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Modulating Function-based Leak Detection, Size Estimation and Localization for a Water Pipe Prototype

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Abstract— **We propose a new method for leak detection and localization in water pipes based on a mathematical model that describes the flow dynamics by two coupled linear first order hyperbolic partial differential equations. Using the modulating function approach, a system of auxiliary PDEs is derived and solved in order to obtain appropriate modulating functions. This allows estimating the leak size and the leak position, resorting to algebraic I/O equations only. For this purpose, no spatial discretization of the PDE model is needed. The theoretical results are validated with experimental data from a water pipe prototype and the performance of the proposed approach is evaluated in comparison to an existing late lumping model-based leak detection system.**

Index Terms— **Leak detection, modulating function, partial differential equation, water pipeline system**

I. Introduction

Pipelines form an indispensable part of critical infrastructure, like e.g., heating systems, water distribution networks or fuel transport. Therefore, a high level of safety and reliability has to be guaranteed. One of the major risks to deal within this context is the appearance of leaks that provoke severe impacts on man and nature like economical costs, environmental damage or social conflicts. As a study of the OECD in 2012 reveals, 48 selected cities from OECD countries have an average water loss rate of 21 % including cities with a water loss rate higher than 50 % [1]. It is estimated that the yearly water losses sum up to economical costs of about USD 39 billion [2].

Hence, the development of efficient leak detection and localization methods is of great research, economic, ecological and social interest. Consequently, there is a wide range of leak detection and localization methods that are typically classified according to their technical nature into three main categories: hardware-based / exterior-based methods, non-technical / biological methods, and software-based / interior-based methods [3], [4]. The leak detection and localization methods proposed in this paper belong to the last category and may be characterized as a model-based, real-time transient modelling (RTTM) approach.

RTTM approaches rely on a distributed parameters model of the pipe flow, derived e.g. from mass and momentum balances. The resulting hyperbolic PDE model is an infinite dimensional system. Reducing it to a finite dimensional system leads to two different observer designs, early lumping and late lumping approaches [5]. Early lumping approaches firstly discretize the PDE system to obtain a set of ODEs [5] what allows utilizing classical approaches of observer theory for finite dimensional systems like sliding mode observers [6], [7], Luenberger type observers [8] or Kalman filters (KF) [9], [10], [11], [12], [13]. Especially, the design of KF-type observers to estimate and localize leaks has been the subject of intense research during the past four decades (see e.g., [9], [14]) and is already used for commercial leak detection algorithms like PipePetrol by the KROHNE Group [15], such that they can be regarded as a benchmark for RTTM leak detection and localization approaches.

Although the KF-type observers show an acceptable accuracy in simulation and real data-based test environments (see e.g., [9], [16], [17]), the early lumping approaches share several drawbacks. Physical information contained in the PDE model of the fluid dynamics may be lost and important conditions for the observer design like the observability of the obtained ODE system will depend on the choice of discretization scheme and the location of the discretization points [5].

These limitations motivate late lumping leak detection and localization techniques, which let design the observer directly from the PDE model and do not require any discretization, neither for deriving nor for implementing the observer algorithm. In [18], [19], a late lumping observer for leak detection, size estimation and localization is proposed using backstepping methods to prove the asymptotic stability of the observation error.

Regarding these existing model-based leak detection and localization approaches, the main contributions of this paper are twofold. On the one hand, we propose a novel, late lumping leak detection and localization method that results in an algebraic observer by applying the modulating function method (MFM). In the context of distributed-parameter systems, modulating functions (MF) have been established by [20] for fault detection tasks. Recently the authors of [21] have extended the approach to PDE state reconstruction. For the first time, we combine both approaches and use the MFM for fault detection where the algebraic PDE state reconstruction step subsequently enables the estimation of the leak size as well as the leak position. Thereby, the research gap regarding model-based late lumping leak detection and localization techniques, that do not require a spatial discretization of the PDE model, is addressed.

Moreover, the performance of the proposed observer is validated with real measurement data taken from a water pipe prototype installed at the laboratory of Advanced Control

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Engineering at the Pontificia Universidad Católica del Perú (PUCP) in Lima. The accuracy of the leak size estimation and localization is compared to the late lumping Adaptive Backstepping Observer (ABSO) presented in [18]. Thus, to the best of the author's knowledge, this is the first publication that studies the validation of a late lumping model-based leak detection and localization system with real experimental data.

The paper is organized as follows: In Section II, we introduce the pipe flow model and a transformation that allows to independently estimate the unknown leak size and the unknown leak position. Section III recapitulates some necessary fundamentals of the modulating function method. We derive the auxiliary problem and the estimator equation for the leak size. Subsequently, the leak localization is performed via an MFM-based state estimation of the transformed model and an approach to solve the auxiliary problems is presented. In Section IV, the pipe prototype installed at the PUCP is described and the designed leak detection and localization observer is validated with measurement data. Finally we draw our conclusions in Section V.

II. Model & Transformation

According to [18], [19], the fluid dynamics of the water flow may be described by the mass and momentum balance resulting in the coupled linear hyperbolic PDE system

$$\Sigma_{\text{phy}}: \begin{cases} p_t(z,t) = -\frac{\beta}{A} q_z(z,t) - \frac{\beta}{A}\chi d(z) \\ q_t(z,t) = -\frac{A}{\rho} p_z(z,t) - \frac{F}{\rho} q(z,t) \\ \qquad - Ag\sin(\theta(z)) - \frac{\eta}{A}\chi d(z) \\ p(\ell,t) = p_\ell(t),\ q(0,t) = q_0(t) \end{cases} \tag{1}$$

where $z \in [0,\ell]$ represents the one-dimensional spatial coordinate [m], ℓ is the length [m] of the pipe, t is the time [s], $q(z,t)$ denotes the volumetric fluid flow rate $[\frac{\mathrm{m}^3}{\mathrm{s}}]$, $p(z,t)$ is the fluid pressure [Pa], β is the bulk modulus [Pa] describing the compressibility of the fluid, A is the cross-sectional area [m^2] of the pipe, ρ is the fluid density $[\frac{\mathrm{kg}}{\mathrm{m}^3}]$, $\theta(z)$ is the inclination angle [rad] of the pipe, g is the gravity acceleration $[\frac{\mathrm{m}}{\mathrm{s}^2}]$ and F is the friction factor $[\frac{\mathrm{kg}}{\mathrm{m}^3\mathrm{s}}]$. Moreover, it is assumed that the pressure and the flow rate at the inlet and the outlet are known and given by $p_0(t), q_0(t), p_\ell(t)$ and $q_\ell(t)$. The factor η describes the additional momentum loss due to the leak and is given by $\eta = 0.8\, q_{\text{nom}}$ for point leaks where q_{nom} denotes the volumetric flow in steady-state [19], [22]. Furthermore, the leak is characterized by the total leak size χ that is assumed to be constant and by the leak distribution function $d(z)$ that satisfies

$$\forall z \in [0,\ell] : d(z) \geq 0,\ \int_0^\ell d(\gamma)\mathrm{d}\gamma = 1\,. \tag{2}$$

We shall point out that the friction losses are assumed to depend linearly on the flow rate $q(z,t)$ such that the PDE model Σ_{phy} is also linear in the states $p(z,t)$, $q(z,t)$. Hence, the presented model Σ_{phy} differs from the classical water hammer equations [14] that include nonlinear friction losses and are widely used for the Kalman filter-based leak detection and localization (see, e.g., [9], [17]). The limitations of this simplified linear pipe model are discussed in Section IV.

Since the unknown leak size χ and the unknown leak distribution $d(z)$ appear in both equations of the pipe model Σ_{phy} as a product, a transformation to decouple the two unknowns is sought. Such invertible transformation is found in [18, Lemma 1] and leads to the transformed model

$$\Sigma_{\text{tr}}: \begin{cases} \begin{bmatrix} u_t(x,t) \\ v_t(x,t) \end{bmatrix} = \Lambda \begin{bmatrix} u_x(x,t) \\ v_x(x,t) \end{bmatrix} + \Sigma(x) \begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} \\ \begin{bmatrix} u(0,t) \\ v(1,t) \end{bmatrix} = \begin{bmatrix} -v(0,t) + q_0(t) - \chi \\ U(t) \end{bmatrix} \end{cases} \tag{3}$$

with $x \in [0,1]$, $\epsilon = \frac{1}{\ell}\sqrt{\frac{\beta}{\rho}}$, $c_1(x) = -\frac{1}{2}\frac{F}{\rho}e^{\frac{\ell F}{\sqrt{\beta\rho}}x}$, $c_2(x) = -\frac{1}{2}\frac{F}{\rho}e^{-\frac{\ell F}{\sqrt{\beta\rho}}x}$, $\Lambda = \begin{bmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{bmatrix}$, $\Sigma(x) = \begin{bmatrix} 0 & c_1(x) \\ c_2(x) & 0 \end{bmatrix}$ and auxiliary variable

$$\delta_d(z) := \ell - z - \int_z^\ell \int_0^\eta d(\gamma)\mathrm{d}\gamma\mathrm{d}\eta\,. \tag{4}$$

The known boundary measurements are given by

$$\begin{aligned} U(t) &= \frac{1}{2}e^{-\frac{\ell F}{2\sqrt{\beta\rho}}}\left(q_\ell(t) - \frac{A}{\sqrt{\beta\rho}}\left(p_\ell(t) + \rho g h\right)\right) \\ Y(t) &= u(1,t) = \frac{1}{2}e^{\frac{lF}{2\sqrt{\beta\rho}}}\left(q_\ell(t) + \frac{A}{\sqrt{\beta\rho}}\left(p_\ell(t) + \rho g h\right)\right) \end{aligned} \tag{5}$$

where $h = \int_0^\ell \sin(\theta(\gamma))\mathrm{d}\gamma$. As a result, the dynamics of the transformed system Σ_{tr} are independent of the leak distribution $d(z)$ such that the leak size χ can be estimated separately from the leak position. Throughout the following section, the MFM is utilized to estimate the leak size χ. Subsequently, an MFM-based state estimation of the transformed system is performed to localize the leak.

III. Modulating Function Method For Leak Size Estimation and Localization

A. Preliminaries

The proposed approach to estimate the leak size and the leak position is based on the modulating function method, whose fundamentals are summarized in [20], [23] as follows.

Definition 1: A function $\varphi \in \mathcal{C}^k([0,1]\times[0,T],\mathbb{R}^n)$ is called MF of order $k \in \mathbb{N}$ if

$$\forall i \in \{0,1,\ldots,k-1\} : \frac{\partial^i\varphi}{\partial t^i}(x,0)\frac{\partial^i\varphi}{\partial t^i}(x,T) = 0\,. \tag{6}$$

Based on this classical definition pointing out the annihilation property of the kernel at a boundary due its support, the modulating functional is introduced.

Definition 2: Let $\varphi \in \mathcal{C}^k([0,1]\times[0,T],\mathbb{R}^n)$ be an MF of order k and $h : [0,1]\times\mathbb{R}_{\geq 0} \to \mathbb{R}^m, m \in \{1,n\}$ an integrable signal. Then, the modulating functional is defined as

$$\mathcal{M}[h](t) := \int_{t-T}^t \int_0^1 \varphi(x,\tau - t + T)^\top h(x,\tau)\mathrm{d}x\mathrm{d}\tau\,. \tag{7}$$

For simplicity, the abbreviation as an inner product

$$\langle\varphi, h\rangle_{\Omega,I} := \mathcal{M}[h] \tag{8}$$

is used where $I := [t-T, t],\ t \geq T$ describes a moving time horizon with receding horizon length $T > 0$ and $\Omega := [0, 1]$ the fixed spatial domain. Furthermore, the abbreviations $\langle \varphi, h\rangle_I$ and $\langle \varphi, h\rangle_\Omega$ are introduced if the integration is realized only w.r.t. the temporal or spatial variable, respectively.

B. Leak Size Estimation

Similar to [20], the leak size χ is regarded as an unknown source term that enters the transformed model Σ_{tr} via the boundary condition. Following the basic procedure of the MFM, the modulating functional as defined in Equation (7) is applied to the transformed system Σ_{tr} to derive the auxiliary problem and the algebraic I/O-relationship between the known boundary measurements and the unknown leak size χ. The following theorem provides the algorithm for the MFM-based leak size estimation.

Theorem 1: Consider the transformed system Σ_{tr} given in (3) with the known boundary measurements defined by (5). Assume that $\varphi : [0,1] \times [0,T] \to \mathbb{R}^2$ is a MF of order 1 that solves the following auxiliary problem

$$\Sigma_{\text{aux},\chi} : \begin{cases} \varphi_\tau(x,\tau) = \Lambda^\top \varphi_x(x,\tau) - \Sigma^\top(x)\varphi(x,\tau) \\ \varphi_1(0,\tau) = -\varphi_2(0,\tau) \\ \varphi_1(1,\tau) = \eta_1(\tau) \\ \varphi(x,0) = 0,\ \varphi(x,T) = 0 \end{cases} \tag{9}$$

with $x \in [0,1], \tau \in [0,T]$ and $\int_0^T \varphi_1(0,\tau)\mathrm{d}\tau \neq 0$ where $\eta_1 : [0,T] \to \mathbb{R}$ represents a remaining degree of freedom. Then, the leak size χ can be estimated by

$$\hat{\chi} = \frac{1}{\int_0^T \varphi_1(0,\tau)\mathrm{d}\tau}\Big(\langle \varphi_1(0), q_0\rangle_I - \langle \varphi_1(1), Y\rangle_I + \langle \varphi_2(1), U\rangle_I\Big). \tag{10}$$

Proof: Applying the modulating functional (7) with a MF $\varphi : [0,1] \times [0,T] \to \mathbb{R}^2$ of order 1 to the transformed system Σ_{tr} leads to

$$\Big\langle \varphi, \begin{bmatrix} u_t \\ v_t \end{bmatrix} \Big\rangle_{\Omega,I} = \Big\langle \varphi, \Lambda \begin{bmatrix} u_x \\ v_x \end{bmatrix} \Big\rangle_{\Omega,I} + \Big\langle \varphi, \Sigma(x) \begin{bmatrix} u \\ v \end{bmatrix} \Big\rangle_{\Omega,I} . \tag{11}$$

Using that $\varphi(x,\tau)$ is a MF of order 1, the left-hand side of (11) is simplified via integration by parts to

$$\Big\langle \varphi, \begin{bmatrix} u_t \\ v_t \end{bmatrix} \Big\rangle_{\Omega,I} = -\Big\langle \varphi_\tau, \begin{bmatrix} u \\ v \end{bmatrix} \Big\rangle_{\Omega,I} . \tag{12}$$

Applying integration by parts for the first term on the right-hand side of (11) gives

$$\langle \varphi, \Lambda \begin{bmatrix} u_x \\ v_x \end{bmatrix} \rangle_{\Omega,I} = -\langle \Lambda^\top \varphi_x, \begin{bmatrix} u \\ v \end{bmatrix} \rangle_{\Omega,I} + \Big(\langle \Lambda^\top \varphi, \begin{bmatrix} u \\ v \end{bmatrix} \rangle_I \Big)\Big|_0^1 . \tag{13}$$

Inserting (12) and (13) into (11) results in

$$-\Big\langle \varphi_\tau, \begin{bmatrix} u \\ v \end{bmatrix} \Big\rangle_{\Omega,I} = -\Big\langle \Lambda^\top \varphi_x, \begin{bmatrix} u \\ v \end{bmatrix} \Big\rangle_{\Omega,I} + \Big(\Big\langle \Lambda^\top \varphi, \begin{bmatrix} u \\ v \end{bmatrix} \Big\rangle_I \Big)\Big|_0^1 + \Big\langle \Sigma^\top(x)\varphi, \begin{bmatrix} u \\ v \end{bmatrix} \Big\rangle_{\Omega,I} . \tag{14}$$

To eliminate the unknown state variables $u(x,t),\ v(x,t)$ in (14), the following auxiliary PDE is imposed:

$$\varphi_\tau(x,\tau) = \Lambda^\top \varphi_x(x,\tau) - \Sigma(x)^\top \varphi(x,\tau) . \tag{15}$$

Thus, (14) simplifies to

$$\begin{aligned} 0 &= \Big(\Big\langle \Lambda^\top \varphi, \begin{bmatrix} u \\ v \end{bmatrix} \Big\rangle_I \Big)\Big|_0^1 \\ &= \Big\langle \Lambda^\top \varphi(1,t), \begin{bmatrix} u(1,t) \\ v(1,t) \end{bmatrix} \Big\rangle_I - \Big\langle \Lambda^\top \varphi(0,t), \begin{bmatrix} u(0,t) \\ v(0,t) \end{bmatrix} \Big\rangle_I . \end{aligned} \tag{16}$$

To eliminate the dependence on the unknown boundary state $v(0,t)$, the boundary condition

$$\varphi_1(0,\tau) = -\varphi_2(0,\tau) \tag{17}$$

is added. Consequently, by inserting the known boundary conditions $u(0,t) = -v(0,t) + q_0(t) - \chi,\ v(1,t) = U(t)$, the known boundary measurement $u(1,t) = Y(t)$, and the imposed boundary condition (17) into (16), we obtain

$$\langle \varphi_1(0), \chi\rangle_I = \langle \varphi_1(0), q_0\rangle_I - \langle \varphi_1(1), Y\rangle_I + \langle \varphi_2(1), U\rangle_I . \tag{18}$$

Since the leak size χ is constant and $\int_0^T \varphi_1(0,\tau)\mathrm{d}\tau \neq 0$, (18) results in the leak size estimation equation

$$\chi = \frac{1}{\int_0^T \varphi_1(0,\tau)\mathrm{d}\tau} \left(\langle \varphi_1(0), q_0\rangle_I - \langle \varphi_1(1), Y\rangle_I + \langle \varphi_2(1), U\rangle_I \right) \tag{19}$$

completing the proof. ■

The leak size estimation equation (10) establishes an algebraic I/O-relation between the measurements taken from the pipe system Σ_{phy} and the leak size χ where the MF $\varphi(x,t)$ works as a filter of the measurements. Moreover, the estimation of the leak size allows detecting a leakage by comparing the estimated leak size with a predefined threshold that is determined, e.g., based on the standard deviation of the measurement noise of the sensors. For a more sophisticated derivation of a detection threshold, which considers possible disturbances inside the MFM framework, see [24].

Besides, the MFM basically transforms the estimation problem into a control problem, i.e., the implementation of the leak size estimation equation (10) requires a solution of the auxiliary problem (9) such that the MF $\varphi(x,t)$ is steered from the initial condition $\varphi(x,0) = 0$ to the final condition $\varphi(x,T) = 0$ over the time horizon $[0,T]$ avoiding the trivial solution $\varphi \equiv 0$. A way to solve the auxiliary problem $\Sigma_{\text{aux},\chi}$ is discussed at the end of this section.

C. Leak Localization

Similar to [18], the leak is localized via an estimation of the states of the transformed system Σ_{tr}. In this publication, a novel approach for reconstructing the states of the system Σ_{tr} is proposed by applying the MFM. This approach can be seen as an extension of the results from [25] where a similar state estimation problem for coupled PDEs is solved by applying the MFM.

We represent the states $u(x,t)$, $v(x,t)$ by the function expansion

$$
\begin{aligned}
u(x,t) &= \sum_{k=0}^{\infty} c_1^k(t)\Psi^k(x) \approx \sum_{k=0}^{N} c_1^k(t)\Psi^k(x) \\
v(x,t) &= \sum_{k=0}^{\infty} c_2^k(t)\Psi^k(x) \approx \sum_{k=0}^{N} c_2^k(t)\Psi^k(x)
\end{aligned} \tag{20}
$$

where N is the approximation order and $\{\Psi^k\}_{k=0}^{\infty}$ denotes the spatial orthonormal basis functions of $\mathcal{L}_2[0,1]$. The orthonormality of the basis functions is defined w.r.t. the scalar product $\langle f,g\rangle_w := \int_0^1 f(x)w(x)g(x)\mathrm{d}x$ where $w(x)$ serves as a weighting function. In view of the function expansion (20), the goal then is the estimation of the basis coefficients $c_1^k(t),\ c_2^k(t)$.

To this end, we apply the modulating functional (7) to the transformed system Σ_{tr}. The following theorem presents the resulting auxiliary problem and the algebraic I/O relationship to determine the basis coefficients $c_1^k(t),\ c_2^k(t)$.

Theorem 2: Let Σ_{tr} be given as defined in (3). Then, the algebraic I/O relation to estimate the $m-$th coefficients of the function expansion (20) is given by

$$
c_{1,2}^m(t) = \epsilon\left(\langle\varphi_2^m(1),U\rangle_I - \langle\varphi_1^m(1),Y\rangle_I + \langle\varphi_1^m(0),q_0\text{-}\chi\rangle_I\right) \tag{21}
$$

where $\varphi(x,\tau):[0,1]\times[0,T]\to\mathbb{R}^2$ is an MF of order 1 that solves the auxiliary problem

$$
\Sigma_{\text{aux},u}:\begin{cases} \varphi_\tau^m(x,\tau) = \Lambda^\top\varphi_x^m(x,\tau) - \Sigma^\top(x)\varphi^m(x,\tau) \\ \varphi(x,0)=0,\ \varphi(x,T) = \begin{bmatrix} w(x)\Psi^m(x) \\ 0\end{bmatrix}\end{cases} \tag{22}
$$

and

$$
\Sigma_{\text{aux},v}:\begin{cases} \varphi_\tau^m(x,\tau) = \Lambda^\top\varphi_x^m(x,\tau) - \Sigma^\top(x)\varphi^m(x,\tau) \\ \varphi(x,0)=0,\ \varphi(x,T) = \begin{bmatrix} 0 \\ w(x)\Psi^m(x)\end{bmatrix},\end{cases} \tag{23}
$$

respectively with $x\in[0,1], \tau\in[0,T]$.

Proof: In the following, the proof will be only shown for the estimation of the m-th coefficient of the function expansion (20) for the state $v(x,t)$.

Applying the modulating functional (7) to the transformed system Σ_{tr} and applying integration by parts regarding time and spatial derivatives similar to (11) and (12) results in

$$
\begin{aligned}
-\langle\varphi_\tau^m,\begin{bmatrix}u\\v\end{bmatrix}\rangle_{\Omega,I} + [\langle\varphi^m,\begin{bmatrix}u\\v\end{bmatrix}\rangle_\Omega]_{t\text{-}T}^t = \left(\langle\Lambda^\top\varphi^m,\begin{bmatrix}u\\v\end{bmatrix}\rangle_I\right)\Big|_0^1 \\
-\langle\Lambda^\top\varphi_x^m,\begin{bmatrix}u\\v\end{bmatrix}\rangle_{\Omega,I} + \langle\Sigma^\top(x)\varphi^m,\begin{bmatrix}u\\v\end{bmatrix}\rangle.
\end{aligned} \tag{24}
$$

Since the MF $\varphi^m(x,\tau)$ fulfills the auxiliary problem $\Sigma_{\text{aux},v}$, (24) simplifies to

$$
\left(\Big\langle\varphi^m,\begin{bmatrix}u\\v\end{bmatrix}\Big\rangle_\Omega\right)\Big|_{t-T}^t = \left(\Big\langle\Lambda^\top\varphi^m,\begin{bmatrix}u\\v\end{bmatrix}\Big\rangle_I\right)\Big|_0^1. \tag{25}
$$

Imposing the initial condition $\varphi(x,0)=0$ and the final condition $\varphi(x,T) = [0\,,\,w(x)\Psi^m(x)]^\top$ leads to

$$
\begin{aligned}
[\Big\langle\varphi^m,\begin{bmatrix}u\\v\end{bmatrix}\Big\rangle_\Omega]_{t-T}^t &= \begin{bmatrix}0\\ \int_0^1 w(x)\Psi^m(x)\sum_{i=0}^\infty c_2^i(t)\Psi^i(x)\mathrm{d}x\end{bmatrix} \\
&= \begin{bmatrix}0\\ \sum_{i=0}^\infty c_2^i(t)\langle\Psi^m(x),\Psi^i(x)\rangle_w\end{bmatrix} = \begin{bmatrix}0\\ c_2^m(t)\end{bmatrix}
\end{aligned} \tag{26}
$$

and thus introduces the m-th coefficient of the function expansion (20) into (25) by

$$
\begin{aligned}
\begin{bmatrix}0\\ c_2^m(t)\end{bmatrix} &= \left(\Big\langle\Lambda^\top\varphi^m(0,t),\begin{bmatrix}u\\v\end{bmatrix}\Big\rangle_I\right)\Big|_0^1 = \\
\epsilon&\begin{bmatrix}-\langle\varphi_1^m(1,t),Y(t)\rangle_I + \langle\varphi_1^m(0,t),-v(0,t)+q_0(t)-\chi\rangle \\ \langle\varphi_2^m(1,t),U(t)\rangle_I - \langle\varphi_2^m(0,t),v(0,t)\rangle_I\end{bmatrix}.
\end{aligned} \tag{27}
$$

To eliminate the dependence on the unknown state $v(0,t)$, we impose the boundary condition $\varphi_1^m(0,\tau) = -\varphi_2^m(0,\tau)$ and sum up the vector entries on both sides of (25). Consequently, the m-th coefficient of the function expansion (20) can be estimated by the algebraic relation

$$
c_2^m(t) = \epsilon\left(\langle\varphi_2^m(1),U\rangle_I - \langle\varphi_1^m(1),Y\rangle_I + \langle\varphi_1^m(0),q_0\text{-}\chi\rangle_I\right). \tag{28}
$$

■

This state estimation algorithm for the transformed system Σ_{tr} enables to take advantage of the leak localization approach derived in [18] as the following theorem shows.

Theorem 3: Consider the pipe model Σ_{phy} with $F>0$ and suppose that a leak occurs, i.e. $\chi>0$. Define the leak localization observer by

$$
\Sigma_\delta:\begin{cases}\hat{p}_0(t) = \frac{\sqrt{\beta\rho}}{A}\left(\hat{u}(0,t)-\hat{v}(0,t)\right) \\ \qquad\qquad + \frac{\rho}{A^2}\eta\hat{\chi} + \frac{F}{A}\hat{\delta}(t)\hat{\chi}(t) \\ \dot{\hat{\delta}} = \underset{[0,\ell]}{\text{proj}}\{\gamma\left(p_0(t)-\hat{p}_0(t)\right)\}\end{cases} \tag{29}
$$

with leak localization observer gain $\gamma>0$ where the projection operator $\text{proj}(\cdot)$ enforces the physical constraint $z^*\in(0,\ell)$. Then, $\lim_{t\to\infty}\hat{\delta}(t) = \delta_d(0)$.

Proof: see [18, Theorem 10]. ■

Finally, the assumption that the leak position is given by a point leak appearing at $z^*\in(0,\ell)$ leads to the following corollary, completing the tasks of leak size estimation and localization.

Corollary 3.1: Consider the pipe model Σ_{phy} with a point leak located at $z=z^*\in(0,\ell)$ of size $\chi>0$, i.e., $d(z)$ is a Dirac-Impulse located at z^*. Then, $\delta_d(0)=z^*$ and with Theorem 3 follows $\lim_{t\to\infty}\hat{\delta}(t) = \delta_d(0) = z^*$.

Proof: see [18, Corollary 12]. ■

The last part of this section addresses the implementation of the observer and the generation of the required MF's by solving the auxiliary problems $\Sigma_{\text{aux},\chi}$ and $\Sigma_{\text{aux},u}$, $\Sigma_{\text{aux},v}$.

D. Solution of the Auxiliary Problem

In order to solve the auxiliary problems $\Sigma_{\text{aux},u}$, $\Sigma_{\text{aux},v}$, we take advantage of the results from [26] where an equivalent problem is solved. By reversing the time t and flipping the coordinates of the MF according to the transformation for $\sigma \in [0, T]$

$$\begin{bmatrix} \xi_1^m(x,\sigma) \\ \xi_2^m(x,\sigma) \end{bmatrix} := \begin{bmatrix} \varphi_2^m(x, T-\sigma) \\ \varphi_1^m(x, T-\sigma) \end{bmatrix}, \tag{30}$$

we bring the auxiliary problems $\Sigma_{\text{aux},u}$, $\Sigma_{\text{aux},v}$ into the form considered in [26]. Then the control law $\eta(\cdot)$ presented in the following theorem realizes the required transition from the origin $\varphi(x, 0)$ to the desired final condition $\varphi(x, T)$.

Theorem 4: Consider the 2×2 hyperbolic PDE system

$$\Sigma_\xi : \begin{cases} \xi_\sigma(x,\sigma) = \begin{bmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{bmatrix} \xi_x(x,\sigma) + \begin{bmatrix} 0 & c_1(x) \\ c_2(x) & 0 \end{bmatrix} \xi(x,\sigma) \\ \xi_1(0,\sigma) = -\xi_2(0,\sigma) \\ \xi(x,0) = \xi_0 \\ \xi_2(1,\sigma) = \eta(\sigma) \end{cases}$$

with $x \in [0,1], c_1(\cdot) \in \mathcal{C}([0,1],\mathbb{R}), c_2(\cdot) \in \mathcal{C}([0,1],\mathbb{R}), \epsilon > 0, \xi_0 \in \mathcal{L}^2([0,1],\mathbb{R}^2)$. Then, the control law

$$\eta(\sigma) = \int_0^1 K^{vu}(1,\zeta)\xi_1(\zeta,\sigma)\mathrm{d}\zeta + \int_0^1 K^{vv}(1,\zeta)\xi_2(\zeta,\sigma)\mathrm{d}\zeta \tag{31}$$

where the control kernels K^{vu}, K^{vv} are given as the solution of (see Equations (24)–(31) in [26])

$$\Sigma_{\mathrm{K}} : \begin{cases} \epsilon \frac{\partial K^{vu}}{\partial x} - \epsilon \frac{\partial K^{vu}}{\partial \zeta} = c_2(\zeta) K^{vv} \\ \epsilon \frac{\partial K^{vv}}{\partial x} - \epsilon \frac{\partial K^{vv}}{\partial \zeta} = c_1(\zeta) K^{vu} \\ K^{vu}(x,x) = -\frac{c_2(x)}{2\epsilon} \\ K^{vv}(x,0) = -K^{vu}(x,0) \end{cases} \tag{32}$$

stabilizes the system Σ_ξ in finite time $t_F = \frac{2}{\epsilon}$.

Proof: see [26, Theorem 1]. ∎

Furthermore, a solution of the auxiliary problem $\Sigma_{\text{aux},\chi}$ is obtained by utilizing control law (31) and the transformation (30) to first steer the MF $\varphi(x,\tau)$ from the origin $\varphi(x, 0) = 0$ to an arbitrary state $\varphi(x, \frac{T}{2}) = \varphi_{\frac{T}{2}}$, and then from $\varphi(x, \frac{T}{2}) = \varphi_{\frac{T}{2}}$ back to the origin $\varphi(x, T) = 0$.

Moreover, a closed form solution of Σ_{K} is found in [18, Lemma 6] where the same kernels Σ_{K} are used to calculate the observer gains of the proposed adaptive backstepping observer for leak size estimation and localization.

IV. Experimental Results

In this section, the proposed MF-based leak detection and localization observer is validated with measurement data, taken from a prototype installed at the laboratory for Advanced Control Engineering at the PUCP. The parameters of the prototype are summarized in Table I. Due to laboratory space limitations, the prototype is not a straight pipe but is arranged in the form of a coil that consists of horizontal sections that are connected by U-shaped parts and elbow joints, as the P & I diagram in Figure 1 illustrates.

TABLE I: Prototype parameters.

Length	Cross-sectional area	Bulk modulus	Density	Gravity
ℓ[m]	A[mm^2]	β[Pa]	$\rho[\frac{\text{kg}}{\text{m}^3}]$	$g[\frac{\text{m}}{\text{s}^2}]$
94.56	2164.8	$2.1 \cdot 10^9$	1000	9.81

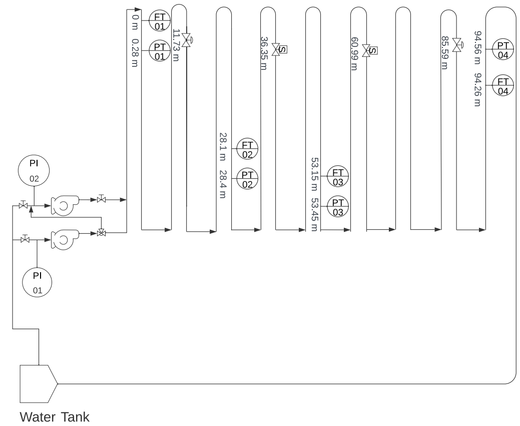
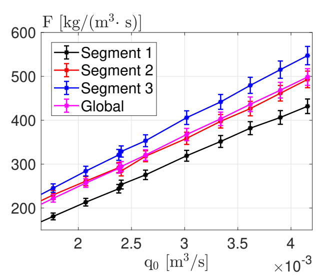


Fig. 1: Schematic P & I Diagram of the prototype.

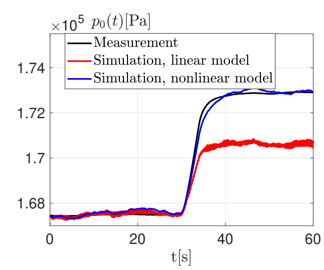
The flow rate and the pressure are measured with magnetic flow meters and differential pressure meters at the inlet and the outlet of the pipe, as well as at two intermediate points, allowing to consider shorter segments of the pipe. Moreover, a leak in the prototype can be provoked by opening one of the four solenoid and proportional valves.

A. Friction Identification & Model Validation

In view of the linear pipe model Σ_{phy}, the friction factor F remains as the only unknown to fully parametrize the model in the nominal case without leakage. The friction factor F is proportional to the steady state pressure gradient between the inlet to the outlet and, hence, can be calculated from the measured pressure and flow rate at the inlet and the outlet for a constant inlet flow rate $q_0(t)$. In Figure 2a the identified friction factor F is displayed over the inlet flow rate $q_0(t)$ for the entire pipe and for each of the three pipe segments by utilizing the intermediate pressure meters. In



(a) Identification of the friction factor F.



(b) Measured and simulated inlet pressure $p_0(t)$.

Fig. 2: Experimental considerations regarding system identification (a) and model validation (b).

opposite to the model assumption, the friction factor F is not constant, but differs significantly from segment to segment and increases for larger flow rates. The spatial variation of the friction factor F is explained by the elbow joints and U-shaped parts which cause additional friction losses in comparison to the horizontal pipe parts (see, e.g., [27], [28]) and whose number varies from pipe segment to pipe segment. Moreover, it is observed that the friction factor F depends approximately linearly on the flow rate and changes by over $1\,\%$ if the inlet flow rate changes by $1\,\%$. Thus, a nonlinear model according to [14] with friction losses that quadratically depend on the flow rate describes the fluid dynamics more accurately, as revealed by the comparison of the measured and the simulated inlet pressure in Figure 2b for a stepwise increase in the inlet flow.

B. Evaluation

To validate the theoretical results from Section III, the proposed MF-based leak detection and localization algorithm is applied to the experimental data from the prototype. Its performance is evaluated in comparison to the observer presented in [18]. To this end, the flow rate and pressure measurements are logged and then filtered with a moving average filter with a time window of length $T = 5$ s to mitigate the fluctuations of the inlet flow rate caused by the pumps. The water pipe prototype operates in steady state with nominal inlet flow rate of $q_0(t) = 3.1\,\frac{\mathrm{l}}{\mathrm{s}}$ and the leak occurs after $t = 30$ s by opening one of the valves.

The required modulating functions for the leak size and state estimation are generated by solving the auxiliary problems $\Sigma_{\text{aux},\chi}$ and $\Sigma_{\text{aux},u}$, $\Sigma_{\text{aux},v}$ using the control law presented in Theorem 4 and first order finite differences for an interval length of $T_\chi = 1$ s and $T_u = T_v = 3.9$ s, respectively. For the function expansion (20), an approximation order of $N = 1$ is selected, and the basis functions $\{\Psi^k\}_{k=0}^{\infty}$ are of polynomial shape w.r.t. weighting function $w(x) = 20x^2(1-x)^2$. The gain for the leak localization according to (29) is chosen as $\gamma = 0.003$. In comparison, the observer presented in [18] is parametrized with $L = -100$, $\gamma = 0.003$ and discretized by using first order finite differences with $n = 90$ spatial nodes. Both observers are implemented with a sampling time of $T_s = 1$ ms. In the following, we examine two different scenarios for the leak position.

In the first scenario, the last valve located at $z^* = 85.59$ m is opened, leading to a leak size of approximately $2.5\,\%$ of the nominal flow rate. The estimated leak size and leak position is shown in Figure 3. The proposed observer accurately reconstructs the leak size with a mean deviation of under $0.01\,\%$. Moreover, the leak is localized with a mean deviation of about $1\,\%$. In comparison, the leak position estimated by the ABSO converges more rapidly, but has a higher mean deviation of about $2.5\,\%$.

In the second scenario, the leak is provoked by opening the first valve located at $z^* = 11.73$ m. As Figure 4 shows, the leak size is still estimated accurately by both observers. However, neither the proposed observer nor the ABSO provide a valid estimate of the leak position and remain at $z^* = 0$. This decrease in accuracy of the leak localization is explained by the observations on the friction factor F in Section IV-A. Since the leak occurs close to the inlet, a large part of the pipe operates in a lower flow regime than in the steady state case. Along with the results from Figure 2a, this part has a significantly lower friction

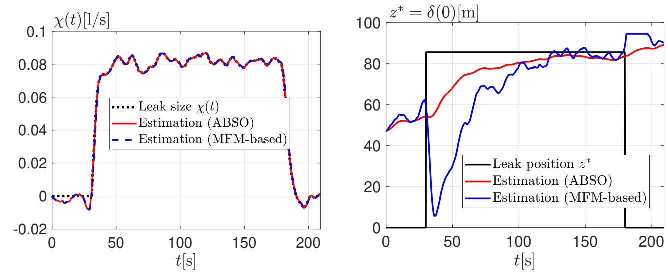


Fig. 3: Estimation of the leak size χ (left) and the leak position z^* (right), leak at $z^* = 85.59$ m.

factor F than the part in front of the pipe. Consequently, the linear pipe model Σ_{phy} assuming a spatially and temporally constant friction factor F in this case does not reflect well the fluid dynamics and both model-based observers fail to localize the leak accurately.

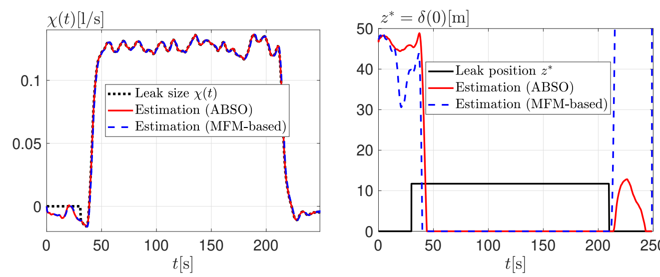


Fig. 4: Estimation of the leak size χ (left) and the leak position z^* (right), leak at $z^* = 11.73$ m.

V. Conclusion

This paper contributes to fault detection and identification in water pipes by developing and validating a new model-based late lumping leak detection approach to estimate the size and the position of leaks. By applying the modulating function method, the leak size estimation and localization problem is mapped into algebraic I/O equations and a system of auxiliary PDEs whose solvability is shown. Since these auxiliary PDEs can be solved offline, the implementation of the presented leak detection and localization observer does not require any spatial discretization of the pipe model, neither for the derivation of the observer algorithm nor for implementation purposes.

Moreover, this publication validates for the first time the theoretical results for the observer design of a late lumping model-based leak detection and localization system with real experimental data from a water pipe prototype. The real-time capability of the proposed methods is confirmed by comparing computing time and time length of the logged measurements, as well as by on-line implementation of the presented leak detection and localization technique for the pilot plant at the PUCP in MATLAB® / Simulink® reading in the pressure and flow rate measurements via an OPC Server. Furthermore, it is demonstrated that the proposed approach provides accurate leak size and leak position estimates compared to an existing late lumping model-based observer, if the leakage occurs near to the outlet. However, the underlying model assumption of linear friction losses is not verified, such that both observers fail to localize leaks that are further away from the outlet.

Consequently, the further development of the presented approach includes the extension of the MF-based observer for a nonlinear friction model that is supposed to cover the flow dynamics more adequately, similar to the extension of the ABSO presented in [29], in order to improve the accuracy of the leak localization.

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REFERENCES

[1] OECD, *Water Governance in Cities (OECD Studies on Water)*. Paris: OECD Publishing, 2016. Available at `https://www.pseau.org/outils/ouvrages/ocde/_water/_governance/_in/_cities/_2016.pdf`.

[2] R. Liemberger and A. Wyatt, "Quantifying the global non-revenue water problem," *Water Science and Technology: Water Supply*, vol. 19, no. 3, pp. 831–837, 2018.

[3] P.-S. Murvay and I. Silea, "A survey on gas leak detection and localization techniques," *Journal of Loss Prevention in the Process Industries*, vol. 25, no. 6, pp. 966–973, 2012.

[4] M. Adegboye, W.-K. Fung, and A. Karnik, "Recent Advances in Pipeline Monitoring and Oil Leakage Detection Technologies: Principles and Approaches," *Sensors*, vol. 19, no. 11, 2019. Article 2548.

[5] I. F. Tello, A. Wouwer, and D. Coutinho, "A concise review of state estimation techniques for partial differential equation systems," *Mathematics*, vol. 9, no. 24, 2021. Article 3180.

[6] A. Navarro, O. Begovich, J. Sanchez-Torres, G. Besançon, and J. Murillo, "Leak detection and isolation using an observer based on robust sliding mode differentiators," in *World Automation Congress 2012*, pp. 1–6, 2012.

[7] W. Rahiman and Z. Ding, "Leak detection in pipeline via sliding motion," in *International Conference on Control, Automation and Systems*, pp. 749–754, 2017.

[8] J. Xie, X. Xu, and S. Dubljevic, "Long range pipeline leak detection and localization using discrete observer and support vector machine," *Sustainable Energy: Process Systems Engineering*, vol. 65, no. 7, 2019. Article e16532.

[9] C. Verde, "Minimal order nonlinear observer for leak detection," *Journal of Dynamic Systems Measurement and Control*, vol. 126, no. 3, pp. 467–472, 2004.

[10] L. Torres, G. Besançon, and D. Georges, "A collocation model for water-hammer dynamics with application to leak detection," in *IEEE Conference on Decision and Control*, pp. 3890–3894, 2008.

[11] B. Arifin, Z. Li, and S. Shah, "Pipeline Leak Detection Using Particle Filters," in *9th IFAC Symposium on Advanced Control of Chemical Processes*, pp. 76–81, 2015.

[12] G. Besançon, "Observer tools for pipeline monitoring," in *Modeling and Monitoring of Pipelines and Networks* (C. Verde and L. Torres, eds.), vol. 7, ch. 5, pp. 83–97, Springer, 1 ed., May 2017.

[13] L. Torres, J. Jiménez-Cabas, O. Gonzalez, and F.-R. López-Estrada, "Kalman filters for leak diagnosis in pipelines: Brief history and future research," *Journal of Marine Science and Engineering*, vol. 8, no. 3, 2020. Article 173.

[14] J.-F. Dulhoste, M. Guillén, G. Besançon, and R. Santos, "One-dimensional modeling of pipeline transients," in *Modeling and Monitoring of Pipelines and Networks* (C. Verde and L. Torres, eds.), vol. 7, ch. 4, pp. 63–81, Springer, 1 ed., May 2017.

[15] G. Geiger, "Principles of leak detection," Reken, 2005. KROHNE Oil and Gas.

[16] J. Delgado-Aguiñaga and O. Begovich, "Water leak diagnosis in pressurized pipelines: A real case study," in *Modeling and Monitoring of Pipelines and Networks* (C. Verde and L. Torres, eds.), vol. 7, ch. 12, p. 235–262, Springer, 1 ed., May 2017.

[17] J. Delgado-Aguiñaga, V. Puig, and F. Becerra-López, "Leak diagnosis in pipelines based on a Kalman filter for Linear Parameter Varying systems," *Control Engineering Practice*, vol. 115, October 2021. Article 104888.

[18] O. Aamo, "Leak detection, size estimation and localization in pipe flows," *IEEE Transactions on Automatic Control*, vol. 61, no. 1, pp. 246–251, 2016.

[19] H. Anfinsen and O. Aamo, "Leak detection, size estimation and localization in branched pipe flows," *Automatica*, vol. 140, June 2022. Article 110213.

[20] F. Fischer, J. Deutscher, and T. Laleg-Kirati, "Source estimation for first order time-varying hyperbolic systems," in *International Symposium on Mathematical Theory of Networks and Systems*, pp. 78–84, 2018.

[21] L. Ghaffour, M. Noack, J. Reger, and T. Laleg-Kirati, "Non-asymptotic state estimation of linear reaction diffusion equation using modulating functions," in *21st IFAC World Congress*, pp. 4196–4201, 2020.

[22] R. A. Bajura, "A Model for Flow Distribution in Manifolds," *Journal of Engineering for Power*, vol. 93, no. 1, pp. 7–12, 1971.

[23] J. Jouffroy and J. Reger, "Finite-time simultaneous parameter and state estimation using modulating functions," in *IEEE International Conference on Control Applications*, pp. 394 – 399, 2015.

[24] F. Fischer and J. Deutscher, "Flatness-based algebraic fault diagnosis for distributed-parameter systems," *Automatica*, vol. 117, July 2020. Article 108987.

[25] D. Pumaricra Rojas, M. Noack, J. Reger, and G. Pérez-Zúñiga, "State Estimation for Coupled Reaction-Diffusion PDE Systems Using Modulating Functions," *Sensors*, vol. 22, no. 13, 2022. Article 5008.

[26] R. Vazquez, M. Krstic, and J.-M. Coron, "Backstepping boundary stabilization and state estimation of a 2×2 linear hyperbolic system," in *IEEE Conference on Decision and Control and European Control Conference*, pp. 4937–4942, 2011.

[27] H. Hassoon, "Pressure drop in $180°$ pipe bends," *Building Services Engineering Research and Technology*, vol. 3, no. 2, pp. 70–74, 1982.

[28] I. Idelchik and M. Steinberg, *Handbook of Hydraulic Resistance*. Begell House, 3 ed., 1996.

[29] N. Wilhelmsen and O. Aamo, "Distributed observer-based leak detection in pipe flow with nonlinear friction." unpublished, Extended Abstract for the 2nd IFAC Workshop on Control Methods for Water Resource Systems, 2022.