2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

Synthesis of constrained robust feedback policies and model predictive control

Dennis Gramlich, Carsten W. Scherer, Hannah Häring and Christian Ebenbauer

***Abstract*— In this work, we develop a method based on robust control techniques to synthesize robust time-varying state-feedback policies for finite, infinite, and receding horizon control problems subject to convex quadratic state and input constraints. To ensure constraint satisfaction of our policy, we employ (initial state)-to-peak gain techniques. Based on this idea, we formulate linear matrix inequality conditions, which are simultaneously convex in the parameters of an affine control policy, a Lyapunov function along the trajectory and multiplier variables for the uncertainties in a time-varying linear fractional transformation model. In our experiments this approach is less conservative than standard tube-based robust model predictive control methods.**

I. INTRODUCTION

In this paper, we deal with the synthesis of robust policies subject to linear time-varying discrete-time dynamics and convex constraints. Solving this type of control problem has applications, for example, in robust model predictive control (MPC) and trajectory planning.

Constraint satisfaction and robustness to uncertainties in dynamical systems are among the most important design objectives in controller synthesis. Controller design methods addressing these objectives are typically ascribed to the fields of optimal and robust control. A key issue in this setting is the complexity of the policy even for linear quadratic control problems if these are subject to constraints [40] or uncertainties [32]. In fact, no method is known today to solve these problems exactly with polynomial complexity (over an infinite horizon). For this reason, relaxation techniques are employed to find suboptimal solutions. Specifically, robust control relies on multiplier relaxations [11], [35], [39] to guarantee robustness against uncertainties and, e.g., MPC relaxes the infinite horizon optimal control problem by approximating it with a finite horizon problem.

Given these facts, a natural approach to address control problems with constraints and uncertainties is to combine both relaxation techniques, i.e., to study the finite horizon robust control problem subject to constraints. This is the topic of the present paper. To avoid overly conservative stability and feasibility results [28], we optimize over affine linear feedback policies instead of feedforward policies. This is by now generally accepted as the preferred approach for problems with uncertainties [12]. Specifically, to deal with uncertainties, we utilize multipliers from robust control. To ensure robust constraint satisfaction, we define the constraints as outputs of our system and use the energy-to-peak gain [34] to bound the maximum of these outputs for a given initial state. The resulting optimization problem can be convexified in all variables as a linear matrix inequality (LMI) problem.

The fusion of robust control with receding horizon optimal control has been studied since at least 1987 [6]. From this perspective, the present paper follows the ideas outlined by Kothare in [21], [22], who proposed to solve time-invariant robust control problems subject to constraints online, taking the current state into account. The time-invariance introduces conservatism, which is why [8], [7], [36] proposed to additionally optimize over a N-step input sequence using Kothares MPC scheme as terminal ingredient. Convex optimization of time-varying feedback policies subject to constraints and uncertainties is then established by minimax MPC as described, e.g., in [25]. All these methods rely on solving LMI optimization problems online, which is often considered costly. On the other hand, as we show in [13] that the structure of LMIs for robust controller synthesis can be exploited to develop faster solvers, which is particularly interesting for online applications like MPC.

After minimax MPC, so-called tube-based MPC methods with fixed feedback terms [24] gained much attention due to their seemingly lower computational complexity. However, this reduced computational effort comes at the price of not being able to optimize the tube and tube controller online and tightening the constraints, resulting in conservatism. Furthermore, optimizing robust performance and incorporating model uncertainty descriptions from robust control is challenging in tube-based MPC. In recent publications, the integration of classical robust control tools, like dynamic IQCs [29], [37], into MPC is explored. In addition, optimizing tube parameters online to reduce conservatism has gained interest [31].

In parallel to robust MPC, finite horizon robust controller design is investigated from a robust control per-

The second author thanks Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - EXC 2075 – 390740016 for funding and acknowledges the support by the Stuttgart Center for Simulation Science (SimTech).

Dennis Gramlich, Hannah Häring and Christian Ebenbauer are with the Chair of Intelligent Control Systems, RWTH Aachen University, 52074 Aachen, Germany {dennis.gramlich, hannah.haering ,christian.ebenbauer} @ic.rwth-aachen.de

Carsten W. Scherer is with the Chair of Mathematical Systems Theory, University of Stuttgart, 70569 Stuttgart, Germany carsten.scherer@imng.uni-stuttgart.de

spective [38], [3], [14] using LMI-based tools. Constraints have been incorporated recently into such finite horizon robust control settings under the umbrella of reachability analysis [33]. This synthesis technique is extended to joint optimization over feedforward and feedback terms in [19] for Lipschitz continuous uncertain nonlinearities.

Table I gives an overview of robust MPC schemes from the literature. We indicate whether the MPC methods are recursively feasible (RF), can incorporate linear fractional transformations (LFTs) as uncertainty models, optimize over feedback policies online (OFPO), involve an LMI optimization problem, and how the number of decision variables grows with the prediction horizon. Integrating LFTs is beneficial as these representations of uncertain systems are capable of handling rational dependencies of system matrices on uncertain parameters. Additionally, optimizing feedback policies online is desirable, since this increases the robustness of MPC schemes. LMI optimization problems should be avoided, because the typical online optimization problem in linear MPC is a less costly to solve quadratically constrained quadratic program. Finally, it is desirable that the number of decision variables grows at most linearly in N.

Among the multitude of tube-based MPC methods in the literature, some allow for flexible uncertainty descriptions such as LFTs. However, the offline calculation of the feedback terms leads to unavoidable conservatism. This is in contrast to the other methods considered, which perform these calculations online. Here, LMI-MPC [25] and SLSMPC can be considered as disturbance feedback MPC methods, since LMI-MPC uses disturbance feedback and SLSMPC optimizes the closed loop transfer function from disturbances to outputs. These methods can achieve any of the desired properties in Table I. However, we mention that LFTs can only be handled when incorporating LMIs. Finally, we also analyzed algorithms [22], which solve time-invariant controller design problems online. Through this, feedback terms are optimized online, LFTs can be considered, and the optimization is recursively feasible. Constraints are taken into account using the (initial state)-to-peak gain. Here, conservatism arises from the assumption of quadratic Value/Lyapunov functions and the time invariance of the controller law. The present approach generalizes [22] to the time-varying case, removing the conservatism caused by time-invariant feedback policies. Compared to disturbance feedback approaches, the quadratic value function still poses a source of conservativeness, but, on the other hand, recursive feasibility is much easier to obtain and the number of decision variables grows only linearly with the prediction horizon. Moreover, LMI-based robust controller design is embedded in our approach.

We start our exposition with Section II demonstrating how the energy-to-peak gain enables the integration of constraints in LMI formulations of robust control. The problem formulation of Section II does not consider uncertainties in which case we establish that our convexification is feasible at an initial state $\bar{x}$, whenever the corresponding open loop optimal control problem is feasible at $\bar{x}$. Uncertainties are included in Section III. We introduce another relaxation step in Section IV to also handle infinite horizon robust optimal control problems with constraints. Finally, in Section V this formulation is shown to be stable and recursively feasible if used in a receding horizon fashion.

TABLE I: This table compares the feastures recursive feasibility (RF), linear fractional representation uncertainty model (LFT), online feedback policy optimization (OFPO), convex program to be solved online (CP) and the asymptotic number of decision variables in the prediction horizon N of the MPC schemes *tube MPC* [24], [26], [27], [20], ∞-*hor. MPC* [22], [9], SLSMPC [10], DFMPC [12] (disturbance feedback MPC), LMI-MinMax MPC [25] and our approach.

method	RF	LFTs	OFPO	CP	# dec. v.
Tube MPC	Yes	(Yes)	No	QP	$O(N)$
∞-hor. MPC	Yes	Yes	Yes	SDP	-
SLSMPC	No	No	Yes	QP	$O(N^2)$
DFMPC	(Yes)	(Yes)	Yes	QP/SDP	$O(N^2)$
LMI-MPC	Yes	Yes	Yes	SDP	$O(N^2)$
Our method	Yes	Yes	Yes	SDP	$O(N)$

II. The uncertainty-free finite-horizon case

In a first step, we restrict ourselves to time-varying affine systems without uncertainties of the form

$$x_{k+1} = f_k + A_k x_k + B_k^1 u_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state and $u_k \in \mathbb{R}^m$ is the control input and $f_k \in \mathbb{R}^n$ is the affine term. We further assume that the control policy satisfies some quadratic constraints, i.e., u_k should be chosen such that

$$v_{ki}^\top v_{ki} \leq 1 \qquad \forall i = 1, \ldots, s, \quad k = 0, \ldots, N \tag{2}$$

holds for affine outputs $v_{ki} = g_{ki}^2 + C_{ki}^2 x_k + D_{ki}^{21} u_k$. Note that the affine terms g_{ki}^2 enable the consideration of any polytopic feasible set.

For systems (1) with constraints (2), an initial state $\bar{x}$, and a positive definite matrix P_f, we study the problem

$$\begin{aligned} \underset{(u_k)_{k=0}^{N-1}}{\text{minimize}} \quad & \sum_{k=0}^{N-1} y_k^\top y_k + \begin{pmatrix} 1 \\ x_N \end{pmatrix}^\top P_f \begin{pmatrix} 1 \\ x_N \end{pmatrix} \\ \text{s.t.} \quad & x_{k+1} = f_k + A_k x_k + B_k^1 u_k, \\ & y_k = g_k^1 + C_k^1 x_k + D_k^{11} u_k, \\ & v_{ki} = g_{ki}^2 + C_{ki}^2 x_k + D_{ki}^{21} u_k, \quad i = 1, \ldots, s, \\ & v_{ki}^\top v_{ki} \leq 1, \quad i = 1, \ldots, s, \\ & x_0 = \bar{x}. \end{aligned} \tag{3}$$

To approach this optimal control problem, we construct a family of functions $V_k : \mathbb{R}^n \to \mathbb{R}$ with

$$V_k(x_k) = \begin{pmatrix} 1 \\ x_k \end{pmatrix}^\top \underbrace{\begin{pmatrix} p_k^{11} & p_k^{12} \\ p_k^{21} & P_k^{22} \end{pmatrix}}_{=:P_k} \begin{pmatrix} 1 \\ x_k \end{pmatrix} \tag{4}$$

defined by a sequence of symmetric matrices $P_k \in \mathbb{R}^{(1+n)\times(1+n)}$, a sequence of affine linear control policies $u_k = \pi_k(x_k) = k_k^1 + K_k^2 x_k = K_k \begin{pmatrix} 1 & x_k^\top \end{pmatrix}^\top$, and a constant $\nu \in \mathbb{R}$ satisfying the conditions

$$
\begin{aligned}
V_k(x) &\geq y^\top y + V_{k+1}(x^+), && \text{(5a)}\\
&\text{for } \; x^+ = f_k + B_k^1 k_k^1 + (A_k + B_k^1 K_k^2)x,\\
&\phantom{\text{for }} \; y = g_k^1 + D_k^{11} k_k^1 + (C_k^1 + D_k^{11} K_k^2)x,\\
V_0(\bar{x}) &\leq \nu, && \text{(5b)}\\
V_N(x) &\geq \begin{pmatrix} 1 \\ x \end{pmatrix}^\top P_f \begin{pmatrix} 1 \\ x \end{pmatrix}, && \text{(5c)}\\
V_k(x) &\leq \nu \Rightarrow v_i^\top v_i \leq 1, && \text{(5d)}\\
&\text{for } \; v_i = g_{ki}^2 + D_{ki}^{21} k_k^1 + (C_{ki}^2 + D_{ki}^{21} K_k^2)x,
\end{aligned}
$$

for all $x \in \mathbb{R}^n$, $k = 0, \ldots, N-1$ and $i = 1, \ldots, s$. Condition (5a) states that V_k decreases at least by the cost $y_k^\top y_k$ at every time-step, (5b) signifies that the initial state is contained in a ν-sublevel set of V_0, (5c) implies that V_N upper bounds the terminal cost. Lastly, (5d) means that states in the ν-sublevel set of V_k satisfy all constraints.

We can formulate (5) as LMI constraints in $(P_k)_{k=0}^N$ and ν.

Proposition 2.1: Let the functions V_k be parametrized as in (4) and define $\Sigma_0 := \begin{pmatrix} 1 & \bar{x}^\top \end{pmatrix}^\top \begin{pmatrix} 1 & \bar{x}^\top \end{pmatrix}$ as well as

$$
\begin{pmatrix} f_k^K & A_k^K \\ g_k^{1K} & C_k^{1K} \\ g_{ki}^{2K} & C_{ki}^{2K} \end{pmatrix} := \begin{pmatrix} f_k & A_k & B_k^1 \\ g_k^1 & C_k^1 & D_k^{11} \\ g_{ki}^2 & C_{ki}^2 & D_{ki}^{21} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & I \\ k_k^1 & K_k^2 \end{pmatrix}.
$$

Then, (5a)-(5c) are equivalent to the conditions

$$
0 \geq (\bullet)^\top \begin{pmatrix} -p_k^{11} & -p_k^{12} & & & \\ -p_k^{21} & -P_k^{22} & & & \\ & & p_{k+1}^{11} & p_{k+1}^{12} & \\ & & p_{k+1}^{21} & P_{k+1}^{22} & \\ & & & & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & I \\ 1 & 0 \\ f_k^K & A_k^K \\ g_k^{1K} & C_k^{1K} \end{pmatrix}, \tag{6a}
$$

$$
\nu \geq \operatorname{trace} P_0 \Sigma_0, \tag{6b}
$$

$$
P_N \geq P_f \tag{6c}
$$

respectively. Moreover, (5d) is implied by the inequality

$$
P_k \geq \nu \begin{pmatrix} g_{ki}^{2K} & C_{ki}^{2K} \end{pmatrix}^\top \begin{pmatrix} g_{ki}^{2K} & C_{ki}^{2K} \end{pmatrix}. \tag{6d}
$$

Proof: Left-multiplication with $\begin{pmatrix} 1 & x^\top \end{pmatrix}$ and right-multiplication with $\begin{pmatrix} 1 & x^\top \end{pmatrix}^\top$ of the matrix inequalities (6a) and (6c) reveals the equivalence to (5a) and (5c), respectively. The same argument shows (6d) $\Rightarrow$ (5d). Lastly, plugging the definition of Σ_0 into (6b) proves the equivalence to (5b). ■

Remark 2.2: Using the lossless S-procedure, it is possible to construct an equivalent LMI constraint for (5d) instead of (6d). However, we could not convexify the lossless constraint in all the parameters.

The matrix inequalities of Proposition 2.1 are linear in the Lyapunov function matrices $(P_k)_{k=0}^N$ and the variable ν. In addition, since we impose the constraint $\nu \geq \operatorname{trace} P_0 \Sigma_0$, we obtain the sequence of estimates

$$
\begin{aligned}
\nu &\geq V_0(x_0) \geq y_0^\top y_0 + V_1(x_1)\\
&\geq y_0^\top y_0 + y_1^\top y_1 + V_2(x_2)\\
&\geq \sum_{k=0}^{N-1} y_k^\top y_k + V_N(x_N)\\
&\geq \sum_{k=0}^{N-1} y_k^\top y_k + \begin{pmatrix} 1 \\ x_N \end{pmatrix}^\top P_f \begin{pmatrix} 1 \\ x_N \end{pmatrix}
\end{aligned}
$$

if we implement the control law $u_k = \pi_k(x_k) = k_k^1 + K_k^2 x_k = K_k \begin{pmatrix} 1 & x_k^\top \end{pmatrix}^\top$ and evaluate the inequalities along the resulting trajectory. Thus, by minimizing ν as objective function, we can try to find a smallest upper bound on the performance of our controller $(K_k)_{k=0}^{N-1}$ while certifying constraint satisfaction. However, the matrix inequalities of Proposition 2.1 are not linear in the controller $(K_k)_{k=0}^{N-1}$ such that they cannot be utilized for convex synthesis. For this reason, we provide a convexification in all variables in the following theorem.

Theorem 2.3: Define the decision variables

$$
\widetilde{P}_k := P_k^{-1}, \quad \widetilde{K}_k := K_k P_k^{-1}, \quad \tilde{\nu} := \nu^{-1}, \quad Z := \nu^{-2} \sqrt{\Sigma_0} P_0 \sqrt{\Sigma_0}
$$

including the slack variable Z and introduce the notation

$$
\begin{pmatrix} \tilde{f}_k^K & \widetilde{A}_k^K \\ \tilde{g}_k^{1K} & \widetilde{C}_k^{1K} \\ \tilde{g}_{ki}^{2K} & \widetilde{C}_{ki}^{2K} \end{pmatrix} := \begin{pmatrix} f_k & A_k & B_k^1 \\ g_k^1 & C_k^1 & D_k^{11} \\ g_{ki}^2 & C_{ki}^2 & D_{ki}^2 \end{pmatrix} \begin{pmatrix} \tilde{p}_k^{11} & \tilde{p}_k^{12} \\ \tilde{p}_k^{21} & \widetilde{P}_k^{22} \\ \tilde{k}_k^1 & \widetilde{K}_k^2 \end{pmatrix}.
$$

Then, for positive definite matrices $\widetilde{P}_k$ and P_k, (6a)-(6d) are equivalent to

$$
0 \leq \begin{pmatrix} \tilde{p}_{k+1}^{11} & \tilde{p}_{k+1}^{12} & 0 & \tilde{p}_k^{11} & \tilde{p}_k^{12} \\ \tilde{p}_{k+1}^{21} & \widetilde{P}_{k+1}^{22} & 0 & \tilde{f}_k^K & \widetilde{A}_k^K \\ 0 & 0 & I & \tilde{g}_k^{1K} & \widetilde{C}_k^{1K} \\ \tilde{p}_k^{11} & (\tilde{f}_k^K)^\top & (\tilde{g}_k^{1K})^\top & \tilde{p}_k^{11} & \tilde{p}_k^{12} \\ \tilde{p}_k^{21} & (\widetilde{A}_k^K)^\top & (\widetilde{C}_k^{1K})^\top & \tilde{p}_k^{21} & \widetilde{P}_k^{22} \end{pmatrix}, \tag{7a}
$$

$$
0 \leq \begin{pmatrix} \widetilde{P}_0 & \tilde{\nu}\sqrt{\Sigma_0} \\ \tilde{\nu}\sqrt{\Sigma_0}^\top & Z \end{pmatrix}, \quad \operatorname{trace} Z \leq \tilde{\nu}, \tag{7b}
$$

$$
\widetilde{P}_N \leq P_f^{-1}, \tag{7c}
$$

$$
0 \leq \begin{pmatrix} \tilde{p}_k^{11} & \tilde{p}_k^{12} & \tilde{g}_{ki}^{2K} \\ \tilde{p}_k^{21} & \widetilde{P}_k^{22} & \widetilde{C}_{ki}^{2K} \\ \tilde{g}_{ki}^{2K} & \widetilde{C}_{ki}^{2K} & \tilde{\nu} I \end{pmatrix}. \tag{7d}
$$

Proof: Multiplying (6a) from both sides by $\widetilde{P}_k$ yields

$$
0 \geq (\bullet)^\top \begin{pmatrix} -\tilde{p}_k^{11} & -\tilde{p}_k^{12} & & & \\ -\tilde{p}_k^{21} & -\widetilde{P}_k^{22} & & & \\ & & p_{k+1}^{11} & p_{k+1}^{12} & \\ & & p_{k+1}^{21} & P_{k+1}^{22} & \\ & & & & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & I \\ \tilde{p}_k^{11} & \tilde{p}_k^{12} \\ \tilde{f}_k^K & \widetilde{A}_k^K \\ \tilde{g}_k^{1K} & \widetilde{C}_k^{1K} \end{pmatrix}.
$$

To arrive at (7a), we apply the Schur complement. To infer (7d) from (6d) we proceed analogously. We multiply from both sides by $\widetilde{P}_k$ to obtain

$$\widetilde{P}_k \geq \nu \begin{pmatrix} \tilde{g}_{ki}^{2K} & \widetilde{C}_{ki}^{2K} \end{pmatrix}^\top \begin{pmatrix} \tilde{g}_{ki}^{2K} & \widetilde{C}_{ki}^{2K} \end{pmatrix}.$$

Now, applying the Schur complement lemma yields the LMI (7d). Next, consider the condition $\nu \geq \operatorname{trace} P_0 \Sigma_0 = \operatorname{trace} \widetilde{P}_0^{-1} \Sigma_0$. To handle this constraint, we add the slack variable $Z \geq \nu^{-2} \sqrt{\Sigma_0} \widetilde{P}_0^{-1} \sqrt{\Sigma_0}$. With this slack variable, we can replace the original constraint by

$$\tilde{\nu} \geq \operatorname{trace} Z, \qquad Z \geq \tilde{\nu}^2 \sqrt{\Sigma_0} \widetilde{P}_0^{-1} \sqrt{\Sigma_0}.$$

By taking the Schur complement, the above is equivalent to (7b). Finally, to convexify the constraint $P_N \geq P_f$, we invert the matrices on both sides leading to (7c). ■

With Theorem 2.3 we arrive at the convex relaxation of problem (3) optimized over control policies

$$\begin{aligned} \underset{\widetilde{P}_k, \widetilde{K}_k, \tilde{\nu}, Z}{\text{maximize}} \quad & \tilde{\nu} \\ \text{s.t.} \quad & \text{(7a) for } k = 0, \dots, N-1, \\ & \text{(7d) for } k = 0, \dots, N-1, \\ & \text{(7b), and (7c).} \end{aligned} \tag{8}$$

In general, such a convex relaxation provides only an upper bound on the optimal value of the considered problem. In the present case without uncertainties, we can nevertheless show that strict feasibility of (3) implies strict feasibility of (8).

Theorem 2.4: There exists an input sequence $(u_k)_{k=0}^{N-1}$ such that all constraints of (3) are satisfied and $v_{ki}^\top v_{ki}$ is strictly smaller than one for $k = 0, \dots, N-1$, $i = 1, \dots, s$, if and only if (8) is strictly feasible.

Proof: The proof can be found in [15]. ■

The optimal values of (3) and (8) are not necessarily equivalent, which is discussed in [15].

III. The finite-horizon case with uncertainty

Building on Section II, we can incorporate an additional disturbance input in our model and work with

$$x_{k+1} = f_k + A_k x_k + B_k^1 u_k + B_k^2 w_k. \tag{9}$$

In this setup, $w_k \in \mathbb{R}^l$ is the disturbance input, which is introduced in addition to the state $x_k \in \mathbb{R}^n$ and the input $u_k \in \mathbb{R}^m$. We assume that both the disturbances and the controller are subject to constraints, i.e., the control input needs to enforce (2). Moreover, we require the disturbance input to satisfy the quadratic constraint

$$\begin{pmatrix} z_k \\ w_k \end{pmatrix}^\top \underbrace{\begin{pmatrix} M_k^{11} & M_k^{12} \\ M_k^{21} & M_k^{22} \end{pmatrix}}_{=: M_k} \begin{pmatrix} z_k \\ w_k \end{pmatrix} \geq 0$$

for all elements M_k of a convex cone $\mathcal{M}$, where $z_k = g_k^3 + C_k^3 x_k + D_k^{31} u_k + D_k^{32} w_k$ is some output of our system. We further assume that M_k^{-1} exists for all $M_k \in \mathcal{M}$, that the inverse is contained in another convex cone $\mathcal{M}'$ and that for all $M_k \in \mathcal{M}$ the block M_k^{11} is positive definite and M_k^{22} is negative definite. Introducing quadratically constrained inputs w_k in this fashion is standard in robust control and can be utilized to describe simple bounded disturbances such as $\|w_k\| \leq 1$, but also parametric uncertainties modeled by LFTs as, e.g., in [16].

In the following dynamic program, we treat the disturbance as an adversary, i.e., we maximize over w_k while minimizing over u_k in the problem

$$\min_{u_0 \in \mathcal{U}_0} \max_{w_0 \in \mathcal{W}_0} \min_{u_1 \in \mathcal{U}_1} \max_{w_1 \in \mathcal{W}_1} \dots \sum_{k=0}^{N-1} y_k^\top y_k + \begin{pmatrix} 1 \\ x_N \end{pmatrix}^\top P_f \begin{pmatrix} 1 \\ x_N \end{pmatrix} \tag{10}$$

$$\begin{aligned} \text{s.t.} \quad & x_{k+1} = f_k + A_k x_k + B_k^1 u_k + B_k^2 w_k, \\ & y_k = g_k^1 + C_k^1 x_k + D_k^{11} u_k + D_k^{12} w_k, \\ & v_{ki} = g_{ki}^2 + C_{ki}^2 x_k + D_{ki}^{21} u_k, \qquad i = 1, \dots, s, \\ & z_k = g_k^3 + C_k^3 x_k + D_k^{31} u_k + D_k^{32} w_k, \\ & x_0 = \bar{x}, \end{aligned}$$

with sets $\mathcal{U}_k = \mathcal{U}_k(x_k)$ and $\mathcal{W}_k = \mathcal{W}_k(z_k)$ defined by

$$u_k \in \mathcal{U}_k(x_k) \Leftrightarrow v_{ki}^\top v_{ki} \leq 1 \qquad i = 1, \dots, s, \tag{11}$$

$$w_k \in \mathcal{W}_k(z_k) \Leftrightarrow \begin{pmatrix} z_k \\ w_k \end{pmatrix} M_k \begin{pmatrix} z_k \\ w_k \end{pmatrix} \geq 0 \quad \forall M_k \in \mathcal{M}. \tag{12}$$

Note that while minimizing over u_k and maximizing over w_k may seem like a symmetric situation, this is not the case from an optimization perspective. In particular, since the cost function is convex, minimizing over $(u_k)_{k=0}^{N-1}$ for fixed $(w_k)_{k=0}^{N-1}$ is a convex quadratically constrained quadratic program. On the other hand, maximizing over $(w_k)_{k=0}^{N-1}$ for fixed $(u_k)_{k=0}^{N-1}$ is a non-convex quadratically constrained quadratic program. For this reason, we cannot carry over the result of Theorem 2.4 from the uncertainty-free case.

Due to this asymmetry we treat constraints on u_k differently from constraints on w_k. Concretely, to handle (11), we once again search for functions V_k whose sublevel sets guarantee the fullfilment of $u_k \in \mathcal{U}_k(x_k)$, while (12) is taken into account through the S-procedure. Hence, to solve (10), we again construct a family of functions $(V_k)_{k=0}^N$ as in (4) and a control policy $\pi_k(x_k) = k_k^1 + K_k^2 x_k = K_k \begin{pmatrix} 1 & x_k^\top \end{pmatrix}^\top$ satisfying the inequalities (5b)-(5d). In contrast to the disturbance-free case, however, instead of (5a), the increment condition

$$\begin{aligned} & V_k(x) \geq y^\top y + V_{k+1}(x^+), \\ & \text{for} \quad x^+ = f_k + B_k^1 k_k^1 + (A_k + B_k^1 K_k^2) x + B_k^2 w, \\ & \qquad y = g_k + D_k^{11} k_k^1 + (C_k^1 + D_k^{11} K_k^2) x + D_k^{12} w, \end{aligned} \tag{13}$$

needs to be satisfied robustly with respect to w, i.e., for all $w \in \mathcal{W}_k(z_k)$ and all $x \in \mathbb{R}^n$. By the S-procedure, if there exists some $M_k \in \mathcal{M}$ such that for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^l$

$$V_k(x) \geq y^\top y + \begin{pmatrix} z \\ w \end{pmatrix}^\top M_k \begin{pmatrix} z \\ w \end{pmatrix} + V_{k+1}(x^+), \tag{14}$$

$$\begin{aligned} \text{for} \quad x^+ &= f_k + B_k^1 k_k^1 + (A_k + B_k^1 K_k^2)x + B_k^2 w, \\ y &= g_k^1 + D_k^{11} k_k^1 + (C_k^1 + D_k^{11} K_k^2)x + D_k^{12} w, \\ z &= g_k^3 + D_k^{31} k_k^1 + (C_k^3 + D_k^{31} K_k^2)x + D_k^{32} w \end{aligned}$$

holds, then this implies (13). Consequently, we can certify robust performance and robust constraint satisfaction using (14) and (5b)-(5d) as follows.

Proposition 3.1: Let the functions V_k be parametrized as in (4) and define

$$\begin{pmatrix} f_k^K & A_k^K \\ g_k^{1K} & C_k^{1K} \\ g_k^{3K} & C_k^{3K} \end{pmatrix} := \begin{pmatrix} f_k & A_k & B_k^1 \\ g_k^1 & C_k^1 & D_k^{11} \\ g_k^3 & C_k^3 & D_k^{31} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & I \\ k_k & K_k \end{pmatrix}.$$

Then, (14) is characterized by

$$(\bullet)^\top \begin{pmatrix} -p_k^{11} & -p_k^{12} & & & & & & \\ -p_k^{21} & -P_k^{22} & & & & & & \\ & & M_k^{22} & & & & & M_k^{21} \\ & & & p_{k+1}^{11} & p_{k+1}^{12} & & & \\ & & & p_{k+1}^{21} & P_{k+1}^{22} & & & \\ & & & & & & I & \\ & & M_k^{12} & & & & & M_k^{11} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 1 & 0 & 0 \\ f_k^K & A_k^K & B_k^2 \\ g_k^{1K} & C_k^{1K} & D_k^{12} \\ g_k^{3K} & C_k^{3K} & D_k^{32} \end{pmatrix} \leq 0. \tag{15}$$

Furthermore, the conditions (15) and (6b)-(6d) imply robust performance in the sense that ν upper bounds the optimal cost of (10) and robust constraint satisfaction in the sense that the policy $(\pi_k)_{k=0}^{N-1}$ chooses feasible inputs for any realization of $(w_k)_{k=0}^{N-1}$ satisfying (12).

Proof: To establish equivalence between (15) and (14), left-multiply $\begin{pmatrix} 1 & x^\top & w^\top \end{pmatrix}$ and right-multiply its transpose to (15). Summing (14) for $k = 0, \ldots, N-1$ with $x = x_k$ and $w = w_k$ chosen as a trajectory of (9) yields

$$V_0(x_0) \geq \sum_{k=0}^{N-1} \left(y_k^\top y_k + \begin{pmatrix} z_k \\ w_k \end{pmatrix}^\top M_k \begin{pmatrix} z_k \\ w_k \end{pmatrix} \right) + V_N(x_N). \tag{16}$$

Here, the terms $\begin{pmatrix} z_k \\ w_k \end{pmatrix}^\top M_k \begin{pmatrix} z_k \\ w_k \end{pmatrix}$ are positive for all admissible values of w_k and, hence, can be omitted without violating (16). This establishes the inequalities

$$V_0(x_0) \geq \sum_{k=0}^{N-1} y_k^\top y_k + V_N(x_N) \quad \text{and} \quad V_k(x_k) \geq V_{k+1}(x_{k+1}).$$

In a next step we use the relations in Proposition 2.1. From (6b) we conclude $V_0(x_0) \leq \nu$ and from (6c) we infer

$$\nu \geq \sum_{k=0}^{N-1} y_k^\top y_k + \begin{pmatrix} 1 \\ x_N \end{pmatrix}^\top P_f \begin{pmatrix} 1 \\ x_N \end{pmatrix},$$

showing the robust performance bound of ν and that $x_k^\top P_k x_k \leq \nu$ holds for all k. The latter fact together with (6d) implies robust constraint satisfaction. ■

As in the disturbance-free case, we are interested in matrix inequalities, which are also linear in the controller parameter. This is provided in the next theorem.

Theorem 3.2: Define the decision variables

$$\widetilde{P}_k := P_k^{-1}, \;\; \widetilde{K}_k := K_k P_k^{-1}, \;\; \tilde{\nu} := \nu^{-1}, \;\; Z := \nu^{-2} \sqrt{\Sigma_0} P_0 \sqrt{\Sigma_0}$$

including the slack variable Z, denote the matrix block

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ B_k^2 & 0 & I & 0 & 0 \\ D_k^{12} & 0 & 0 & I & 0 \\ D_k^{32} & 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} \widetilde{M}_k^{22} & & & & \widetilde{M}_k^{21} \\ & \tilde{p}_{k+1}^{11} & \tilde{p}_{k+1}^{12} & & \\ & \tilde{p}_{k+1}^{21} & \widetilde{P}_{k+1}^{22} & & \\ & & & I & \\ \widetilde{M}_k^{12} & & & & \widetilde{M}_k^{11} \end{pmatrix} (\bullet)^\top$$

by $\widetilde{Q}_k$ and introduce the notation

$$\begin{pmatrix} \tilde{f}_k^K & \widetilde{A}_k^K \\ \tilde{g}_k^{1K} & \widetilde{C}_k^{1K} \\ \tilde{g}_k^{3K} & \widetilde{C}_k^{3K} \end{pmatrix} := \begin{pmatrix} f_k & A_k & B_k^1 \\ g_k^1 & C_k^1 & D_k^{11} \\ g_k^3 & C_k^3 & D_k^{31} \end{pmatrix} \begin{pmatrix} \tilde{p}_k^{11} & \tilde{p}_k^{12} \\ \tilde{p}_k^{21} & \widetilde{P}_k^{22} \\ \tilde{k}_k^1 & \widetilde{K}_k^2 \end{pmatrix}.$$

Then the matrix inequalities (15), (6b), (6c) and (6d) are equivalent to the linear matrix inequalities

$$\begin{pmatrix} \tilde{q}_k^{11} & \tilde{q}_k^{12} & \tilde{q}_k^{13} & \tilde{q}_k^{14} & \tilde{p}_k^{11} & \tilde{p}_k^{12} \\ \tilde{q}_k^{21} & \widetilde{Q}_k^{22} & \widetilde{Q}_k^{23} & \widetilde{Q}_k^{24} & \tilde{f}_k^K & \widetilde{A}_k^K \\ \tilde{q}_k^{31} & \widetilde{Q}_k^{32} & \widetilde{Q}_k^{33} & \widetilde{Q}_k^{34} & \tilde{g}_k^{1K} & \widetilde{C}_k^{1K} \\ \tilde{q}_k^{41} & \widetilde{Q}_k^{42} & \widetilde{Q}_k^{43} & \widetilde{Q}_k^{44} & \tilde{g}_k^{3K} & \widetilde{C}_k^{3K} \\ \tilde{p}_k^{11} & (\tilde{f}_k^K)^\top & (\tilde{g}_k^{1K})^\top & (\tilde{g}_k^{3K})^\top & \tilde{p}_k^{11} & \tilde{p}_k^{12} \\ \tilde{p}_k^{21} & (\widetilde{A}_k^K)^\top & (\widetilde{C}_k^{1K})^\top & (\widetilde{C}_k^{3K})^\top & \tilde{p}_k^{21} & \widetilde{P}_k^{22} \end{pmatrix} \geq 0, \tag{17}$$

(7b), (7c) and (7d) for positive definite P_k and $\widetilde{P}_k$.

Proof: Applying the dualization lemma (Lemma 4, [18]) to (15) proves that (15) holds if and only if

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & I & 0 & 0 & f_k^K & A_k^K & B_k^2 \\ 0 & 0 & I & 0 & g_k^{1K} & C_k^{1K} & D_k^{12} \\ 0 & 0 & 0 & I & g_k^{2K} & C_k^{2K} & D_k^{22} \end{pmatrix} \begin{pmatrix} -\tilde{p}_k^{11} & -\tilde{p}_k^{12} & & & & & & \\ -\tilde{p}_k^{21} & -\widetilde{P}_k^{22} & & & & & & \\ & & \widetilde{M}_k^{22} & & & & & \widetilde{M}_k^{21} \\ & & & \tilde{p}_{k+1}^{11} & \tilde{p}_{k+1}^{12} & & & \\ & & & \tilde{p}_{k+1}^{21} & \widetilde{P}_{k+1}^{22} & & & \\ & & & & & & I & \\ & & \widetilde{M}_k^{12} & & & & & \widetilde{M}_k^{11} \end{pmatrix} (\bullet)^\top$$

is positive semi-definite. By invoking the Schur complement we obtain (17). The remaining matrix inequalities are identical to those in Theorem 2.3 and proven analogously. ■

With this theorem, we can perform LMI synthesis of robust state-feedback controllers subject to constraints for finite horizon problems by solving the SDP

$$\begin{aligned} \underset{\widetilde{P}_k, \widetilde{K}_k, \tilde{\nu}, Z, \widetilde{M}_k \in \mathcal{M}'}{\text{maximize}} \quad & \tilde{\nu} \\ \text{s.t.} \quad & (17) \text{ for } k = 0, \ldots, N-1, \\ & (7\text{d}) \text{ for } k = 0, \ldots, N-1, \\ & (7\text{b}) \text{ and } (7\text{c}). \end{aligned} \tag{18}$$

IV. The infinite horizon case with uncertainty

Now, we consider the infinite horizon problem

$$
\begin{aligned}
\underset{u_0\in\mathcal{U}_0}{\min.}\ \underset{w_0\in\mathcal{W}_0}{\max.}\ \underset{u_1\in\mathcal{U}_1}{\min.}\ \underset{w_1\in\mathcal{W}_1}{\max.}\ \dots\ & \sum_{k=0}^{\infty} y_k^\top y_k && (19)\\
\text{s.t.}\quad x_{k+1} &= f_k + A_k x_k + B_k^1 u_k + B_k^2 w_k,\\
y_k &= g_k^1 + C_k^1 x_k + D_k^{11} u_k + D_k^{12} w_k,\\
v_{ki} &= g_{ki}^2 + C_{ki}^2 x_k + D_{ki}^{21} u_k, \qquad i = 1,\dots,s,\\
z_k &= g_k^2 + C_k^2 x_k + D_k^{21} u_k + D_k^{22} w_k,\\
x_0 &= \bar{x},
\end{aligned}
$$

where the constraint $u_k \in \mathcal{U}_k(x_k)$ is described by (11) and the constraint $w_k \in \mathcal{W}_k(z_k)$ is described by (12). To obtain a tractable relaxation of this problem, we assume that for $k \geq N \in \mathbb{N}_0$ the problem parameters are not changing in k. Denoting the parameters at time k by

$$
G_k := \begin{pmatrix} f_k & A_k & B_k^1 & B_k^2 \\ g_k^1 & C_k^1 & D_k^{11} & D_k^{12} \\ g_{ki}^2 & C_{ki}^2 & D_{ki}^{21} & 0 \\ g_k^3 & C_k^3 & D_k^{31} & D_k^{32} \end{pmatrix},
$$

this means $G_k = G_N\ \forall k \geq N$. The assumption of constant problem parameters for $k \geq N$ can be well justified, e.g., if the data $(G_k)_{k=0}^{\infty}$ results from the linearization of a nonlinear system around a reference trajectory, which is at an equilibrium for $k \geq N$. Accordingly, our strategy for solving (19) consists of searching for a function parametrized by $(P_k)_{k=0}^{\infty}$ and a controller parametrized by $(K_k)_{k=0}^{\infty}$ which are stationary for $k \geq N$, i.e.,

$$
\begin{aligned}
(P_0, P_1, P_2, \dots) &= (P_0, \dots, P_{N-1}, P_N, P_N, P_N, \dots),\\
(K_0, K_1, K_2, \dots) &= (K_0, \dots, K_{N-1}, K_N, K_N, K_N, \dots).
\end{aligned}
$$

Refer to *dual mode prediction* [23] for details on this parametrization of $(P_k)_{k=0}^{\infty}$, $(K_k)_{k=0}^{\infty}$. In the infinite horizon case we require (14), (5b) for all $k \geq N$ instead of enforcing the terminal condition (5c). To render this requirement tractable, we exploit the stationarity assumption to reduce this infinite family to the two constraints

$$
\begin{aligned}
V_N(x) &\geq y^\top y + \begin{pmatrix} z \\ w \end{pmatrix}^\top M_N \begin{pmatrix} z \\ w \end{pmatrix} + V_N(x^+) && (20)\\
\text{for}\quad x^+ &= f_N + B_N^1 k_N^1 + (A_N + B_N^1 K_N^2)x + B_N^2 w,\\
y &= g_N^1 + D_N^{11} k_N^1 + (C_N^1 + D_N^{11} K_N^2)x + D_N^{12} w,\\
z &= g_N^3 + D_N^{31} k_N^1 (C_N^3 + D_N^{31} K_N)x + D_N^{32} w,\\
V_N(x) &\leq \nu \Rightarrow v_i^\top v_i \leq 1, && (21)\\
\text{for}\quad v_i &= g_{Ni}^2 + D_{Ni}^2 k_N + (C_{Ni}^2 + D_{Ni}^2 K_N)x,
\end{aligned}
$$

for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^l$. The constraint (21) corresponds to (5d) for $k = N$. However, (20) differs from (13), since the same function V_N appears on both sides of the inequality. Nonetheless, we can still express (20) as matrix inequality and convexify it, as follows.

Proposition 4.1: If the functions V_k are parametrized as in (4) with stationary sequences as above, then the conditions (14),(5d) are satisfied for all $k \in \mathbb{N}_0$, if (15) holds for $k = 0,\dots,N-1$, (6d) holds for $k = 0,\dots,N$, and P_N, M_N satisfy the matrix inequality

$$
(\bullet)^\top \begin{pmatrix} -p_N^{11} & -p_N^{12} & & & & & \\ -p_N^{21} & -P_N^{22} & & & & & \\ & & M_N^{22} & & & & M_N^{21} \\ & & & p_N^{11} & p_N^{12} & & \\ & & & p_N^{21} & P_N^{22} & & \\ & & & & & I & \\ & & M_N^{12} & & & & M_N^{11} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 1 & 0 & 0 \\ f_N^K & A_N^K & B_N^2 \\ g_N^{1K} & C_N^{1K} & D_N^{12} \\ g_N^{3K} & C_N^{3K} & D_N^{32} \end{pmatrix} \leq 0. \tag{22}
$$

Proof: The proof relies on the same arguments as the proof of Proposition 3.1 and is thus omitted. ■

Essentially, the matrix inequality (22) replaces the terminal condition (6c). It ensures an upper bound on the infinite tail cost and robust constraint satisfaction for $k \geq N$. Next, we derive a linear formulation of this matrix inequality providing a convex relaxation of (19).

Theorem 4.2: Define the decision variables

$$
\widetilde{P}_k := P_k^{-1}, \quad \widetilde{K}_k := K_k P_k^{-1}, \quad \tilde{\nu} := \nu^{-1}, \quad Z := \nu^{-2}\sqrt{\Sigma_0}P_0\sqrt{\Sigma_0}
$$

including the slack variable Z, denote the matrix block

$$
\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ B_N^2 & 0 & I & 0 & 0 \\ D_N^{12} & 0 & 0 & I & 0 \\ D_N^{22} & 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} \widetilde{M}_N^{22} & & & & \widetilde{M}_N^{21} \\ & \tilde{p}_N^{11} & \tilde{p}_N^{12} & & \\ & \tilde{p}_N^{21} & \widetilde{P}_N^{22} & & \\ & & & \tilde{\nu} I & \\ \widetilde{M}_N^{12} & & & & \widetilde{M}_N^{11} \end{pmatrix} (\bullet)^\top
$$

by $\widetilde{Q}_N$ and introduce the new notation

$$
\begin{pmatrix} \tilde{f}_N^K & \widetilde{A}_N^K \\ \tilde{g}_N^{1K} & \widetilde{C}_N^{1K} \\ \tilde{g}_N^{2K} & \widetilde{C}_N^{2K} \end{pmatrix} := \begin{pmatrix} f_N & A_N & B_N^1 \\ g_N^1 & C_N^1 & D_N^{11} \\ g_N^2 & C_N^2 & D_N^{21} \end{pmatrix} \begin{pmatrix} \tilde{p}_N^{11} & \tilde{p}_N^{12} \\ \tilde{p}_N^{21} & \widetilde{P}_N^{22} \\ \tilde{k}_N^1 & \widetilde{K}_N^2 \end{pmatrix}.
$$

Then the matrix inequalities (15), (6b), (22) and (6d) are equivalent to the LMIs (17), (7b), (7d) and

$$
\begin{pmatrix} \tilde{q}_N^{11} & \tilde{q}_N^{12} & \tilde{q}_N^{13} & \tilde{q}_N^{14} & \tilde{p}_N^{11} & \tilde{p}_N^{12} \\ \tilde{q}_N^{21} & \widetilde{Q}_N^{22} & \widetilde{Q}_N^{23} & \widetilde{Q}_N^{24} & \tilde{f}_N^K & \widetilde{A}_N^K \\ \tilde{q}_N^{31} & \widetilde{Q}_N^{32} & \widetilde{Q}_N^{33} & \widetilde{Q}_N^{34} & \tilde{g}_N^{1K} & \widetilde{C}_N^{1K} \\ \tilde{q}_N^{41} & \widetilde{Q}_N^{42} & \widetilde{Q}_N^{43} & \widetilde{Q}_N^{44} & \tilde{g}_N^{2K} & \widetilde{C}_N^{2K} \\ \tilde{p}_N^{11} & (\tilde{f}_N^K)^\top & (\tilde{g}_N^{1K})^\top & (\tilde{g}_N^{2K})^\top & \tilde{p}_N^{11} & \tilde{p}_N^{12} \\ \tilde{p}_N^{21} & (\widetilde{A}_N^K)^\top & (\widetilde{C}_N^{1K})^\top & (\widetilde{C}_N^{2K})^\top & \tilde{p}_N^{21} & \widetilde{P}_N^{22} \end{pmatrix} \geq 0. \tag{23}
$$

Proof: The proof follows the same lines as the one of Theorem 3.2 and is omitted. ■

Using Theorem 4.2 we now formulate a convex relaxation of (19). In this problem, we maximize over the variable $\tilde{\nu}$, whose inverse provides a robust upper bound on the cost in (19). The resulting convex program is

$$
\begin{aligned}
\underset{\widetilde{P}_k, \widetilde{K}_k, \tilde{\nu}, Z, \widetilde{M}_k \in \mathcal{M}'}{\text{maximize}}\quad & \tilde{\nu} && (24)\\
\text{s.t.}\quad & (17) \text{ for } k = 0,\dots,N-1,\\
& (7\text{d}) \text{ for } k = 0,\dots,N,\\
& (7\text{b}) \text{ and } (23).
\end{aligned}
$$

V. Receding horizon control with uncertainty

Resulting from the controller synthesis problem (24) we can define the receding horizon controller

$$\pi_j^{\mathrm{MPC}}(\bar{x}_j) = \widetilde{K}_{0,j}\widetilde{P}_{0,j}^{-1}\begin{pmatrix}1\\ \bar{x}_j\end{pmatrix} = K_{0,j}\begin{pmatrix}1\\ \bar{x}_j\end{pmatrix}$$

where $\widetilde{K}_{0,j}$ and $\widetilde{P}_{0,j}$ are obtained by solving (24) for the problem data $(G_{k,j})_{k=0}^{N} := (G_{j+k})_{k=0}^{N}$. Our final result is the recursive feasibility, robust convergence and robust constraint satisfaction of π_j^{MPC}. We highlight that π_j^{MPC} is defined simply as the solution to problem (24), which is recursively feasible by default. This property is a consequence of the constraint (23), which can be interpreted as a terminal ingredient for π_j^{MPC}.

Theorem 5.1 (Recursive feasibility and convergence): If (24) is feasible at $\bar{x}_0$, then (24) is feasible at all states $\bar{x}_j$ with $j \in \mathbb{N}$ defined by the closed loop

$$\bar{x}_{j+1} = f_j + A_j\bar{x}_j + B_j^1\pi_j^{\mathrm{MPC}}(\bar{x}_j) + B_j^2 w_j \qquad (25)$$

for any realization of $(w_j)_{j=0}^{\infty}$ satisfying (12). Furthermore, the constraints (11) are robustly satisfied and the signal $(y_j)_{j=0}^{\infty}$ converges to zero.

Proof: The proof can be found in [15]. ∎

In Theorem 5.1, we interpret the convergence of $(y_j)_{j=0}^{\infty}$ to zero as stability result given the fact that an appropriate choice of $(g_k^1, C_k^1, D_k^{11}, D_k^{12})$ yields $y_j^\top = \begin{pmatrix}\bar{x}_j^\top & u_j^\top\end{pmatrix}$ and implies $\bar{x}_j \to 0$ and $u_j \to 0$ for $j \to \infty$.

VI. Numerical example

We adopt an example from [10], where the LTI system

$$x_{k+1} = \begin{pmatrix}1+\epsilon_1 & 0.15\\ 0.1 & 1\end{pmatrix}x_k + \begin{pmatrix}0.1\\ 1.1+\epsilon_2\end{pmatrix}u_k$$

with $x_k \in \mathbb{R}^2$, $u_k \in \mathbb{R}$ and time-varying uncertain parameters ϵ_1 and ϵ_2 is considered. The states and inputs are constrained by

$$x_k^\top\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}x_k \le 8^2, \qquad x_k^\top\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}x_k \le 8^2, \qquad u_k^2 \le 4^2.$$

In our git repository `https://github.com/SphinxDG/ConstrainedRobustControl`, we derive G_k and a family of multipliers $\mathcal{M}$ that describe this system, its constraints, and the uncertainties in the parameters ϵ_1 and ϵ_2. As a result, we can test the feasibility of the infinite horizon robust optimal control problem (19) at some initial condition $\bar{x}$ by attempting to solve the relaxation (24). We can further determine the feasibility of (19) at $\bar{x}$ exactly using a dynamic programming method described in [17]. Thus, to evaluate our relaxation (24), we sample initial conditions $\bar{x}$ from the equidistant 10×10 grid $\mathcal{X}_0$ at $[-7.9, 7.9]^2$ and compute the fraction of initial conditions certified as feasible by (24) over the truly feasible initial conditions determined by the dynamic programming method of [17]. To solve (24) we use the solver Mosek [2] and the parser CVXPY [1]. The fraction of initial conditions correctly classified as feasible is plotted in Figure 1 for different prediction horizons N over an uncertainty magnitude $\gamma \in [0.05, 0.45]$ with $\epsilon_1 \in [-\gamma, \gamma]$ and $\epsilon_2 \in [-0.1, 0.1]$. As expected, a longer prediction horizon monotonically increases the fraction of feasible initial conditions and provides an improvement over Kothares' method [22], which is obtained for $N = 0$.

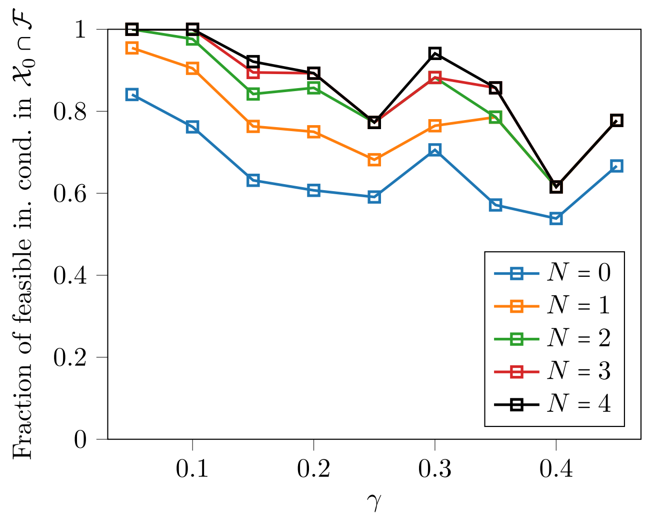

Fig. 1: This figure shows the fraction of initial conditions from $\mathcal{X}_0 \cap \mathcal{F}$, for which we found a solution $\tilde{\nu} > 0$ to (24). Here, $\mathcal{F}$ is the set of feasible initial conditions of the original problem (19). The parameter ϵ_1 is assumed in $[-\gamma, \gamma]$ with γ sampled from $\{0.05p \mid p = 1, \ldots, 9\}$ and ϵ_2 is assumed to be in $[-0.1, 0.1]$. The plots show the fraction of feasible initial conditions for different prediction horizons N.

As we mentioned, a similar experiment has been carried out in [10], where an additional norm bounded disturbance has been considered. If we test the methods benchmarked in [10] on our modified setup and compare them to our solution, we obtain the results depicted in Figure 2. We observe that SLSMPC achieves higher feasibility ratios, while our method performs at least as good as the best competitors considered in [10]. Recall, however, that the method from [10] cannot be applied to systems in LFT-form directly and it is not shown to be recursively feasible in [10]. Furthermore, all methods considered in [10] were supplied with the exact feasible set of (19) as terminal region, which has been shown to be a significant advantage [30] but can be expensive to compute in higher dimensions; we did not make use of this set in (24).

VII. Conclusion

In this article, we demonstrate how classical techniques from robust control theory can be exploited to solve dynamic programs with uncertainties and inequality constraints for time-varying linear systems. Since we invoke the standard relaxations from robust control for convexification, the solutions resulting from our method are inexact. We apply these solutions for MPC and highlight advantages over, e.g., tube-based MPC, which stems

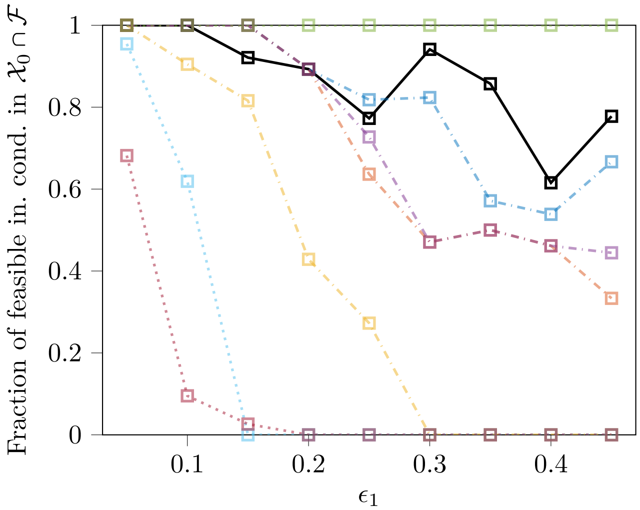

Fig. 2: This figure compares the feasibility of (24) to other methods for robust MPC. The y-axis is the fraction of initial conditions from $\mathcal{X}_0 \cap \mathcal{F}$, for which feasibility of (19) is certified. Here, $\mathcal{F}$ is the set of feasible initial conditions for the original problem (19). The parameter ϵ_1 is assumed in $[-\gamma, \gamma]$ with γ sampled from $\{0.05p \mid p = 1, \ldots, 9\}$ and ϵ_2 is assumed to be in $[-0.1, 0.1]$. The methods compared are our method (—⊟—), the tube based MPC methods [24] (-·⊟·-), [26] (-·⊟·-), [20] (-·⊟·-), [27] (-·⊟·-), the disturbance feedback methods [5] (··⊟··), [4] (··⊟··), and SLSMPC [10] (··⊟··).

from the ability of our method to optimize feedback policies online and to incorporate flexible uncertainty descriptions in the form of LFT models. In future work, we aim to extend the method presented to nonlinear systems incorporating ideas from [14] and to reduce the computational burden due to solving LMIs making use of structure exploiting SDP solvers as in [13].

References

[1] A. Agrawal, R. Verschueren, S. Diamond, and S. Boyd. A rewriting system for convex optimization problems. *Journal of Control and Decision*, 5(1):42–60, 2018.

[2] M. ApS. *Mosek. Version 10.1.*, 2023.

[3] J. Buch and P. Seiler. Finite horizon robust synthesis using integral quadratic constraints. *International Journal of Robust and Nonlinear Control*, 31(8):3011–3035, 2021.

[4] M. Bujarbaruah, U. Rosolia, Y. R. Stürz, X. Zhang, and F. Borrelli. Robust MPC for linear systems with parametric and additive uncertainty: A novel constraint tightening approach. *arXiv preprint arXiv:2007.00930*, 2020.

[5] M. Bujarbaruah, U. Rosolia, Y. R. Stürz, X. Zhang, and F. Borrelli. Robust MPC for LPV systems via a novel optimization-based constraint tightening. *Automatica*, 143:110459, 2022.

[6] P. J. Campo and M. Morari. Robust model predictive control. In *1987 American control conference*, pages 1021–1026. IEEE, 1987.

[7] A. Casavola, D. Famularo, and G. Franzé. Robust constrained predictive control of uncertain norm-bounded linear systems. *Automatica*, 40(11):1865–1876, 2004.

[8] A. Casavola, M. Giannelli, and E. Mosca. Min–max predictive control strategies for input-saturated polytopic uncertain systems. *Automatica*, 36(1):125–133, 2000.

[9] H. Chen and C. W. Scherer. Moving horizon H_∞ control with performance adaptation for constrained linear systems. *Automatica*, 42(6):1033–1040, 2006.

[10] S. Chen, V. M. Preciado, M. Morari, and N. Matni. Robust model predictive control with polytopic model uncertainty through system level synthesis. *arXiv preprint arXiv:2203.11375*, 2022.

[11] Y.-S. Chou, A. L. Tits, and V. Balakrishnan. Stability multipliers and/spl mu/upper bounds: connections and implications for numerical verification of frequency domain conditions. *IEEE transactions on automatic control*, 44(5):906–913, 1999.

[12] P. J. Goulart, E. C. Kerrigan, and J. M. Maciejowski. Optimization over state feedback policies for robust control with constraints. *Automatica*, 42(4):523–533, 2006.

[13] D. Gramlich, T. Holicki, C. W. Scherer, and C. Ebenbauer. A structure exploiting SDP solver for robust controller synthesis. *IEEE Control Systems Letters*, 2023.

[14] D. Gramlich, C. W. Scherer, and C. Ebenbauer. Robust Differential Dynamic Programming. In *2022 IEEE 61st Conference on Decision and Control (CDC)*, pages 1714–1721. IEEE, 2022.

[15] D. Gramlich, C. W. Scherer, H. Häring, and C. Ebenbauer. Synthesis of constrained robust feedback policies and model predictive control. *arXiv extended version arXiv:2310.11404*, 2023.

[16] M. Green and D. J. Limebeer. *Linear robust control.* Courier Corporation, 2012.

[17] P. Grieder, P. A. Parrilo, and M. Morari. Robust receding horizon control-analysis & synthesis. In *42nd IEEE International Conference on Decision and Control (IEEE Cat. No. 03CH37475)*, volume 1, pages 941–946. IEEE, 2003.

[18] T. Iwasaki and S. Hara. Well-posedness of feedback systems: Insights into exact robustness analysis and approximate computations. *IEEE Transactions on Automatic Control*, 43(5):619–630, 1998.

[19] T. Kim, P. Elango, and B. Acikmese. Joint synthesis of trajectory and controlled invariant funnel for discrete-time systems with locally Lipschitz nonlinearities. *arXiv preprint arXiv:2209.03535*, 2022.

[20] J. Köhler, E. Andina, R. Soloperto, M. A. Müller, and F. Allgöwer. Linear robust adaptive model predictive control: Computational complexity and conservatism. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 1383–1388. IEEE, 2019.

[21] M. V. Kothare, V. Balakrishnan, and M. Morai. Robust constrained model predictive control using linear matrix inequalities. In *Proceedings of 1994 American Control Conference-ACC'94*, volume 1, pages 440–444. IEEE, 1994.

[22] M. V. Kothare, V. Balakrishnan, and M. Morari. Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 32(10):1361–1379, 1996.

[23] B. Kouvaritakis and M. Cannon. Model predictive control. *Switzerland: Springer International Publishing*, 38:13–56, 2016.

[24] W. Langson, I. Chryssochoos, S. Raković, and D. Q. Mayne. Robust model predictive control using tubes. *Automatica*, 40(1):125–133, 2004.

[25] J. Löfberg. *Minimax approaches to robust model predictive control*, volume 812. Linköping University Electronic Press, 2003.

[26] M. Lorenzen, M. Cannon, and F. Allgöwer. Robust MPC with recursive model update. *Automatica*, 103:461–471, 2019.

[27] X. Lu and M. Cannon. Robust adaptive tube model predictive control. In *2019 American Control Conference (ACC)*, pages 3695–3701. IEEE, 2019.

[28] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, 2000.

[29] M. M. Morato, T. Holicki, and C. W. Scherer. Stabilizing model predictive control synthesis using integral quadratic constraints and full-block multipliers. *Authorea Preprints*, 2023.

[30] B. Pluymers, J. Rossiter, J. Suykens, and B. De Moor. A simple algorithm for robust MPC. *IFAC Proceedings Volumes*, 38(1):257–262, 2005.

[31] S. V. Rakovic, B. Kouvaritakis, M. Cannon, C. Panos, and R. Findeisen. Parameterized tube model predictive control. *IEEE Transactions on Automatic Control*, 57(11):2746–2761, 2012.

[32] D. Ramirez and E. Camacho. Characterization of min-max MPC with bounded uncertainties and a quadratic criterion. In *Proceedings of the 2002 American Control Conference (IEEE Cat. No. CH37301)*, volume 1, pages 358–363. IEEE, 2002.

[33] T. Reynolds, D. Malyuta, M. Mesbahi, B. Acikmese, and J. M. Carson. Funnel synthesis for the 6-DOF powered descent guidance problem. In *AIAA Scitech 2021 Forum*, page 0504, 2021.

[34] M. A. Rotea. The generalized H2 control problem. *Automatica*, 29(2):373–385, 1993.

[35] C. W. Scherer. LMI relaxations in robust control. *European Journal of Control*, 12(1):3–29, 2006.

[36] J. Schuurmans and J. Rossiter. Robust predictive control using tight sets of predicted states. *IEE proceedings-Control theory and applications*, 147(1):13–18, 2000.

[37] L. Schwenkel, J. Köhler, M. A. Müller, and F. Allgöwer. Model predictive control for linear uncertain systems using integral quadratic constraints. *IEEE Transactions on Automatic Control*, 68(1):355–368, 2022.

[38] K. Uchida and M. Fujita. Finite horizon H_∞ control problems with terminal penalties. *IEEE Transactions on Automatic control*, 37(11):1762–1767, 1992.

[39] J. Veenman, C. W. Scherer, and H. Köroğlu. Robust stability and performance analysis based on integral quadratic constraints. *European Journal of Control*, 31:1–32, 2016.

[40] C. Wen, X. Ma, and B. E. Ydstie. Analytical expression of explicit MPC solution via lattice piecewise-affine function. *Automatica*, 45(4):910–917, 2009.