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Stability test for some classes of linear time-delay systems: A Legendre polynomial approximation-based approach*

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***Abstract*— In this paper, we present necessary and sufficient stability tests for two classes of linear delay systems: neutral-type linear time-delay systems and linear time-delay systems with distributed delays. In both cases, they are based on the Lyapunov-Krasovskii functionals with prescribed derivative expressed in terms of the delay Lyapunov matrix. The stability tests amount to verify the semi-positivity of a matrix resulting from the substitution of Legendre polynomial approximation of the functional argument. As illustrated in two examples, the low matrix dimension for establishing sufficiency makes the test numerically efficient.**

I. INTRODUCTION

In recent research, the Lyapunov-Krasovskii functionals with prescribed derivatives introduced in [1] have become crucial in achieving necessary and sufficient stability conditions for time-delay systems such as those presented in [2], [3], [4] to mention a few. These functionals depend on the delay Lyapunov matrix, which is the solution of three equations, called algebraic, dynamic, and symmetry [1].

The above-mentioned necessary and sufficient stability conditions are essentially based on approximation theory, where the functional argument is approximated via the fundamental matrix [5], [3], piece-wise constant or linear approximations [4], [6], [7] or the Legendre polynomials approximation [8].

For the case of fundamental matrix-based approximations, a stability criterion was delivered in terms of point-wise values of the delay Lyapunov matrix [2], [3] with overlarge orders of approximation. However, this dimensional issue was overcome with the help of polynomial approximations, either piece-wise as in [4], [7] or the complete segment of the functional argument in [8]. In these cases, the stability criteria are given in terms of integrals of the delay Lyapunov matrix, but the orders of approximation are now reduced. In particular, the resulting stability criterion presented in [8] considerably reduced the dimension of the test due to the convergence properties of Legendre orthogonal polynomials approximation adapted to the sets of functions under consideration [9]. It is worthy of mention that a recursive method to compute the integral matrices was presented in each case.

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This work aims to extend to neutral-type linear and distributed delays systems the results for retarded type systems presented in [8], [10]. We decided to use the first projections onto Legendre polynomials of the argument of the Lyapunov functional in order to benefit from the following underlying properties: rapid convergence for smooth arguments; orthogonality with respect to the Lebesgue measure; and second-order recurrence satisfied by the coefficients. The necessary and sufficient conditions stem from this quadratic approximation of the functional and take the mathematical form of a semi-positiveness (neutral-type systems) or positivity (distributed time-delay systems) test.

This work is organized as follows. In Section II, the Legendre projection for the functional argument and an estimate of the convergence rate toward a special set of functions are introduced. Some preliminaries, concepts, and results on the Lyapunov-Krasovskii framework for neutral-type systems and distributed time-delay systems are reminded in Section III and Section IV, respectively. Then, for each class of time-delay system, necessary and sufficient stability conditions via Legendre projections are presented. Finally, the results are validated through two examples in Section V, and some conclusions are given in Section VI.

Notation: The spaces of $\mathbb{R}^n$-valued piece-wise continuously differentiable, smooth functions on $[-h,0]$ and continuously differentiable functions are considered and denoted by $PC^1\big([-h,0],\mathbb{R}^n\big)$, $C_\infty\big([-h,0],\mathbb{R}^n\big)$ and $\mathcal{C}([-h,0],\mathbb{R}^n)$, respectively. They are equipped with the uniform norm

$$\|\varphi\|_h = \sup_{\theta\in[-h,0]} \|\varphi(\theta)\|,$$

where $\|\cdot\|$ stands for the Euclidean norm for vectors and the spectral norm for matrices. $\Re(s)$ denotes the real part of a complex value s; $\lambda_{\min}(W)$ is the smallest eigenvalue of a matrix W; notation $k=\overline{n_1,n_2}$, where $n_1,\ n_2\in\mathbb{Z}$, $n_1<n_2$, means that k is an integer between n_1 and n_2; I_n stands for the $n\times n$ identity matrix; $\lceil\cdot\rceil$ denotes the ceiling function. For a symmetric matrix Λ, the notation $\Lambda>0$ ($\Lambda\geq 0$) means that Λ is a positive definite (positive semidefinite) matrix; $M^\top$ denotes the transpose $M\in\mathbb{R}^{m\times m}$. $\mathcal{W}(z)$ denotes the Lambert function given by $\mathcal{W}: z\to y,\ z\in\mathbb{R}_+,\ y\in\mathbb{R}_+$, which is uniquely defined by the relation $y\mathrm{e}^y=z$.

II. LEGENDRE POLYNOMIAL APPROXIMATION

In this section, we introduce a Legendre polynomial approximation-based expression for functions $\varphi\in PC^1\big([-h,0],\mathbb{R}^n\big)$, as well as an estimate of its super-geometric convergence rate toward a special set of functions. It is worth mentioning that quantifying this convergence rate is crucial to obtain sufficient stability conditions.

Let us consider a function $\varphi \in PC^1([-h,0],\mathbb{R}^n)$ and a N-order Legendre polynomial approximation of this function given by

$$\begin{aligned}\varphi(\theta) &= \varphi_N(\theta) + \tilde{\varphi}_N(\theta),\\ \varphi_N(\theta) &= \ell_N^\top(\theta)\Phi_N,\ \theta \in [-h,0].\end{aligned} \tag{1}$$

Here,

$$\Phi_N = \left(\int_{-h}^{0} \ell_N(s)\ell_N^\top(s)\mathrm{d}s\right)^{-1} \int_{-h}^{0} \ell_N(s)\varphi(s)\mathrm{d}s \ \in \mathbb{R}^{Nn},$$

$$\ell_N(\theta) = [l_0(\theta)I_n\ l_1(\theta)I_n\ \cdots\ l_{N-1}(\theta)I_n]^\top,\ \theta \in [-h,0],$$

$\{l_k\}_{k\in\overline{0,N-1}}$ is the set of Legendre polynomials of order lower than N and, vector $\Phi_N \in \mathbb{R}^{nN}$ collects the projection coefficients associated to the Legendre polynomials.

Now, introduce the following compact set in the space of continuously differentiable functions [11]

$$\begin{aligned}\mathcal{S}_r = \Big\{\varphi \in \mathcal{C}_\infty([-h,0],\mathbb{R}^{n_x}) \mid \|\varphi\|_h = \|\varphi(0)\| = 1,\\ \|\varphi^{(k)}\|_h \leqslant r^k,\ \forall\, k \in \mathbb{N}\Big\},\end{aligned} \tag{2}$$

where the spectral radius r will be provided in the sequel depending on the class of delay system.

Next, a crucial lemma on the Legendre convergence rate is presented.

Lemma 1: Consider $\varphi \in \mathcal{S}_r$. The Legendre approximation error $\tilde{\varphi}_N$ satisfies the following inequality

$$\|\tilde{\varphi}_N\|_h \leqslant \frac{5\max\left(1, (\frac{2hr}{3})^N\right)}{N!}, \tag{3}$$

Proof: For $N = \{1,2\}$, we have roughly

$$\begin{aligned}\|\tilde{\varphi}_1\|_h &= \left\|\varphi - \tfrac{1}{h}\int_{-h}^{0}\varphi(s)\mathrm{d}s\right\|_h \leqslant 2\|\varphi\|_h = 2,\\ \|\tilde{\varphi}_2\|_h &= \left\|\varphi - \tfrac{1}{h}\int_{-h}^{0}\varphi(s)\mathrm{d}s - \tfrac{3}{h}\int_{-h}^{0}\tfrac{2s+h}{h}\varphi(s)\mathrm{d}s\right\|_h \leqslant 5\|\varphi\|_h = 5.\end{aligned} \tag{4}$$

For any $N \geqslant 3$ and $2 \leq k \leq N-1$, according to [10, Lemma 2.2], an upper bound of the Legendre approximation error is given by

$$\|\tilde{\varphi}_N\|_h \leqslant \frac{(hr)^{k+1}}{2^k(k-1)(N-\frac{3}{2})\dots(N-k+\frac{1}{2})}\|\varphi\|_h. \tag{5}$$

Here, k is introduced in the set $\mathcal{S}_r$ and, with $k = N-1$, we have that

$$\begin{aligned}\|\tilde{\varphi}_N\| &\leqslant \frac{2(\frac{hr}{2})^N}{N!}\frac{N(N-1)\dots 2}{(N-2)(N-\frac{3}{2})\dots(\frac{3}{2})}\|\varphi\|_h\\ &\leqslant \frac{2(\frac{hr}{2})^N 3(\frac{4}{3})^{N-1}}{N!}\|\varphi\|_h = \frac{\frac{9}{2}(\frac{2hr}{3})^N}{N!}.\end{aligned} \tag{6}$$

Taking the worst-case scenario between (4) and (6) completes the proof. ■

III. Neutral-type linear time-delay systems

Consider the neutral-type time-delay system

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - Dx(t-h)] = A_0x(t) + A_1x(t-h), \quad t \geq 0, \tag{7}$$

where $h > 0$, A_0, A_1, and D are given real $n \times n$ matrices. The solution $x(t) = x(t,\varphi)$, $\varphi \in PC^1([-h,0],\mathbb{R}^n)$, is a piece-wise continuous function that satisfies system (7) almost everywhere for $t \geqslant 0$, and the difference $x(t) - Dx(t-h)$ is continuous for $t \geqslant 0$, except for possibly a countable number of points. The restriction of the solution $x(t,\varphi)$ to the interval $[t-h,t]$, $t \geqslant 0$, is denoted by

$$x_t(\varphi) : \theta \mapsto x(t+\theta,\varphi),\ \theta \in [-h,0].$$

Definition 1: System (7) is exponentially stable if there exist $\gamma > 0$ and $\sigma > 0$ such that for any initial function $\varphi \in PC^1([-h,0],\mathbb{R}^n)$,

$$\|x(t,\varphi)\| \leqslant \gamma e^{-\sigma t}\|\varphi\|_h,\ t \geqslant 0.$$

The following assumption is essential to ensure the exponential stability of system (7).

Assumption 1: The matrix D is a given $n\times n$ Schur stable matrix, i.e. $|\lambda_j(D)| < 1$, $j = \overline{1,n}$, where $\lambda_j(D)$ are the eigenvalues of D.

Lemma 2: [1]. If the matrix D is Schur stable, then there exist $\rho \in (0,1)$ and $d \geqslant 1$ such that for any integer $k \geq 0$,

$$\|D^k\| \leqslant d\rho^k.$$

A. Lyapunov functionals and matrices

The main results in this work are obtained via functionals with prescribed derivative expressed in terms of the delay Lyapunov matrix, whose definition is introduced next.

Definition 2: Let $W \in \mathbb{R}^{n\times n}$ be a positive definite matrix. The delay Lyapunov matrix $U : [-h,h] \to \mathbb{R}^{n\times n}$ is a continuous matrix function, which satisfies the following properties.

1) Dynamic property

$$U'(\tau) - U'(\tau-h)D = U(\tau)A_0 + U(\tau-h)A_1,\ \tau \in (0,h).$$

2) Symmetry property

$$U^\top(\tau) = U(-\tau),\ \tau \in [-h,h].$$

3) Algebraic property

$$P - D^\top PD = -W,\ P = \lim_{\tau\to 0,\tau>0}\left(\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} - \frac{\mathrm{d}U(-\tau)}{\mathrm{d}\tau}\right).$$

In order to guarantee the existence and uniqueness of the delay Lyapunov matrix U, the following result is introduced.

Theorem 1: [1] System (7) admits a unique Lyapunov matrix if and only if the system satisfies the Lyapunov condition, i.e., if there exists $\varepsilon > 0$ such that any two points s_1 and s_2 of the spectrum of system (7) satisfy $|s_1 + s_2| > \varepsilon$.

Under Assumption 1 and Definition 2, we introduce the following functional with prescribed derivative:

$$v_0(\varphi) = (\varphi(0) - D\varphi(-h))^\top U(0)(\varphi(0) - D\varphi(-h)) + \sum_{j=1}^{6} I_j, \tag{8}$$

where

$$I_1 = 2(\varphi(0) - D\varphi(-h))^\top \int_{-h}^{0} U^\top(h+\theta)A_1\varphi(\theta)\mathrm{d}\theta$$

$$I_2 = -2(\varphi(0) - D\varphi(-h))^\top \int_{-h}^{0} U'^\top(h+\theta)D\varphi(\theta)\mathrm{d}\theta,$$

$$I_3 = \int_{-h}^{0}\int_{-h}^{0} \varphi^\top(\theta_1)A_1^\top U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1,$$

$$I_4 = 2\int_{-h}^{0}\int_{-h}^{0} \varphi^\top(\theta_1)A_1^\top U'(\theta_1-\theta_2)D\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1$$

$$I_5 = -\int_{-h}^{0}\int_{-h}^{\theta_1} \varphi^\top(\theta_1)D^\top U''(\theta_1-\theta_2)D\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 - \int_{-h}^{0}\int_{\theta_1}^{0} \varphi^\top(\theta_1)D^\top U''(\theta_1-\theta_2)D\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1,$$

$$I_6 = -\int_{-h}^{0} \varphi^\top(\theta)D^\top PD\varphi(\theta)\mathrm{d}\theta,$$

whose time derivative along the solution of system (7) satisfies

$$\frac{\mathrm{d}v_0(x_t)}{\mathrm{d}t} = -x^\top(t)Wx(t), \quad t \geq 0. \tag{9}$$

B. Necessary stability conditions

The Legendre orthogonal projections of the functional argument $\varphi \in PC^1\big([-h,0],\mathbb{R}^n\big)$ is substituted into functional (8), resulting in the functional $v_0(\varphi_N)$, which serves as a functional approximation of (8). Observe that the approximation of each summand yields

$$\begin{aligned} v_0(\varphi_N) = &(\varphi(0) - D\varphi(-h))^\top J_{0N}(\varphi(0) - D\varphi(-h)) \\ &+ 2(\varphi(0) - D\varphi(-h))^\top (J_{1N} - J_{2N})\Phi_N \\ &+ \Phi_N^\top (J_{3N} + 2J_{4N} - J_{5N} - J_{6N})\Phi_N, \end{aligned} \tag{10}$$

where

$$J_{0N} = U(0),$$

$$J_{1N} = \int_{-h}^{0} U^\top(h+\theta)A_1\ell_N^\top(\theta)\mathrm{d}\theta,$$

$$J_{2N} = \int_{-h}^{0} U'^\top(h+\theta)D\ell_N^\top(\theta)\mathrm{d}\theta$$

$$J_{3N} = \int_{-h}^{0}\int_{-h}^{0} \ell_N(\theta_1)A_1^\top U(\theta_1-\theta_2)A_1\ell_N^\top(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1,$$

$$J_{4N} = \int_{-h}^{0}\int_{-h}^{0} \ell_N(\theta_1)A_1^\top U'(\theta_1-\theta_2)D\ell_N^\top(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1,$$

$$J_{5N} = \int_{-h}^{0}\left(\int_{-h}^{\theta_1} \ell_N(\theta_1)D^\top U''(\theta_1-\theta_2)D\ell_N^\top(\theta_2)\mathrm{d}\theta_2 + \int_{\theta_1}^{0} \ell_N(\theta_1)D^\top U''(\theta_1-\theta_2)D\ell_N^\top(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1,$$

$$J_{6N} = \int_{-h}^{0} \ell_N(\theta)D^\top PD\ell_N^\top(\theta)\mathrm{d}\theta.$$

The above expression can be rewritten in quadratic form as follows

$$v_0(\varphi_N) = p_N^\top \mathbf{P}_N p_N, \tag{11}$$

where $p_N = \begin{bmatrix}\varphi(0) - D\varphi(-h)\\ \Phi_N\end{bmatrix} \in \mathbb{R}^{Nn}$,

$$\mathbf{P}_N = \begin{bmatrix} J_{0N} & J_{1N} + J_{2N} \\ [J_{1N} + J_{2N}]^\top & J_{3N} + 2J_{4N} - J_{5N} - J_{6N}\end{bmatrix},$$

Necessary stability conditions based on the functional approximation (11) follow immediately:

Theorem 2: If system (7) is exponentially stable, then the matrix $\mathbf{P}_N \geq 0$, for all integer $N \geqslant 1$.

Proof: The proof follows from the fact that, for any $\varphi \in PC^1\big([-h,0],\mathbb{R}^n\big)$, $v_0(\varphi) \geqslant 0$ if system (7) is exponentially stable. In particular, for $\varphi = \varphi_N$, the positive semi-definiteness of $\mathbf{P}_N$ is a necessary condition of exponential stability for any value of N. ■

C. Necessary and sufficient stability conditions

In this section, let $r = \dfrac{d(\|A_0\| + \|A_1\|)}{1-\rho}$ where d, ρ are given by Lemma 2 and recall the following instability result.

Lemma 3: [11] If system (7) is unstable, then there exists a function $\varphi \in \mathcal{S}_r$ such that

$$v_0(\varphi) < -\beta, \quad \beta = \frac{\lambda_{\min}(W)}{4\hat{\alpha}},$$

where $\hat{\alpha}$ is such that $\Re(s) \leqslant \hat{\alpha}$ for any eigenvalue s with strictly positive real part.

The sufficiency of Theorem 2 is achieved in two steps. First, an estimation of the functional approximation error with respect to $v_0(\varphi)$ is required [2], [7]. Denote $\Psi(\theta) = A_1^\top U(\theta)A_1 + 2A_1^\top U'(\theta)D$ and intoduce the functional approximation error $\Upsilon_{0N} = v_0(\varphi) - v_0(\varphi_N)$. Thus

$$\begin{aligned} \Upsilon_{0N} = {}& v_0(\varphi) - v_0(\varphi_N) = 2[\varphi(0) - D\varphi(-h)]^\top \\ &\times \int_{-h}^{0}\left(U^\top(h+\theta)A_1 - U'^\top(h+\theta)D\right)\tilde{\varphi}_N(\theta)\mathrm{d}\theta \\ &+ \int_{-h}^{0}\int_{-h}^{0} \varphi^\top(\theta_1)\Psi(\theta_1-\theta_2)\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 \\ &- \int_{-h}^{0}\int_{-h}^{0} \varphi_N^\top(\theta_1)\Psi(\theta_1-\theta_2)\varphi_N(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 \\ &- \int_{-h}^{0}\int_{-h}^{\theta_1} \varphi^\top(\theta_1)D^\top U''(\theta_1-\theta_2)D\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 \\ &+ \int_{-h}^{0}\int_{-h}^{\theta_1} \varphi_N^\top(\theta_1)D^\top U''(\theta_1-\theta_2)D\varphi_N(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 \\ &- \int_{-h}^{0}\int_{\theta_1}^{0} \varphi^\top(\theta_1)D^\top U''(\theta_1-\theta_2)D\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 \\ &+ \int_{-h}^{0}\int_{\theta_1}^{0} \varphi_N^\top(\theta_1)D^\top U''(\theta_1-\theta_2)D\varphi_N(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 \\ &- \int_{-h}^{0} \varphi^\top(\theta)D^\top PD\varphi(\theta)\mathrm{d}\theta + \int_{-h}^{0} \varphi_N^\top(\theta)D^\top PD\varphi_N(\theta)\mathrm{d}\theta. \end{aligned}$$

As $U(s)$ is guaranteed to be bounded if the Lyapunov condition holds, we introduce the constants

$$M_1 = \sup_{\theta\in(0,h)} \|U^\top(\theta)A_1 - U'^\top(\theta)D\|,$$

$$M_2 = \sup_{\theta\in(0,h)} \|A_1^\top U(\theta)A_1 + 2A_1^\top U'(\theta)D - D^\top U''(\theta)D\|,$$

$$M_3 = |\lambda_{\max}(D^T P D)|.$$

Now, considering $\varphi \in \mathcal{S}_r$, $\|\varphi\|_h = \|\varphi(0)\| = 1$ and applying Lemma 1, the error Υ_{0N} is over-estimated as follows

$$\begin{aligned} |\Upsilon_{0N}| &\leqslant \kappa_2\|\tilde{\varphi}_N\|_h^2 + 2\kappa_1\|\tilde{\varphi}_N\|_h, \\ &\leqslant \kappa_2\Big(\frac{5\bar{r}^N}{N!}\Big)^2 + 2\kappa_1\Big(\frac{5\bar{r}^N}{N!}\Big). \end{aligned} \tag{12}$$

where

$$\begin{aligned} &\kappa_1 = h((1+\|D\|)M_1 + hM_2 + M_3), \\ &\kappa_2 = h(hM_2 + M_3), \quad \bar{r} = \max\Big(1, (\tfrac{2hr}{3})\Big). \end{aligned} \tag{13}$$

The second step is to compute the approximation order $N^\star$ such that if the positive semi-definiteness of $v_0(\varphi_{N^\star})$ is ensured, so is the exponential stability of system (7). The estimate (12) and Lemma 3 lead to the following result.

Theorem 3: System (7) is exponentially stable, if and only if the Lyapunov condition holds and the matrix $\mathbf{P}_{N_0^\star} \geq 0$, with the order of approximation $N^\star = \mathcal{N}(\mathcal{E}(\beta))$ where

$$\begin{aligned} \mathcal{N}(\mathcal{E}(\beta)) &= \left\lceil \bar{r}\exp\left[1 + \mathcal{W}\left((e\bar{r})^{-1}\log\left(\frac{5}{\mathcal{E}(\beta)}\right)\right)\right]\right\rceil, \\ \mathcal{E}(\beta) &= -\frac{\kappa_1}{\kappa_2} + \sqrt{\left(\frac{\kappa_1}{\kappa_2}\right)^2 + \frac{\beta}{\kappa_2}}, \end{aligned} \tag{14}$$

Here, $\kappa_1, \kappa_2, \bar{r}$ are given by (13) and the scalar β is determined in Lemma 3.

Proof: The necessity follows from Theorem 2 with $N = N^\star$. For sufficiency, take into account that $v_0(\varphi) = v_0(\varphi_N) + \Upsilon_{0N}$ and observe that the order of approximation $N^\star$ implies that $|\Upsilon_{0N^\star}| \leqslant \beta_0$. Now, assuming that $\mathbf{P}_{N^\star} \geq 0$ and that system (7) is unstable, then Lemma 3 leads to a contradiction. Thus system (7) is exponentially stable. ■

IV. Linear distributed time-delay systems

We now analyze the stability of the following linear time-invariant retarded type time-delay system:

$$\begin{cases} \dot{x}(t) = A_0x(t) + A_1x(t-h) + \displaystyle\int_{-h}^{0} G(\theta)x(t+\theta)d\theta, \ \forall t \geqslant 0, \\ x(t) = \varphi(t), \ \forall\, t \in [-h, 0]. \end{cases} \tag{15}$$

Here A_0 and A_1 are given real $n \times n$ matrices, $h > 0$, $\varphi \in PC^1\big([-h,0],\mathbb{R}^n\big)$ represent the initial conditions and $G(\theta)$, kernel of the distributed delay, is a real piece-wise continuous matrix functions defined for $\theta \in [-h, 0]$.

A. Lyapunov functionals and matrices

Here, the delay Lyapunov matrix $U : [-h, h] \to \mathbb{R}^{n\times n}$ of system (15) associated with $W > 0$ satisfies the following properties.

1) Dynamic property for $\tau > 0$

$$\frac{d}{d\tau}U(\tau) = U(\tau)A_0 + U(\tau-h)A_1 + \int_{-h}^{0} U(\tau+\theta)G(\theta)d\theta.$$

2) Symmetry property

$$U(\tau) = U^\top(-\tau), \quad \tau \in \mathbb{R}.$$

3) Algebraic property

$$\begin{aligned} -W &= U(0)A_0 + U(-h)A_1 \\ &\quad + \int_{-h}^{0}\Big[G^\top(\theta)U^\top(\theta)d\theta + U(\theta)G(\theta)\Big]d\theta. \end{aligned}$$

As in Section III, the existence and uniqueness of the delay Lyapunov matrix [1] is ensured by the Lyapunov condition assumption.

In the Lyapunov-Krasovskii functionals converse framework is widely used to derivated stability conditions [3]

$$v_1(\varphi) = \varphi^\top(0)U(0)\varphi(0) + \sum_{j=1}^{6} I_j \tag{16}$$

where

$$\begin{aligned} I_1 &= 2\varphi^\top(0)\int_{-h}^{0} U^\top(\theta+h)A_1\varphi(\theta)d\theta, \\ I_2 &= \int_{-h}^{0}\int_{-h}^{0} \varphi^\top(\theta_1)A_1^\top U(\theta_1-\theta_2)A_1\varphi(\theta_2)d\theta_2 d\theta_1, \\ I_3 &= 2\varphi^\top(0)\int_{-h}^{0}\int_{-h}^{\theta} U^\top(\theta-\xi)G(\xi)d\xi\varphi(\theta)d\theta, \\ I_4 &= 2\iiint_{\mathcal{R}_1} \varphi^\top(\theta_1)A_1^\top \mathcal{U}(\theta_1,\theta_2,-h,\xi)G(\xi)\varphi(\theta_2)d\mathcal{R}_1, \\ I_5 &= \iiiint_{\mathcal{R}_2} \varphi^\top(\theta_1)G^\top(\xi_1)\mathcal{U}(\theta_1,\theta_2,\xi_1,\xi_2)G(\xi_2)\varphi(\theta_2)d\mathcal{R}_2, \\ I_6 &= \int_{-h}^{0} \varphi^\top(\theta)W\varphi(\theta)d\theta, \end{aligned}$$

with the function $\mathcal{U}(\theta_1,\theta_2,\xi_1,\xi_2) = U(\theta_1-\theta_2-\xi_1+\xi_2)$, and the integration regions

$$\begin{aligned} &\mathcal{R}_1 = \{\theta_1 \in [-h,0], \theta_2 \in [-h,0], \xi \in [-h,\theta_2]\}, \\ &\mathcal{R}_2 = \{\theta_1 \in [-h,0], \theta_2 \in [-h,0], \xi_1 \in [-h,\theta_1], \xi_2 \in [-h,\theta_2]\}, \\ &d\mathcal{R}_1 = d\xi d\theta_2 d\theta_1, \quad d\mathcal{R}_2 = d\xi_2 d\xi_1 d\theta_2 d\theta_1. \end{aligned}$$

Its time derivative along system (15) trajectories is

$$\frac{d}{dt}v_1(x_t(\varphi)) = -x^\top(t-h,\varphi)Wx(t-h,\varphi). \tag{17}$$

B. Necessary stability conditions

In this section, a stability result for systems with distributed delays is recalled: when the system is stable, the functional v_1 admits a quadratic lower bound.

Lemma 4 (see [12]): If system (15) is exponentially stable, then there exist positive numbers α_0 and α_1 such that

$$v_1(\varphi) \geqslant \alpha_0\|\varphi(0)\|^2 + \alpha_1\int_{-h}^{0}\|\varphi(\theta)\|^2 d\theta, \tag{18}$$

for all $\varphi \in PC^1\big([-h,0],\mathbb{R}^n\big)$.

The Legendre orthogonal projections of the functional argument $\varphi \in PC^1\big([-h,0],\mathbb{R}^n\big)$ is substituted into functional (16), resulting in the functional $v_1(\varphi_N)$, which serves as a functional approximation of (16). For any integer $N \geqslant 1$, the approximated Lyapunov-Krasovskii functional is given by

$$v_1(\varphi_N) = q_N^\top \mathbf{Q}_N q_N, \tag{19}$$

where $q_N = [\varphi^T(0)\ \Phi_N^T]^T \in \mathbb{R}^{Nn}$,

$$\mathbf{Q}_N = \begin{bmatrix} U(0) & \mathbf{J}_{1N} + \mathbf{J}_{2N} \\ [\mathbf{J}_{1N} + \mathbf{J}_{2N}]^\top & \mathbf{J}_{3N} + \mathbf{J}_{4N} + \mathbf{J}_{4N}^\top + \mathbf{J}_{5N} + \mathbf{J}_{6N} \end{bmatrix}.$$

Here,

$$\begin{aligned}
\mathbf{J}_{1N} &= \int_{-h}^{0} U^\top(\theta+h)A_1\ell_N^\top(\theta)d\tau,\\
\mathbf{J}_{2N} &= \int_{-h}^{0}\int_{-h}^{0} \ell_N(\theta_1)A_1^\top U(\theta_1-\theta_2)A_1\ell_N^\top(\theta_2)d\theta_2 d\theta_1,\\
\mathbf{J}_{3N} &= \int_{-h}^{0}\int_{-h}^{\theta} U^\top(\theta-\xi)G(\xi)\ell_N^\top(\theta)d\xi d\theta,\\
\mathbf{J}_{4N} &= \iiint_{\mathcal{R}_1} \ell_N(\theta_1)A_k^\top \mathcal{U}(\theta_1,\theta_2,-h,\xi)G(\xi)\ell_N^\top(\theta_2)d\mathcal{R}_1,\\
\mathbf{J}_{5N} &= \iiiint_{\mathcal{R}_2} \ell_N(\theta_1)G^\top(\xi_1)\mathcal{U}(\theta_1,\theta_2,\xi_1,\xi_2)G(\xi_2)\ell_N^\top(\theta_2)d\mathcal{R}_2,\\
\mathbf{J}_{6N} &= \int_{-h}^{0} \ell_N(\tau)W\ell_N^\top(\tau)d\tau,
\end{aligned} \tag{20}$$

with $\mathcal{U}(\cdot)$ and $\mathcal{R}_1$, $\mathcal{R}_2$ defined in Section IV-A.

Next, we present a necessary exponential stability condition for systems with distributed delays.

Theorem 4: If system (15) is exponentially stable, then the matrix $\mathbf{Q}_N > 0$, for all integer $N \geqslant 1$.

Proof: It follows from Lemma 4 that if system (15) is exponentially stable, then $v_1(\varphi) > 0$ for any non null $\varphi \in PC^1([-h,0],\mathbb{R}^n)$. Considering $\varphi = \varphi_N$ implies that $v_1(\varphi_N) = q_N^\top \mathbf{Q}_N q_N > 0$ for any non null vector q_N. Hence, the positive definiteness of the matrix $\mathbf{Q}_N$ is a necessary exponential stability condition for any value of N. ■

C. Necessary and sufficient stability conditions

In this section, let $r = \|A_0\| + \|A_1\| + h\|G\|_h$. An instability result for systems with distributed delays is recalled: when the system is unstable, then functional v_1 does not admit a non-negative lower bound.

Lemma 5 (see [12]): If system (15) is unstable and the Lyapunov condition holds, there exists $\widehat{\varphi} \in \mathcal{S}_r$ such that

$$v_1(\widehat{\varphi}) \leq -\beta, \quad \beta = \frac{\lambda_{\min}(W)}{8re^{2hr}}(1+\cos b), \tag{21}$$

where $b \in (0,\pi)$ is a unique solution of the equation

$$((2hr)^2 + b^2)(1-\cos b)^2 = 4(2hr)^2. \tag{22}$$

To obtain the sufficiency part from the necessary condition presented in Theorem 4, an approximation order must be computed as described in Section III. An estimate of the functional approximation error $\Upsilon_{1N} = v_1(\varphi) - v_1(\varphi_N)$ is given next.

$$\begin{aligned}
\Upsilon_{1N} &= v_1(\varphi) - v_1(\varphi_N),\\
&= 2\varphi^\top(0)\int_{-h}^{0} U^\top(h+\theta)A_1\tilde{\varphi}_N(\theta)\mathrm{d}\theta\\
&+ 2\varphi^\top(0)\int_{-h}^{0}\int_{-h}^{\theta} U^\top(\theta-\xi)G(\xi)d\xi\tilde{\varphi}_N(\theta)d\theta\\
&+ \int_{-h}^{0} \tilde{\varphi}_N^\top(\theta)W\tilde{\varphi}_N(\theta)d\theta\\
&+ \int_{-h}^{0}\int_{-h}^{0} \varphi^\top(\theta_1)A_1^\top U(\theta_1-\theta_2)A_1\varphi(\theta_2)d\theta_2 d\theta_1\\
&- \int_{-h}^{0}\int_{-h}^{0} \varphi_N^\top(\theta_1)A_1^\top U(\theta_1-\theta_2)A_1\varphi_N(\theta_2)d\theta_2 d\theta_1\\
&+ 2\iiint_{\mathcal{R}_1} \varphi^\top(\theta_1)A_1^\top \mathcal{U}(\theta_1,\theta_2,-h,\xi)G(\xi)\varphi(\theta_2)d\mathcal{R}_1\\
&- 2\iiint_{\mathcal{R}_1} \varphi_N^\top(\theta_1)A_1^\top \mathcal{U}(\theta_1,\theta_2,-h,\xi)G(\xi)\varphi_N(\theta_2)d\mathcal{R}_1\\
&+ \iiiint_{\mathcal{R}_2} \varphi^\top(\theta_1)G^\top(\xi_1)\mathcal{U}(\theta_1,\theta_2,\xi_1,\xi_2)G(\xi_2)\varphi(\theta_2)d\mathcal{R}_2\\
&- \iiiint_{\mathcal{R}_2} \varphi_N^\top(\theta_1)G^\top(\xi_1)\mathcal{U}(\theta_1,\theta_2,\xi_1,\xi_2)G(\xi_2)\varphi_N(\theta_2)d\mathcal{R}_2.
\end{aligned}$$

Using the same arguments presented in Section III, the error Υ_{1N} is over-estimated as follows. For any $\varphi \in \mathcal{S}_r$, we have

$$\begin{aligned}
\|\Upsilon_{1N}\| &\leqslant \kappa_2\|\tilde{\varphi}_N\|_h^2 + 2\kappa_1\|\tilde{\varphi}_N\|_h\\
&\leqslant \kappa_2\Big(\frac{5\bar{r}^N}{N!}\Big)^2 + 2\kappa_1\Big(\frac{5\bar{r}^N}{N!}\Big),
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
\kappa_1 &= \|U\|_h\Big(h\|A_1\| + \tfrac{1}{2}\|G\|_h h^2\Big)\Big(1 + h\|A_1\| + \tfrac{1}{2}\|G\|_h h^2\Big),\\
\kappa_2 &= \|U\|_h\Big(h\|A_1\| + \tfrac{1}{2}\|G\|_h h^2\Big)^2 + h\|W\|,\\
\|U\|_h &= \max_{\theta\in[0,h]}\|U(\theta)\|, \quad \bar{r} = \max\Big(1, (\tfrac{2hr}{3})\Big).
\end{aligned} \tag{24}$$

Necessary and sufficient stability conditions for systems with distributed delays, verifiable in a finite number of mathematical operations, are now given.

Theorem 5: System (15) is exponentially stable if and only if the Lyapunov condition holds and $\mathbf{Q}_{N^\star} > 0$ with the order of approximation $N^\star = \mathcal{N}(\mathcal{E}(\beta))$, where functions $\mathcal{N}$ and $\mathcal{E}$ are those in (14) but where $\kappa_1, \kappa_2, \bar{r}$ are given by (24) and the scalar β is determined in Lemma 5.

Proof: The necessity follows from Theorem 4, for $N = N^\star$. For sufficiency, note that functional (16) is written as $v_1(\varphi) = v_1(\varphi_N) + \Upsilon_{1N}$, and that for $N^\star$ we obtain $\|\Upsilon_{1N}\| \leqslant \beta$. By contradiction, we assume that system (15) is not exponentially stable, but $\mathbf{Q}_{N^\star} > 0$. Then, Lemma 5 constradicts the positive definiteness of $\mathbf{Q}_{N^\star}$. ■

V. Illustrative Examples

In this section, we validate the necessary and sufficiency condition of Theorem 3 and Theorem 5 through some examples. Since the stability conditions are in terms of the delay Lyapunov matrix, this is computed via a semi-analytic method introduced in [1] for $W = I_n$. In each example, the stability/instability boundaries, depicted by solid lines, are obtained using the D-subdivision method [13]. The positive semi-definiteness of $\mathbf{P}_N$ and the positivity of $\mathbf{Q}_N$ are verified through the function "cholcov" in MATLAB.

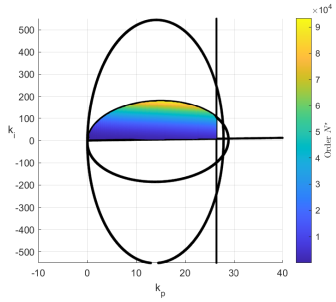

Fig. 1. Map of the order $N^\star$ in the space of parameter (k_p, k_i).

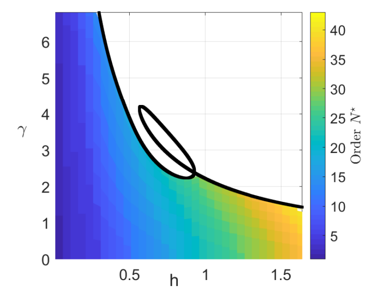

Fig. 2. Map of the orden $N^\star$ in the space of parameters (h, γ).

A. Example 1: neutral-type linear delay system

The σ-stability analysis of the proportional-integral control of a passive linear system leads to studying a quasipolynomial of neutral-type [14]. Its time domain representation is of the form (7), with matrices $D = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\alpha_2}{\alpha_1} \end{pmatrix}$,

$$A_0 = \frac{1}{\alpha_1}\begin{pmatrix} 0 & \alpha_1 \\ -\sigma^2\alpha_1 + \sigma\beta_1 - \gamma_1 & -\beta_1 + 2\sigma\alpha_1 \end{pmatrix},$$

$$A_1 = \frac{1}{\alpha_1}\begin{pmatrix} 0 & 0 \\ -\sigma^2\alpha_2 + \sigma\beta_2 - \gamma_2 & -\beta_2 + 2\sigma\alpha_2 \end{pmatrix},$$

where

$$\begin{aligned} \alpha_1 &= d + k_p, \quad \gamma_1 = bk_i d + ak_i, \\ \alpha_2 &= (d - k_p)e^{\sigma h}, \quad \gamma_2 = (bk_i d - ak_i)e^{\sigma h}, \\ \beta_1 &= (bk_p + a)d + bd^2 + ak_p + k_i, \\ \beta_2 &= ((bk_p + a)d - bd^2 - ak_p - k_i)e^{\sigma h}. \end{aligned}$$

The stability of the difference operator imposes in the D-subdivision map the additional condition $|k_p| < 26.67$. For parameter values $a = 0.4$, $b = 50$, $h = 0.2$, $d = 0.8$ and $\sigma = 0.3$, the approximation order $N^\star$ for the space of parameter (k_p, k_i) is depicted in Figure 1.

B. Example 2: linear distributed time-delay systems

A linearized version of the dynamic equation of the feeding system and combustion chamber of a liquid monopropellant rocket motor [15] is given by

$$\dot{x}(t) = \begin{pmatrix} \gamma-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 \\ -\kappa & 0 & -\kappa & \kappa \\ 0 & 1 & -1 & 0 \end{pmatrix} x(t) - \begin{pmatrix} \frac{\gamma}{h} & 0 & \frac{1}{h} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \int_{-h}^{0} x(t+\theta)d\theta \quad (25)$$

where $\kappa = 0.5556$, $\gamma \in \mathbb{R}$, and $h > 0$ are system parameters. The estimation order $N^\star$ of Theorem 5 is illustrated in Figure 2 for pairs (h, γ) in the space of parameters.

VI. Conclusions

This contribution is dedicated to the development and application of the ideas introduced in [8] on the Lyapunov-Krasovskii functional argument approximation via Legendre polynomials projections. New necessary and sufficient stability conditions for neutral-type linear and linear time-delay systems with distributed delays are formulated. These results benefit from the super-geometric convergence property of Legendre polynomials approximation which allows reducing of the numerical complexity of the criteria.

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