre manifold based reduced order sliding-mode state observer is described in Section IV. Its efficiency and convergence are analyzed in Section V by considering the friction-induced vibrations

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(FIV) estimation problem. Conclusion is given at the end of the paper.

II. Problem Formulation

Let,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_x(\mu)\boldsymbol{x}(t) + \boldsymbol{f}_x(\boldsymbol{x}(t), \mu) \tag{1}$$

be a dynamical system with a linear part modeled by the state matrix $\boldsymbol{A}_x \in R^{n\times n}$ and a nonlinear part $\boldsymbol{f}_x$ which is assumed to be smooth and supposed to be locally Lipchitz so that for each initial state $x(0)$ at $t = 0$ the System (1) has the unique solution $x(t, \mu)$. Otherwise $\boldsymbol{f}_x$ is considered as a polynomial vector field and the system output $y(t)$ is supposed to be given by a linear combination of the state variables that is $y(t) = \boldsymbol{C}_x\boldsymbol{x}(t)$, $\boldsymbol{C}_x$ being a row observation matrix. Furthermore, the origin $\boldsymbol{x}_e = \boldsymbol{0}$ is assumed to be the equilibrium of System (1) and is not influenced by the parameter μ.

Let,

$$\dot{\boldsymbol{x}}_r(t) = \boldsymbol{A}_r(\mu)\boldsymbol{x}_r(t) + \boldsymbol{f}_r(\boldsymbol{x}_r(t), \mu) \tag{2}$$

a reduced order model which is supposed to be an enough accurate representation of the dynamical behavior of the full order version (1) where $\boldsymbol{A}_r \in R^{r\times r}$, $\boldsymbol{f}_r$ is a reduced nonlinear vector field. The associated output $y_r(t) = \boldsymbol{C}_r\boldsymbol{x}_r(t)$ is required to be an approximation of y with a suitable accuracy. Then, the main issue the present study addresses can be formulated in a more general form which consists in the starting from the reduced order model (2) to define a reduced order state observer given as:

$$\begin{cases} \dot{\hat{\boldsymbol{x}}}_r(t) = \boldsymbol{A}_r(\mu)\hat{\boldsymbol{x}}_r(t) + \boldsymbol{f}_r(\hat{\boldsymbol{x}}_r(t), \mu) + \chi(\hat{y}_r(t) - y(t)) \\ \hat{y}_r(t) = \boldsymbol{C}_r\hat{\boldsymbol{x}}_{\boldsymbol{r}}(t), \hat{\boldsymbol{x}}_r(0) = \boldsymbol{x}_r^0 \end{cases} \tag{3}$$

such that $\hat{\boldsymbol{x}}(t) = \boldsymbol{\psi}(\hat{\boldsymbol{x}}_r(t)) \to \boldsymbol{x}(t)$ as $t \to \infty$ with $\boldsymbol{\psi}$ is a smooth coordinate transformation mapping the reduced state estimation $\hat{\boldsymbol{x}}_r(t)$ to the full order state vector estimation $\hat{\boldsymbol{x}}(t)$ which is then required to converge to the real full order system state $\boldsymbol{x}(t)$.

III. Centre Manifold

The centre manifold theory is here considered to derive the reduced order model (2). More specifically, this study focuses on a particular region near the Hopf bifurcation points $\boldsymbol{\mu}_c$ of System (1), [2]. At this point and near the equilibrium, System (1) loses its asymptotic stability property and may converge to a stationary regime of oscillation named limit cycle. It can be put into a canonical form by using a linear basis transformation $\boldsymbol{x}(t) = \boldsymbol{T}\boldsymbol{z}(t)$ where $\boldsymbol{T}$ is the matrix of the generalized eigenvectors associated to the eigenvalues of the state matrix $\boldsymbol{A}_x$ at $\boldsymbol{\mu} = \boldsymbol{\mu}_c$. This canonical form is given by:

$$\begin{cases} \dot{\boldsymbol{z}}_c(t) = \boldsymbol{A}_c(\mu_c)\boldsymbol{z}_c(t) + \boldsymbol{f}_c(\boldsymbol{z}_c(t), \boldsymbol{z}_s(t), \mu_c) \\ \dot{\boldsymbol{z}}_s(t) = \boldsymbol{A}_s(\mu_c)\boldsymbol{z}_s(t) + \boldsymbol{f}_s(\boldsymbol{z}_c(t), \boldsymbol{z}_s(t), \mu_c) \end{cases} \tag{4}$$

where the new state vector $\boldsymbol{z}(t)$ is partitioned into two state sub-vectors namely $\boldsymbol{z_c}(t) \in R^{n_c}$ and $\boldsymbol{z_s}(t) \in R^{n_s}$ such that $n_c + n_s = n$, which refer to the centre and stable manifolds characterized by the system' modes with zero real parts and strictly negative real parts, respectively. The diagonal matrix $\boldsymbol{A}$ is defined by the mentioned eigenvalues that is, $\boldsymbol{A}(\mu_c) = \begin{bmatrix} \boldsymbol{A}_c & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_s \end{bmatrix} = \mathrm{diag}(\boldsymbol{\lambda}_\mathrm{c}, \boldsymbol{\lambda}_\mathrm{s})$ where $\boldsymbol{\lambda}_\mathrm{c} = \begin{bmatrix} \lambda_1 & \dots & \lambda_{n_c} \end{bmatrix}^\mathrm{T}$ is the vector of the imaginary eigenvalues while $\boldsymbol{\lambda}_\mathrm{s} = \begin{bmatrix} \lambda_{n_c+1} & \dots & \lambda_{n_c+n_s} \end{bmatrix}^\mathrm{T}$ is the vector of the stable eigenvalues. Moreover, the polynomial vector field $\boldsymbol{f}_z = \begin{bmatrix} \boldsymbol{f}_c & \boldsymbol{f}_s \end{bmatrix}$ is also partitioned according to the two distinguished sets of the state variables with $\boldsymbol{f}_c(\boldsymbol{0}, \boldsymbol{0}, \mu_c) = \boldsymbol{0}$, $\boldsymbol{f}_s(\boldsymbol{0}, \boldsymbol{0}, \mu_c) = \boldsymbol{0}$ and the associated jacobian matrices $\boldsymbol{J}_{\boldsymbol{f}_c}$ and $\boldsymbol{J}_{\boldsymbol{f}_s}$ are null at the equilibrium point. Otherwise, the system output is also rewritten and given in the new coordinates:

$$y(t) = \begin{bmatrix} \boldsymbol{C}_c & \boldsymbol{C}_s \end{bmatrix}\begin{bmatrix} \boldsymbol{z}_c(t) \\ \boldsymbol{z}_s(t) \end{bmatrix} \tag{5}$$

Then,it is also possible to define, in some neighborhood of the Hopf bifurcation point given by $\tilde{\mu} = (1 + \epsilon)\mu_c$ with $\epsilon << 1$, the following augmented dynamics:

$$\begin{cases} \dot{\boldsymbol{z}}_c = \boldsymbol{A}_c(\tilde{\mu})\boldsymbol{z}_c(t) + \boldsymbol{f}_c(\boldsymbol{z}_c(t), \boldsymbol{z}_s(t), \tilde{\mu}) \\ \dot{\boldsymbol{z}}_s(t) = \boldsymbol{A}_s(\tilde{\mu})\boldsymbol{z}_s(t) + \boldsymbol{f}_s(\boldsymbol{z}_c, \boldsymbol{z}_s(t), \tilde{\mu}) \\ \dot{\tilde{\mu}} = 0 \end{cases} \tag{6}$$

Hence, the centre manifold theorem stated that there exist a local centre manifold within some neighborhood such that $\|\boldsymbol{z}_c(t)\| < \delta$ and small $\|\tilde{\mu}\|$, by means of which the stable variable $\boldsymbol{z}_s(t)$ can be determined by some nonlinear function $\boldsymbol{\phi}$ in the centre variables $(\boldsymbol{z}_c(t), \tilde{\mu})$ as: $\boldsymbol{z}_s(t) = \boldsymbol{\phi}(\boldsymbol{z}_c(t), \tilde{\mu})$ where $\boldsymbol{\phi}$ verifies $\boldsymbol{\phi}(\boldsymbol{0}, 0) = 0$ and the associated Jacobian $\boldsymbol{J}_{\boldsymbol{\phi}}(\boldsymbol{0}, 0)$ is a null matrix [1], [2].

Consequently, a reduced order model can be simply obtained by suppressing the stable dynamics in System (6). The resulting reduced dynamics and the corresponding output are then determined by the following equations, respectively.

$$\begin{cases} \dot{\boldsymbol{z}}_c(t) = \boldsymbol{A}_c(\tilde{\mu})\boldsymbol{z}_c(t) + \boldsymbol{f}_c(\boldsymbol{z}_c(t), \boldsymbol{\phi}(\boldsymbol{z}_c(t), \tilde{\mu}), \tilde{\mu}) \\ \dot{\tilde{\mu}} = 0 \end{cases} \tag{7}$$

$$y_r(t) = \boldsymbol{C}_c\boldsymbol{z}_c(t) \tag{8}$$

The main step is then to calculate the centre manifold $\boldsymbol{\phi}$. This is carried out from the solution of the following algebraic equation:

$$\begin{aligned} (\boldsymbol{J}_{\boldsymbol{\phi}}(\boldsymbol{z}_c(t), \tilde{\mu}))\left\{\boldsymbol{A}_c(\tilde{\mu})\boldsymbol{z}_c(t) + \boldsymbol{f}_c(\boldsymbol{z}_c(t), \boldsymbol{\phi}(\boldsymbol{z}_c(t), \tilde{\mu}), \tilde{\mu})\right\} \\ = \left\{\boldsymbol{A}_s(\tilde{\mu})\boldsymbol{\phi}(\boldsymbol{z}_c(t), \tilde{\mu}) + \boldsymbol{f}_s(\boldsymbol{z}_c, \boldsymbol{\phi}(\boldsymbol{z}_c, \tilde{\mu}), \tilde{\mu})\right\} \end{aligned} \tag{9}$$

A simple way to determine an approximation of the centre manifold $\boldsymbol{\phi}$ is to choose it as a polynomial function with a fixed degree. The associated coefficients are then identified by using equation (9). The reduced model is required to be enough accurately representative of the dynamical behaviour described by the original model, which involves that $y_r \approx y$ for a given perturbation of the equilibrium system state.

IV. Reduced Order Sliding Mode Observer

For sake of clarity, the following formulation is considered by using the $\boldsymbol{z}(t)$ state variable. The formulation in the original coordinates can be written by introducing the inverse basis transformation of $\boldsymbol{T}$. Then, the full order sliding mode observer of the full order system is defined by considering the sliding surface S as the gap between the system and observer outputs, namely :

$$\begin{cases} \dot{\hat{\boldsymbol{z}}}(t) = \boldsymbol{A}(\tilde{\boldsymbol{\mu}})\hat{\boldsymbol{z}}(t) + \boldsymbol{f}(\hat{\boldsymbol{z}}(t), \tilde{\boldsymbol{\mu}}) - \boldsymbol{\Delta}\text{Sign}\left(\hat{y}(t) - y(t)\right) \\ \hat{\boldsymbol{z}}(0) = \hat{\boldsymbol{z}}_0 \\ \hat{y}(t) = \boldsymbol{C}\hat{\boldsymbol{z}}(t) \end{cases} \tag{10}$$

where $\boldsymbol{\Delta} \in R^{(n\times 1)}$ is a gain vector weighting the sign function of the difference between the output of the system and that of the observer given by:

$$\text{Sign}\left(\hat{y} - y\right) = \begin{cases} 1 & \text{If } \hat{y} - y > 0 \\ -1 & \text{If } \hat{y} - y < 0 \end{cases} \tag{11}$$

As previously mentioned, the main idea that the present study develops is to determine a reduced order sliding-mode state observer for the system (6) by using the corresponding reduced order representation given by the centre manifold formulation (7). It is also worth-mentioned that the reduced observer is forced with a function in the gap between the output of the reduced order observer and that of the full order model, which is supposed to represent the measured output of the real system. This idea is formulated by the following proposition.

Proposition 1: Under the asymptotic stability of the $\boldsymbol{z}_s$-dynamics according to the centre manifold, the reduced order dynamical system given by

$$\begin{cases} \dot{\hat{\boldsymbol{z}}}_c(t) = \boldsymbol{A}_c(\tilde{\mu})\hat{\boldsymbol{z}}_c(t) + \boldsymbol{f}_c(\hat{\boldsymbol{z}}_c(t), \boldsymbol{\phi}(\hat{\boldsymbol{z}}_c(t), \tilde{\mu}), \tilde{\mu}) - \\ \boldsymbol{\Delta}_c\text{sign}(\hat{y}_r(t) - y(t)), \hat{\boldsymbol{z}}_c(0) = \hat{\boldsymbol{z}}_{c_0} \\ \dot{\tilde{\mu}} = 0 \\ \hat{y}_r(t) = \boldsymbol{C}_c(t)\hat{\boldsymbol{z}}_c(t) \end{cases} \tag{12}$$

is a convergent state observer of the full order system (6) that is $\hat{\boldsymbol{z}}_c(t) \to \boldsymbol{z}_c(t)$ as $t \to \infty$ and $\hat{\boldsymbol{z}}_s(t) = \boldsymbol{\phi}(\hat{\boldsymbol{z}}_c(t), \tilde{\boldsymbol{\mu}}) \to \boldsymbol{z}_s(t) = \boldsymbol{\phi}(\boldsymbol{z}_c(t), \tilde{\boldsymbol{\mu}})$, where $\boldsymbol{\Delta}_c \in R^{(n_c\times 1)}$ is a gain vector and $\hat{\boldsymbol{y}}_r(t)$ is the output of the reduced order observer.

Proof of Proposition 1: Let

$$S_{\text{or}} = \hat{y}_r - y = \boldsymbol{C}_c\hat{\boldsymbol{z}}_c - \boldsymbol{C}\boldsymbol{z} \tag{13}$$

be the sliding surface based on which we would like to make convergent the dynamics of the reduced order observer, where $\hat{y}_r$ is the output of the reduced observer. The latter is required to well approximate the measured system output which is modeled by $\boldsymbol{C}\boldsymbol{z} = \boldsymbol{C}_c\boldsymbol{z}_c + \boldsymbol{C}_s\boldsymbol{z}_s$.

The method is then used for the analysis of the attractiveness of the defined sliding surface by considering the Lyapunov function candidate $V_{\text{ob}} = \frac{1}{2}S_{\text{or}}^{\text{T}}S_{\text{or}}$. Then, the surface $S_{\text{or}} = 0$ will be attractive if the time derivative of the Lyapunov function is negative definite that is:

$$S_{\text{or}}^T\dot{S}_{\text{or}} < 0 \tag{14}$$

If so, then $S_{\text{or}} = \boldsymbol{C}_c\left(\hat{\boldsymbol{z}}_c - \boldsymbol{z}_c\right) - \boldsymbol{C}_s\boldsymbol{z}_s = 0$ which will involve the convergence of $\hat{\boldsymbol{z}}_c$ to $\boldsymbol{z}_c$ starting from the considered asymptotic stability of $\boldsymbol{z}_s$ in accordance to the centre manifold assumptions.

The time derivative of the considered Lyapunov function can be expressed as follows:

$$\dot{V}_{\text{ob}} = \hat{\boldsymbol{z}}_c^{\text{T}}\boldsymbol{C}_c^{\text{T}}\boldsymbol{C}_c\dot{\hat{\boldsymbol{z}}}_c - \hat{\boldsymbol{z}}_c^{\text{T}}\boldsymbol{C}_c^{\text{T}}\boldsymbol{C}\dot{\boldsymbol{z}} - \boldsymbol{z}^{\text{T}}\boldsymbol{C}^{\text{T}}\boldsymbol{C}_c\dot{\hat{\boldsymbol{z}}}_c + \boldsymbol{z}^{\text{T}}\boldsymbol{C}^{\text{T}}\boldsymbol{C}\dot{\boldsymbol{z}} \tag{15}$$

After the substitution of $\dot{\boldsymbol{z}}$ and $\dot{\hat{\boldsymbol{z}}}_c$ by their expressions in (15) and by considering the asymptotically stable behaviour of the $\boldsymbol{z}_s$ dynamics, the previous Lyapunov function derivative becomes as follows:

$$\dot{V}_{\text{ob}} = \tilde{\boldsymbol{z}_c}^{\text{T}}\boldsymbol{C}_c^{\text{T}}\boldsymbol{C}_c\dot{\tilde{\boldsymbol{z}}}_c \tag{16}$$

where $\tilde{\boldsymbol{z}_c} = \hat{\boldsymbol{z}_c} - \boldsymbol{z}_c$ is the estimation error the dynamics of which is given by:

$$\dot{\tilde{\boldsymbol{z}}}_c = \boldsymbol{A}_c(\tilde{\boldsymbol{\mu}})\tilde{\boldsymbol{z}}_c + \tilde{\boldsymbol{f}}_c - \boldsymbol{\Delta}_c\text{Sign}\left(S_{\text{or}}\right) \tag{17}$$

with $\tilde{\boldsymbol{f}}_c = \boldsymbol{f}_c\left(\hat{\boldsymbol{z}}_c, \boldsymbol{\phi}(\hat{\boldsymbol{z}}_c, \tilde{\boldsymbol{\mu}})\right) - \boldsymbol{f}_c\left(\boldsymbol{z}_c, \boldsymbol{\phi}(\boldsymbol{z}_c, \tilde{\boldsymbol{\mu}})\right)$.

Thus it follows that $\hat{z}_c(t)$ is guaranteed to be convergent to $\boldsymbol{z}_c$ for $t \to \infty$ by fixing the matrix gain $\boldsymbol{\Delta}_c$ such that:

$$\begin{aligned} &\boldsymbol{\Delta}_c\text{sign}\left(S_{or}\right) > \\ &\boldsymbol{A}_c(\tilde{\mu})\tilde{\boldsymbol{z}}_c + \boldsymbol{f}_c\left(\hat{\boldsymbol{z}}_c, \boldsymbol{\phi}(\hat{\boldsymbol{z}}_c, \tilde{\mu}\right) - \boldsymbol{f}_c\left(\boldsymbol{z}_c, \boldsymbol{\phi}(\boldsymbol{z}_c, \tilde{\mu})\right) \end{aligned} \tag{18}$$

which completes the proof of Proposition 1.

It worth-mentioned that in addition to the fact that the defined state observer is of reduced dimension $n_c < n$, its dynamics is further reduced by the number of system outputs during the sliding step to reach the sliding surface $S_{\text{or}} = 0$. In this study, only the single output case is considered so the dynamics of the state observer is reduced to $n_c - 1$.

Otherwise, the error dynamics of the state observer can be obtained by using the control equivalent method as described in [8] and by considering the asymptotic stability of the $\boldsymbol{z}_s$-dynamics, as follows:

$$\begin{cases} \dot{\tilde{\boldsymbol{z}}}_c = \left(\boldsymbol{I} - \boldsymbol{\Delta}_c\left(\boldsymbol{C}_c\boldsymbol{\Delta}_c\right)^{-1}\boldsymbol{C}_c\right)\tilde{\boldsymbol{f}}_c \\ \boldsymbol{C}_c\tilde{\boldsymbol{z}}_c = \boldsymbol{C}_s\boldsymbol{z}_s = 0 \end{cases} \tag{19}$$

V. Application to Estimating Friction-Induced Vibration

In order to analyse the feasibility and the efficiency of the proposed reduced order sliding-mode observer, it is considered in the sequel for the estimation of friction-induced vibrations in a simplified drum brake defined by Hultèn in [14]. The system is shown in FIG.1. It consists of a mass which is assumed in a permanent contact with a band supposed to be in move with a constant velocity. The contact between the mass and the moving band is modeled by springs with linear and nonlinear stiffness's. The relative velocity between the band and the velocities $\dot{X}_1$ and $\dot{X}_2$ is assumed positive which makes the direction of the friction force constant. Its tangential component is supposed to be proportional to the normal force according to the Coulomb law $F_t = \mu F_n$, μ being the friction coefficient assumed to be constant.

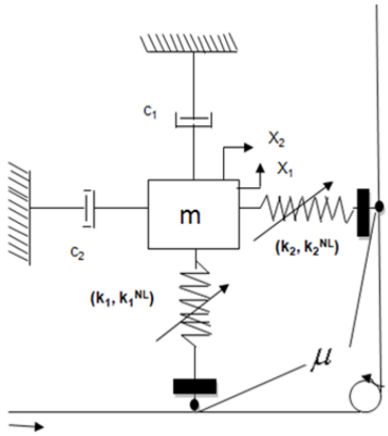

Fig. 1: Hultèn System

The state space representation like-(1) of the Hultèn system is obtained by considering the state vector

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} X_1 \\ \dot{X}_1 \\ X_2 \\ \dot{X}_2 \end{bmatrix}$$

which yields:

$$\boldsymbol{A}_x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -w_1^2 & -\eta_1 w_1 & +\mu w_2^2 & 0 \\ 0 & 0 & 0 & 1 \\ -\mu w_1^2 & 0 & -w_2^2 & -\eta_2 w_2 \end{bmatrix} \quad (20)$$

$$\boldsymbol{f}(\boldsymbol{x}, \mu) = \begin{bmatrix} 0 \\ -\psi_1^{\mathrm{NL}} x_1^3 + \mu \psi_2^{\mathrm{NL}} x_3^3 \\ 0 \\ -\mu \psi_1^{\mathrm{NL}} x_1^3 - \psi_2^{\mathrm{NL}} x_3^3 \end{bmatrix} \quad (21)$$

where $w_i = \sqrt{k_i/m}$ are the natural pulsations, $\eta_i = c_i/\sqrt{mk_i}$ are the relative damping and $\psi_i^{\mathrm{NL}} = k_i^{\mathrm{NL}}$, for $i = 1, 2$.
Let $y = x_1$ be the measurable output.
Otherwise, for numerical simulations, all magnitudes are given in SI by: $w_1 = 2\pi \times 100$ rad/s, $w_2 = 2\pi \times 75$ rad/s $\eta_1 = \eta_2 = 0.02$, $\psi_1^{\mathrm{NL}} = w_1^2$, $\psi_2^{\mathrm{NL}} = 0$, $m = 1$ Kg.
It can be verified that $(\boldsymbol{x}_e, u_e) = (\boldsymbol{0}, 0)$ is the equilibrium of System (21). Then the next step is to determine the Hopf bifurcation point μ_c at which the linear basis transformation $\boldsymbol{T}$ putting System (21) into the canonical form (4), will be calculated. As the equilibrium point is not influenced by the μ parameter, the Hopf bifurcation point μ_c can be determined by carrying out a parametric stability analysis which consists in the calculation of the eigenvalues of the linearized system for a set of values of the friction coefficient μ. Hence, the Hopf bifurcation point μ_c is obtained when the following conditions are fulfilled [2].The obtained value is $\mu_c = 0.2893$.

$$\begin{cases} \text{Real}\left(\boldsymbol{\lambda}_{\text{centre}}(\boldsymbol{A}(\mu))\right)|_{\mu=\mu_c} = 0 \\ \text{Real}\left(\boldsymbol{\lambda}_{\text{no-centre}}(\boldsymbol{A}(\mu))\right)|_{\mu=\mu_c} \neq 0 \\ \frac{d\boldsymbol{\lambda}(\boldsymbol{A}(\mu))}{d\mu}|_{\mu=\mu_c} \neq 0 \end{cases} \quad (22)$$

A. Centre Manifold based Reduced Model of the Hultèn System

The centre manifold defined by the nonlinear coordinate transformation mapping the stable manifold to the centre manifold according to the relation $\boldsymbol{z}_s = \boldsymbol{\phi}(\boldsymbol{z}_c, \tilde{\mu})$ is approximated by considering a third order polynomial form for ϕ, then by solving the corresponding algebraic equation (9). The reduced order model is finally generated by replacing the stable manifold by the centre manifold in the canonical form associated to the system (21). The reduced model is expected to exhibit the same dynamical behaviour as the original one in some neighborhood of the Hopf bifurcation point. Two limit cycles (the velocity $\dot{X}_1$ against the displacement X_1) are predicted from the reduced order model and plotted in FIG. 2 together with the ones determined from the original full order model. They correspond to two different values of the friction coefficient which are taken at different distances ϵ from the Hopf bifurcation point such that $\mu = (1+\epsilon)\mu_c$. The

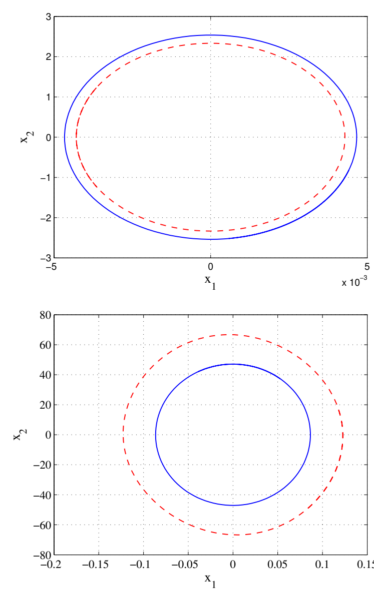

Fig. 2: The limit cycle (x_1, x_2) in the phase plan for different value of the friction coefficient $\tilde{\mu} = (1+\epsilon)\mu_c$. (a):$\epsilon = 10^{-4}$, (b):$\epsilon = 10^{-2}$. Solid line: original model, dashed line: reduced model.

reduced order model presents a similar oscillatory dynamical behaviour as the original model but, as expected, with an accuracy depending on the distance ϵ of the considered value from the Hopf bifurcation point. Indeed, the relative errors between the amplitude of predicted oscillations given by both reduced and original models increase while going far from the Hopf bifurcation point. It passes from 8 per cent for $\epsilon = 10^{-4}$ to almost 30 per cent for $\epsilon = 10^{-2}$. Augmenting the order of the center manifold can enhance the accuracy of the reduced order model. This point is not considered in this study since the main aim is to determine a state observer having some robustness with respect to model inaccuracy and/or parameter dispersion.

B. Reduced order sliding mode observer

The obtained reduced model corresponding to $\tilde{\mu} = (1 + \epsilon)\mu_c$ with $\epsilon = 10^{-4}$ is now used for the synthesis of a reduced order state observer (12) for the full order Hultèn system.

The corresponding estimations are plotted in FIG. 3 and the associated estimation errors are shown in FIG. 4. It can be observed that the estimated states exhibit the same oscillatory behaviours as the system real states. The involved estimation errors present transients with oscillations the amplitudes of which are decaying to reach the origin. The carried simulations point out the stability of the dynamics of the defined reduced order state observer. This stability is in fact strongly influenced by the gain $\mathbf{\Delta}_c$ of the observer as shown by the convergence condition (18) and the error dynamics (19). Furthermore, this gain tunes the speed of the convergence of the state observer and thus the damping property of the error dynamics of the state observer. It also contributes to define the robustness level of the observer.

C. Robustness Analysis of the reduced state observer

First of all, it worth-mentioned that the obtained reduced order observer was calculated by using a reduced order model which is basically an approximation of the full order system. So, the previous reduced observer already presented a robustness with respect to the gap between the used reduced model (the one calculated with $\tilde{\mu} = (1 + \epsilon), \epsilon = 10^{-4}$) and the full order model. In the sequel, the main aim is to further analyze via numerical simulations the robustness of the reduced state observer. For this goal, the performance of the latter is assessed when the friction coefficient is further from the Hopf bifurcation point, which involves a higher gap between the reduced and full order models. The reduced state observer parameters are kept unchanged while the system output is submitted to variations induced by changes in the friction coefficient value. Otherwise, the state observer gain is taken higher than the previously considered values in order to start from a reduced order state observer with faster convergence properties. So a higher observer gain is considered for this analysis.

Based on the simulation results plotted in FIG. 5, it can be observed that the state observer dynamics is influenced by the considered dispersion. The estimation error grows with ϵ. More robustness involves higher values for the gain $\mathbf{\Delta}_c$ of the state observer.

VI. Conclusion

A new paradigm for reduced nonlinear state observer design was proposed in this study. This paradigm proposes the construction of a reduced order state observer of a nonlinear dynamical system by using its reduced order model derived from the centre manifold theorem. The sliding-mode approach is then applied in order to define the reduced order state observer. A convergence condition was stated for the defined reduced order sliding mode observer and verified through numerical simulations within the estimation of friction-induced vibration framework. Optimal characterization of the state observer parameters require more investigations. More particularly, the determination of the observer gain ensuring optimal convergence properties with optimal robustness characteristics is a work in progress.

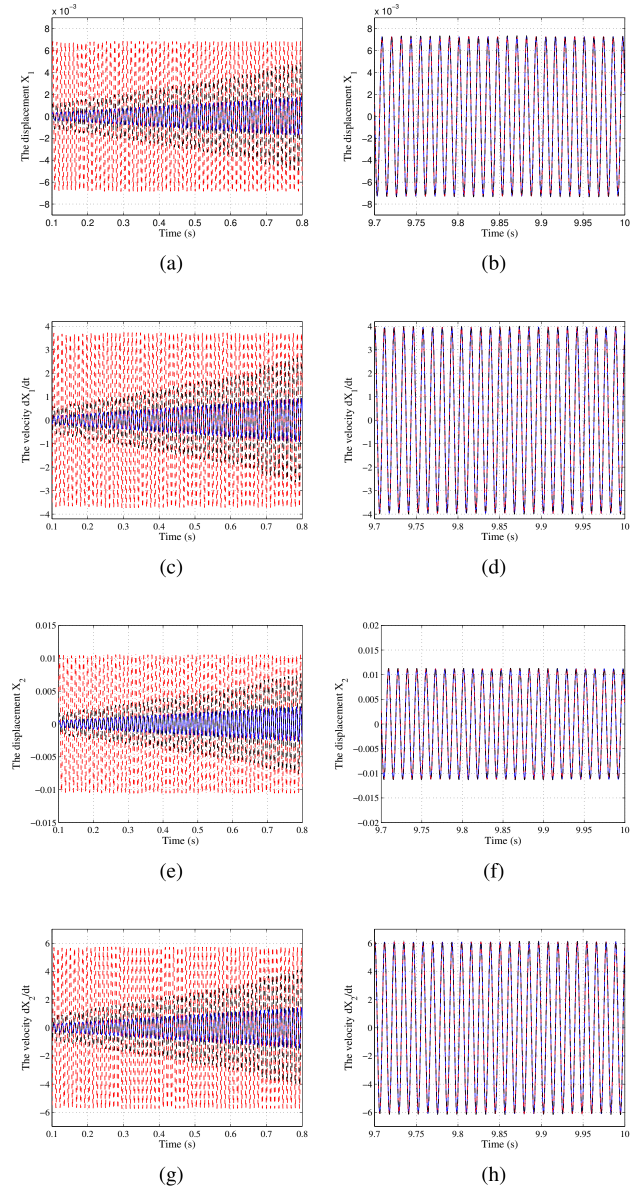

Fig. 3: The system states corresponding to $\tilde{\mu} = (1 + \epsilon)\mu_c$. with $\epsilon = 10^{-4}$ and to $\mathbf{\Delta}_c^1 = [2, 100]$ and $\mathbf{\Delta}_c^2 = [5, 500]$; Solid blue line: real state, dashed red line: estimated states with $\mathbf{\Delta}_c^1$, dashed-dot black line: estimated states with $\mathbf{\Delta}_c^2$. (a,c,e,g): Zoom on the transient of the estimations; (b,d,f,h): zoom on reached stationary behaviour.

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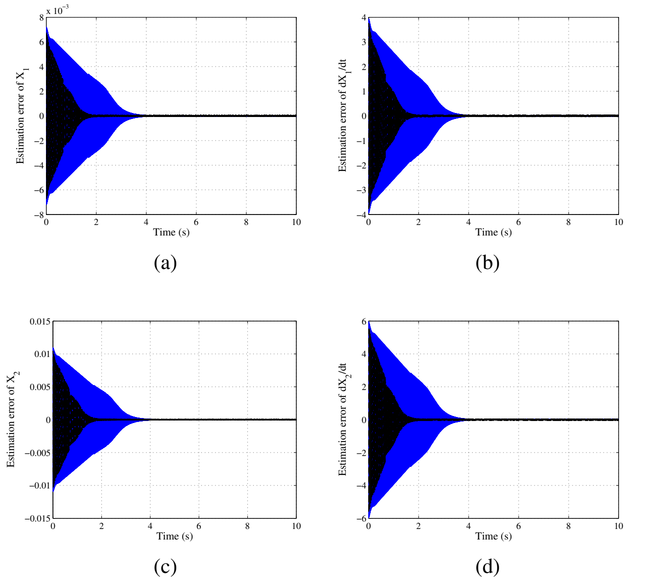

Fig. 4: The time evolution of the estimation errors $(\tilde{x}_i - x_i)_{i=1,\dots,4}$ corresponding to the observer gains. Solid blue line: $\mathbf{\Delta}_c^1 = [2, 100]$, dashed black line: $\mathbf{\Delta}_c^2 = [5, 500]$

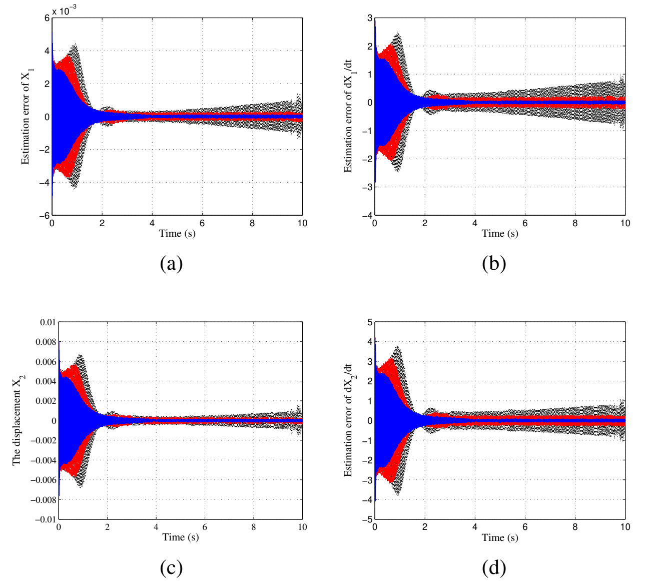

Fig. 5: The time evolution of the estimation errors $(\tilde{x}_i - x_i)_{i=1,\dots,4}$ of the reduced state observer for different values of the friction coefficient ($\mu = (1+\epsilon)\mu_c$) near the Hopf bifurcation point corresponding with $\mathbf{\Delta}_c = [50, 2000]$. Solid blue line:$\epsilon = 1-4$; Dot-dashed red line: $\epsilon = 5e-3$; Dot black line: $\epsilon = 1e-3$

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