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Distributed Containment Control of Multi-Agent Systems under Markovian Randomly Switching Topologies and Infinite Communication Delays

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Abstract— **In this article, the distributed containment control problem of heterogeneous multi-agent systems (MASs) under Markovian randomly switching topologies and infinite communication delays is studied. A novel distributed containment observer is first proposed to estimate the convex hull formed by the states of multiple leaders in the presence of Markovian randomly switching topologies and infinite communication delays. Then a distributed containment controller is further developed based on the proposed distributed observer. It is shown that the output of each follower converges to the convex hull spanned by those of leaders under the proposed controller. Moreover, our findings encompass those results on containment control of MASs with bounded distributed delays or constant delays as special cases. Ultimately, we present a simulation example to illustrate the effectiveness of the proposed controller.**

I. INTRODUCTION

Cooperative control of multi-agent systems (MASs) has drawn considerable interest in recent decades due to their applications in diverse areas, such as robotics, autonomous vehicles, social networks, and biological systems [1], [2], [3]. Containment control, a fundamental problem in MASs, aims to guide the states of a subset of agents, referred to as followers, to asymptotically converge to the convex hull spanned by those of a group of leaders while ensuring system stability and robustness against uncertainties. For instance, to preclude a group of vehicles from entering hazardous areas, specific agents have been designated as leaders, facilitating vehicle access into the safe zone formed by leaders [4]. Recently, containment control problems have been investigated for MASs with single/double-integrator, general linear MASs, and heterogeneous MASs [5], [6], [7].

In real-world applications, the system dynamics often exhibit time-varying network structures and the connections between agents undergo stochastic transitions. In such scenarios, the Markovian randomly switching topologies are often utilized to model these changing communication topologies. During the past years, many researchers have studied containment control problem of MASs under Markovian randomly switching topologies and some fundamental results have been given in [8], [9], [10].

On the other hand, the presence of communication delays is an inherent characteristic that often leads to performance degradation or even system instability. Consequently, the containment control problems of MASs with communication delays have been extensively investigated [10], [11], [12]. However, these works only addressed bounded communication delays. Infinite communication delays (also called unbounded communication delays) exist in many practical systems, such as coupled oscillators and neural networks [13], [14]. Various cooperative control problems of heterogeneous MASs with infinite communication delays have been solved [15], [16]. For instance, the output containment control problem of heterogeneous MASs with infinite delays and fixed topologies was solved in [15]. The robust cooperative output regulation problem of heterogeneous MASs over deterministic switching topologies and infinite delays was considered in [16]. However, to the best of our knowledge, the case of Markovian randomly switching topologies and infinite delay has not been explored for the output containment control problem, which motivates this study.

In this work, we investigate the distributed output containment control problem of heterogeneous MASs under Markovian randomly switching topologies and infinite communication delays. We propose a novel distributed observer in the presence of Markovian randomly switching topologies and infinite delays. Then, a novel distributed controller is developed to ensure the outputs of followers converge to the convex hull spanned by those of leaders. Furthermore, our findings include leader-following consensus problems of MASs under Markovian randomly switching topologies and bounded distributed/constant delays as special cases.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Preliminaries

Consider a heterogeneous MAS consisting of N followers and M leaders. If the agent has no neighbor, it is called a leader; otherwise, it is a follower. Denote the follower set as $\mathcal{R} = \{1, \dots, N\}$ and the leader set as $\mathcal{D} = \{N+1, \dots, N+M\}$, respectively. Denote $(\Omega, \mathcal{F}, \mathfrak{F}, \mathbb{P})$ as a complete probability space where $\mathfrak{F} = \{\mathcal{F}_t; t \geq 0\}$ is a filtration. Denote $\sigma(t)$ as the switch signal, which is determined by a continuous-time Markov process that takes values within $\mathbb{S} = \{1, 2, \dots, s\}$. Let the generator of the Markov process $\{\sigma(t), t \geq 0\}$ be $\Omega = (\gamma_{kr}) \in \mathbb{R}^{s\times s}$, which satisfies $\mathbb{P}\{\sigma(t+h) = r \mid \sigma(t) = k\} = \gamma_{kr}h + o(h)$, if $k \neq r$, otherwise, it equals to $1 + \gamma_{kk}h + o(h)$, where $\lim_{h\to 0} o(h)/h = 0$. Here, $\gamma_{kr} \geq 0$ is the transition rate from k to r if $k \neq r$ while $\gamma_{kk} = -\sum_{r\neq k} \gamma_{kr} \leq 0$. The sum of each row in the transition rate matrix Ω is zero, expressed as $\Omega\mathbf{1} = \mathbf{0}$. Let

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the digraph $\mathcal{G}_{\sigma(t)}=\{\mathcal{V},\mathcal{E}_{\sigma(t)},\mathcal{A}_{\sigma(t)}\}$ be described the time-varying topology among N followers and M leaders, where the vertex set $\mathcal{V}=\{1,\cdots,N+M\}$. The set edge $\mathcal{E}_{\sigma(t)}=\{(j,i)\mid i,j\in\mathcal{V}\}$ and the adjacency matrix $\mathcal{A}_{\sigma(t)}$ exhibit time-varying characteristics. Let $\mathcal{G}_F^{\sigma(t)}=(\mathcal{R},\mathcal{E}_F^{\sigma(t)},\mathcal{A}_F^{\sigma(t)})$ be a digraph among followers, where $\mathcal{E}_F^{\sigma(t)}\subseteq\mathcal{R}\times\mathcal{R}$ and $\mathcal{A}_F^{\sigma(t)}=(a_{ij}^{\sigma(t)})_{N\times N}$ with $a_{ij}^{\sigma(t)}>0\Leftrightarrow(j,i)\in\mathcal{E}_F^{\sigma(t)}$ and $a_{ij}^{\sigma(t)}=0$, otherwise. Here, we assume $a_{ii}^{\sigma(t)}=0,\ \forall i\in\mathcal{R}$. The pinning gains from the jth leader to each follower i is $c_{ij}^{\sigma(t)}$, where $c_{ij}^{\sigma(t)}>0$ if the follower i can receive the signal from the leader j; otherwise, $c_{ij}^{\sigma(t)}=0$. The Laplacian matrix $L(\sigma(t))$ corresponding to $\mathcal{G}_{\sigma(t)}$ can be written as: $L(\sigma(t))=\begin{pmatrix}L_1(\sigma(t)) & L_2(\sigma(t))\\ 0_{M\times N} & 0_{M\times M}\end{pmatrix}$, where $L_1(\sigma(t))$ and $L_2(\sigma(t))$ are the same as in [10]. For k in $\mathbb{S}$, let $\mathcal{G}_{un}=\bigcup_{k=1}^{s}\mathcal{G}_k=(\mathcal{V},\bigcup_{k=1}^{s}\mathcal{E}_k)$ be denoted as the *union graph* of $\mathcal{G}_k$.

B. Problem Statement

The dynamics of the N followers are described as follows:

$$\begin{cases}\dot{x}_i=A_ix_i+B_iu_i,\\ y_i=C_ix_i, i\in\mathcal{R},\end{cases}\tag{1}$$

where $x_i\in\mathbb{R}^{n_i}$, $u_i\in\mathbb{R}^{m_i}$, and $y_i\in\mathbb{R}^m$ are the ith follower's state, input, and output, respectively. The M leaders dynamics are described by:

$$\begin{cases}\dot{z}_j=Sz_j,\\ y_j=Rz_j, j\in\mathcal{D},\end{cases}\tag{2}$$

where $z_j\in\mathbb{R}^q$ and $y_j\in\mathbb{R}^m$ are the state and output of the jth leader, respectively.

This goal of this work is to give a distributed controller for heterogeneous MAS (1)-(2) under Markovian randomly switching topologies and infinite communication delays, such that the output of each follower can converge, in mean square sense, to the convex hull formed by those of the leaders. Next, a definition and some assumptions are shown.

Definition 1: Consider the heterogeneous MAS (1)-(2) under Markovian randomly switching topologies $\mathcal{G}_{\sigma(t)}$, the distributed output containment control problem can be achieved in mean square sense, if there exist nonnegative constants $\beta_{ij},\ i\in\mathcal{R},j\in\mathcal{D}$ satisfying $\sum_{j\in\mathcal{D}}\beta_{ij}=1$ such that

$$\lim_{t\to+\infty}\mathbb{E}\Big[\big\|y_i-\sum_{j\in\mathcal{D}}\beta_{ij}y_j\big\|^2\Big]=0,\ i\in\mathcal{R}.\tag{3}$$

Infinite distributed communication delays among agents are considered in this work. Let the signals to be transmitted from agent $j\in\mathcal{R}\cup\mathcal{D}$ in $t\geq0$ be $\vartheta_j\in\mathbb{R}^q, j\in\mathcal{R}$ or $z_j\in\mathbb{R}^q, j\in\mathcal{D}$ and the initial information in $t<0$ is $\vartheta_j^0\in\mathbb{R}^q, j\in\mathcal{R}$ or $z_j^0\in\mathbb{R}^q, j\in\mathcal{D}$. Note that $\vartheta_j,\vartheta_j^0$ and z_j,z_j^0 are required to be the same as in [15] while this strict assumption is not required in this work. Because of the existence of infinite delays, the information that the ith follower obtains from its neighboring agent j is $\int_0^t\omega_{ij}(\eta)\vartheta_j(t-\eta)d\eta+\int_t^{+\infty}\omega_{ij}(\eta)\vartheta_j^0(t-\eta)d\eta, j\in\mathcal{R}$ or $\int_0^t\omega_{ij}(\eta)z_j(t-\eta)d\eta+\int_t^{+\infty}\omega_{ij}(\eta)z_j^0(t-\eta)d\eta, j\in\mathcal{D}$, where $\omega_{ij}(\eta):[0,+\infty)\to[0,+\infty)$ is the delay kernel function satisfying $\int_0^{+\infty}\omega_{ij}(\eta)d\eta=1$.

Assumption 1: For any $i\in\mathcal{R}$, (A_i,B_i) is stabilizable and (A_i,C_i) is detectable.

Assumption 2: The linear matrix equations

$$A_iX_i+B_i\Xi_i=X_iS\tag{4a}$$
$$C_iX_i=R,\tag{4b}$$

have a solution (X_i,Ξ_i) for each $i\in\mathcal{R}$.

Assumption 3: The leader set $\mathcal{D}$ is globally reachable within $\mathcal{G}_{un}$.

Assumption 4: The Markov process $\{\sigma(t),t\geq0\}$ is ergodic.

Assumption 5: The delay kernel functions $\omega_{ij},t\geq0,i\in\mathcal{R},j\in\mathcal{R}\cup\mathcal{D}$ satisfy

$$\omega_{ij}(t)\leq\omega(t),\tag{5}$$
$$\int_0^{+\infty}\omega(\eta)\|\vartheta_i^0(-\eta)\|d\eta<+\infty,\tag{6}$$
$$\int_0^{+\infty}\omega(\eta)\|z_j^0(-\eta)\|d\eta<+\infty,\tag{7}$$

where $\omega(t)>0,t\in[0,+\infty)$ is a non-increasing function satisfying $\int_0^{+\infty}\omega(\eta)d\eta<+\infty$ and $\omega(\eta+s)\leq\omega(\eta)\omega(s),\forall\eta,s\geq0$.

Remark 1: Assumptions 1-4 are frequently employed for containment control problems of heterogeneous MASs [7], [10]. Assumption 5 are limitations on infinite distributed delays and initial conditions. Condition (5) is a restriction on delay kernel functions and conditions (6) and (7) are limitations on initial conditions and have been used in [16].

As shown in [16], Assumption 5 implies $\lim_{t\to\infty}\omega(t)=0$ exponentially, i.e., there exist two positive constants μ_1 and μ_2 such that $\omega(t)\leq\mu_1e^{-\frac{\mu_2}{2}t}$. Ergodic Markov process is characterized by a sole stationary distribution described as $\alpha=(\alpha_1,\ldots,\alpha_s)$. Consequently, it is reasonable to posit that the Markov process $\{\sigma(t),t\geq0\}$ initiates from the stationary distribution, then the adjacency matrix of $\mathcal{G}_{un}$ can be expressed as $\sum_{k=1}^{s}\alpha_k\mathcal{A}_k$. Here, we define $L_1=\mathbb{E}[L_1(\sigma(t))]$, $L_2=\mathbb{E}[L_2(\sigma(t))]$. It is observed that the *expectation graph* $\mathbb{E}[\mathcal{G}_{\sigma(t)}]$ shares an identical structure structure with that of $\mathcal{G}_{un}$. The Laplacian matrix $\mathbb{E}[L(\sigma(t))]$ corresponding to $\mathbb{E}[\mathcal{G}_{\sigma(t)}]$ can be expressed as : $\mathbb{E}[L(\sigma(t))]=\begin{pmatrix}L_1 & L_2\\ 0_{M\times N} & 0_{M\times M}\end{pmatrix}$.

III. Main Results

The distributed observer design, distributed controller design, and analysis of the resulting closed-loop system are presented in this section.

A. Distributed Observer Design

Inspired by [15], [10], we propose the distributed observer as follows,

$$\dot{\vartheta}_i(t)=S\vartheta_i(t)-\varpi\sum_{j\in\mathcal{R}}a_{ij}^{\sigma(t)}\big(\vartheta_i(t)-T_{ij}(S,\vartheta_j,\vartheta_j^0)\big)$$

$$-\varpi \sum_{j\in\mathcal{D}} c_{ij}^{\sigma(t)}\big(\vartheta_i(t) - T_{ij}(S, z_j, z_j^0)\big), \tag{8}$$

where $\varpi > 0$ is a real number, ϑ_i is the state of the distributed observer; $a_{ij}^{\sigma(t)}$ is the element of the adjacency matrix $\mathcal{A}_{\sigma(t)}$; $T_{ij}(S,\vartheta_j,\vartheta_j^0) = \int_0^t \omega_{ij}(\eta)e^{S\eta}\vartheta_j(t-\eta)d\eta + e^{St}\int_t^{+\infty}\omega_{ij}(\eta)\vartheta_j^0(t-\eta)d\eta$ and $T_{ij}(S,z_j,z_j^0) = \int_0^t \omega_{ij}(\eta)e^{S\eta}z_j(t-\eta)d\eta + e^{St}\int_t^{+\infty}\omega_{ij}(\eta)z_j^0(t-\eta)d\eta$ are the delayed signal obtained by agent i from its neighbor j.

Remark 2: The information transmission framework is outlined in the following. Initially, the observer state ϑ_j of follower $j\in\mathcal{R}$ and the leader state z_j of leader $j\in\mathcal{D}$ are multiplied e^{-St} and then sent $e^{-St}\vartheta_j$ and $e^{-St}z_j$ to follower i. Owing to the presence of infinite delays, the information received by follower i is $\int_0^t \omega_{ij}(\eta)e^{-S(t-\eta)}\vartheta_j(t-\eta)d\eta + \int_t^{+\infty}\omega_{ij}(\eta)\vartheta_j^0(t-\eta)d\eta$ and $\int_0^t\omega_{ij}(\eta)e^{-S(t-\eta)}z_j(t-\eta)d\eta + \int_t^{+\infty}\omega_{ij}(\eta)z_j^0(t-\eta)d\eta$. Multiplying by e^{St} with the received information, one can have $T_{ij}(S,\vartheta_j,\vartheta_j^0)$ and $T_{ij}(S,z_j,z_j^0)$. Under such a communication framework, it is not necessary to possess prior knowledge of the delay kernel $\omega_{ij}(\vartheta)$.

Denote

$$-L_1^{-1}L_2 \triangleq \begin{pmatrix} \beta_{M+1,1} & \cdots & \beta_{M+1,M} \\ \vdots & \ddots & \vdots \\ \beta_{M+N,1} & \cdots & \beta_{M+N,M} \end{pmatrix}, \tag{9}$$

where $\sum_{j\in\mathcal{D}}\beta_{ij} = 1$ and $\beta_{ij}\geq 0, i\in\mathcal{R}, j\in\mathcal{D}$ from [10, Lemma 1]. Denote $\vartheta = (\vartheta_1^T,\ldots,\vartheta_N^T)^T$ and $z = (z_{N+1}^T,\ldots,z_{N+M}^T)^T$. The state containment error is denoted as $\zeta = \vartheta + (L_1^{-1}L_2\otimes I_q)z$, where $\zeta = (\zeta_1^T,\ldots,\zeta_N^T)^T$ and $\zeta_i = \vartheta_i - \sum_{j\in\mathcal{D}}\beta_{ij}z_j, i\in\mathcal{R}$. Then, we can get the following compact form:

$$\begin{aligned} \dot{z} =& \left(I_M\otimes S\right)z, \qquad\qquad (10)\\ \dot{\vartheta} =& \left(I_N\otimes S\right)\vartheta - \varpi\Bigg(\left(D_{\sigma(t)}\otimes I_q\right)\vartheta \\ &+ \int_0^t \left(L_1^{\sigma(t)}(\eta)\otimes e^{S\eta}\right)\vartheta(t-\eta)d\eta \\ &+ \int_0^t \left(L_2^{\sigma(t)}(\eta)\otimes e^{S\eta}\right)z(t-\eta)d\eta \\ &+ e^{(I_N\otimes S)t}\int_t^{+\infty}\left(L_1^{\sigma(t)}(\eta)\otimes I_q\right)\vartheta^0(t-\eta)d\eta \\ &+ e^{(I_N\otimes S)t}\int_t^{+\infty}\left(L_2^{\sigma(t)}(\eta)\otimes I_q\right)z^0(t-\eta)d\eta, \end{aligned}$$

where $D_{\sigma(t)} = \mathrm{diag}\{d_i^{\sigma(t)}\}_{N\times N}, d_i^{\sigma(t)} = \sum_{j\in\mathcal{R}}a_{ij}^{\sigma(t)} + \sum_{j\in\mathcal{D}}c_{ij}^{\sigma(t)}$ for $i\in\mathcal{R}$; $L_1^{\sigma(t)}(\eta) = (-a_{ij}^{\sigma(t)}\omega_{ij}(\eta))_{N\times N}, i,j\in\mathcal{R}$; and $L_2^{\sigma(t)}(\eta) = (-c_{ij}^{\sigma(t)}\omega_{ij}(\eta))_{N\times M}, i\in\mathcal{R}, j\in\mathcal{D}$.

Denote $\tilde{z} = e^{-(I_M\otimes S)t}z$ and $\tilde{\vartheta} = e^{-(I_N\otimes S)t}\vartheta$. It then follows from (10) that $\dot{\tilde{z}} = 0$ and

$$\begin{aligned} \dot{\tilde{\vartheta}} =& -\varpi(I_N\otimes e^{-St})\Bigg(\left(D_{\sigma(t)}\otimes I_q\right)\vartheta \\ &+ \int_0^t\left(L_1^{\sigma(t)}(\eta)\otimes e^{S\eta}\right)\vartheta(t-\eta)d\eta \\ &+ \int_0^t\left(L_2^{\sigma(t)}(\eta)\otimes e^{S\eta}\right)z(t-\eta)d\eta\Bigg) \\ &- \varpi\int_t^{+\infty}\left(L_1^{\sigma(t)}(\eta)\otimes I_q\right)\vartheta^0(t-\eta)d\eta \\ &- \varpi\int_t^{+\infty}\left(L_2^{\sigma(t)}(\eta)\otimes I_q\right)z^0(t-\eta)d\eta \\ =& -\varpi\Bigg(\tilde{D}_{\sigma(t)}\tilde{\vartheta} + \int_0^t \tilde{L}_1^{\sigma(t)}(\eta)\tilde{\vartheta}(t-\eta)d\eta \\ &+ \int_0^t\tilde{L}_2^{\sigma(t)}(\eta)\tilde{z}(t-\eta)d\eta + \Psi_{\sigma(t)}\Bigg), \end{aligned} \tag{11}$$

where $\tilde{D}_{\sigma(t)} = D_{\sigma(t)}\otimes I_q$, $\tilde{L}_1^{\sigma(t)}(\eta) = L_1^{\sigma(t)}(\eta)\otimes I_q$, $\tilde{L}_2^{\sigma(t)}(\eta) = L_2^{\sigma(t)}(\eta)\otimes I_q$, and $\Psi_{\sigma(t)} = \int_t^{+\infty}(L_1^{\sigma(t)}(\eta)\otimes I_q)\vartheta^0(t-\eta)d\eta + \int_t^{+\infty}(L_2^{\sigma(t)}(\eta)\otimes I_q)z^0(t-\eta)d\eta$.

Define $\delta = \tilde{\vartheta} + \left(L_1^{-1}L_2\otimes I_q\right)\tilde{z}$. It follows from (11) that

$$\begin{aligned} \dot{\delta} =& -\varpi\Bigg(\tilde{D}_{\sigma(t)}\tilde{\vartheta} + \int_0^t\tilde{L}_1^{\sigma(t)}(\eta)\tilde{\vartheta}(t-\eta)d\eta \\ &+ \int_0^t\tilde{L}_2^{\sigma(t)}(\eta)\tilde{z}(t-\eta)d\eta + \Psi_{\sigma(t)}\Bigg) \\ =& -\varpi\Bigg(\tilde{D}_{\sigma(t)}\delta + \int_0^t\tilde{L}_1^{\sigma(t)}(\eta)\delta(t-\eta)d\eta \\ &+ \int_0^t\tilde{L}_2^{\sigma(t)}(\eta)\tilde{z}(t-\eta)d\eta - \tilde{D}_{\sigma(t)}\left(L_1^{-1}L_2\otimes I_q\right)\tilde{z} \\ &- \int_0^t\tilde{L}_1^{\sigma(t)}(\eta)\left(L_1^{-1}L_2\otimes I_q\right)\tilde{z}(t-\eta)d\eta + \Psi_{\sigma(t)}\Bigg) \\ =& -\varpi\left(\tilde{D}_{\sigma(t)}\delta + \int_0^t\tilde{L}_1^{\sigma(t)}(\eta)\delta(t-\eta)d\eta + \Upsilon_{\sigma(t)}\right), \end{aligned} \tag{12}$$

where $\Upsilon_{\sigma(t)} = \int_0^t\tilde{L}_2^{\sigma(t)}(\eta)\tilde{z}(t-\eta)d\eta - \tilde{D}_{\sigma(t)}\left(L_1^{-1}L_2\otimes I_q\right)\tilde{z} - \int_0^t\tilde{L}_1^{\sigma(t)}(\eta)\left(L_1^{-1}L_2\otimes I_q\right)\tilde{z}(t-\eta)d\eta + \Psi_{\sigma(t)}$. Moreover, it follows that $\Upsilon_{\sigma(t)} = \int_t^{+\infty}\tilde{L}_1^{\sigma(t)}(\eta)\left(L_1^{-1}L_2\otimes I_q\right)\tilde{z}d\eta - \int_t^{+\infty}\tilde{L}_2^{\sigma(t)}(\eta)\tilde{z}d\eta + \int_t^{+\infty}\tilde{L}_1^{\sigma(t)}(\eta)\vartheta^0(t-\eta)d\eta + \int_t^{+\infty}(L_2^{\sigma(t)}(\eta)\otimes I_q)z^0(t-\eta)d\eta$. Under Assumption 5, there exist constants $c_1 - c_4$ and $b_1 - b_4$, such that $\|\int_t^{+\infty}\tilde{L}_2^{(k)}(\eta)\tilde{z}d\eta\| \leq c_1\int_0^{+\infty}\omega(t+\eta)d\eta\|\tilde{z}\| \leq b_1\omega(t)$, $\|\int_t^{+\infty}\tilde{L}_2^{(k)}(\eta)\tilde{z}(t-\eta)d\eta\| \leq c_2\int_0^{+\infty}\omega(t+\eta)\|\tilde{z}(-\eta)\|d\eta \leq b_2\omega(t)$, $\|\int_t^{+\infty}\tilde{L}_1^{(k)}(\eta)(L_1^{-1}L_2\otimes I_q)\tilde{z}d\eta\| \leq c_3\int_0^{+\infty}\omega(t+\eta)d\eta\|\tilde{z}\| \leq b_3\omega(t)$, and $\|\int_t^{+\infty}\tilde{L}_1^{(k)}(\eta)\vartheta^0(t-\eta)d\eta\| \leq c_4\int_0^{+\infty}\omega(t+\eta)\|\vartheta^0(-\eta)\|d\eta \leq b_4\omega(t)$. Thus, it follows that $\Upsilon_{\sigma(t)}$ tends to 0 exponentially, i.e., there exists constants c and b such that $\lim_{t\to\infty}\mathbb{E}[\|\Upsilon_{\sigma(t)}\|^2] \leq ce^{-bt}$.

Rewrite (12) for every follower i, $i\in\mathcal{R}$ as follows:

$$\begin{aligned} \dot{\delta}_i =& -\varpi d_i^{\sigma(t)}\delta_i + \varpi\sum_{i,j\in\mathcal{R}}a_{ij}^{\sigma(t)}\int_0^t\omega_{ij}(\eta)\delta_j(t-\eta)d\eta \\ &- \varpi\phi_i^{\sigma(t)}, \end{aligned} \tag{13}$$

where $\phi_i^{\sigma(t)}$ is the ith element of $\Upsilon_{\sigma(t)} = (\Upsilon_{M+1}^{\sigma(t)},\cdots,\Upsilon_{M+N}^{\sigma(t)})^T$.

Lemma 1: Consider the distributed observer (8) under Markovian randomly switching topologies and infinite delays. Let Assumptions 3, 4, and 5 hold. If eigenvalues of S fall on the imaginary axis, then $\lim_{t\to+\infty}\mathbb{E}[\|\vartheta_i - \sum_{j\in\mathcal{D}}\beta_{ij}\vartheta_j\|^2] = 0,\ i\in\mathcal{R}$, exponentially if constant ϖ is sufficiently large.

Proof: Define the following novel Lyapunov functional,

$$V = \sum_{k\in\mathbb{S}} V(t,k), \tag{14}$$

where $V(t,k) = V_1(t,k) + V_2(t,k) + V_3(t,k)$, $V_1(t,k) = \mathbb{E}[\sum_{i\in\mathcal{R}}\|\delta_i\|^2\mathbf{1}_{\{\sigma(t)=k\}}]$, $V_2(t,k) = \varpi\mathbb{E}[\sum_{i,j\in\mathcal{R}} a_{ij}^{(k)}\int_0^t\|\delta_j(\eta)\|^2\int_{t-\eta}^{+\infty}\omega_{ij}(s)dsd\eta\mathbf{1}_{\{\sigma(t)=k\}}]$, $V_3(t,k) = \nu\mathbb{E}[\sum_{i\in\mathcal{R}}\int_{-\infty}^t\|\delta_i(\eta)\|^2\int_{t-\eta}^{+\infty}\omega(s)dsd\eta\mathbf{1}_{\{\sigma(t)=k\}}]$, with ν being a positive constant to be determined, and $\mathbf{1}_{\{\sigma(t)=k\}}$ representing the indicator function. Along the trajectories of (13), taking derivative of $V_1(t,k)$, we have

$$\begin{aligned}\dot V_1(t,k) =& -2\varpi\mathbb{E}\left[\sum_{i\in\mathcal{R}} d_i^{\sigma(t)}\|\delta_i\|^2\mathbf{1}_{\{\sigma(t)=k\}}\right]\\ &+2\varpi\mathbb{E}\left[\sum_{i,j\in\mathcal{R}} a_{ij}^{\sigma(t)}\delta_i^T\int_0^t\omega_{ij}(\eta)\delta_j(t-\eta)d\eta\mathbf{1}_{\{\sigma(t)=k\}}\right]\\ &+2\varpi\mathbb{E}\left[\sum_{i\in\mathcal{R}}\delta_i^T\phi_i^{\sigma(t)}\mathbf{1}_{\{\sigma(t)=k\}}\right]+\sum_{r=1}^{s}\gamma_{rk}V_1(t,r).\end{aligned} \tag{15}$$

Bear in mind that the process $\{\sigma(t), t\ge 0\}$ initiates from its stationary distribution α. Consequently, one has

$$\begin{aligned}\dot V_1 =& \sum_{k\in\mathbb{S}}\dot V_1(t,k)\\ \le& -2\varpi\mathbb{E}\left[\sum_{i\in\mathcal{R}}\rho_i\|\delta_i\|^2\right]\\ &+2\varpi\mathbb{E}\left[\sum_{i,j\in\mathcal{R}}\rho_{ij}\delta_i^T\int_0^t\omega_{ij}(\eta)\delta_j(t-\eta)d\eta\right]\\ &+\frac{1}{\varpi}\mathbb{E}\left[\sum_{i\in\mathcal{R}}\|\delta_i\|^2\right]+\varpi^3\mathbb{E}\left[\sum_{i\in\mathcal{R}}\|\phi_i^{\sigma(t)}\|^2\right],\end{aligned} \tag{16}$$

where $\rho_i = \sum_{k\in\mathbb{S}}\alpha_k d_i^{(k)}$, $\rho_{ij} = \sum_{k\in\mathbb{S}}\alpha_k a_{ij}^{(k)}$. By utilizing the Cauchy–Schwarz inequality, we have

$$\begin{aligned}&2\varpi\rho_{ij}\delta_i^T\int_0^t\omega_{ij}(\eta)\delta_j(t-\eta)d\eta\\ &\le\varpi\rho_{ij}\|\delta_i\|^2+\varpi\rho_{ij}\int_0^t\omega_{ij}(\eta)\|\delta_j(t-\eta)\|^2d\eta.\end{aligned} \tag{17}$$

Combining (16) and (17) leads to

$$\begin{aligned}\dot V_1 \le& -2\varpi\mathbb{E}\left[\sum_{i\in\mathcal{R}}\rho_i\|\delta_i\|^2\right]+\varpi\mathbb{E}\left[\sum_{i,j\in\mathcal{R}}\rho_{ij}\|\delta_i\|^2\right]\\ &+\varpi\mathbb{E}\left[\sum_{i,j\in\mathcal{R}}\rho_{ij}\int_0^t\omega_{ij}(\eta)\|\delta_j(t-\eta)\|^2d\eta\right]\\ &+\frac{1}{\varpi}\mathbb{E}\left[\sum_{i\in\mathcal{R}}\|\delta_i\|^2\right]+\varpi^3ce^{-bt}.\end{aligned} \tag{18}$$

Moreover, it follows from the definition of $V_2(t,k)$ and $V_3(t,k)$ that

$$\begin{aligned}\dot V_2 =& \varpi\mathbb{E}\left[\sum_{i\in\mathcal{R}}\hat\varrho_i\|\delta_i\|^2\right]\\ &-\varpi\mathbb{E}\left[\sum_{i,j\in\mathcal{R}}\rho_{ij}\int_0^t\omega_{ij}(\eta)\|\delta_j(t-\eta)\|^2d\eta\right],\end{aligned} \tag{19}$$

and

$$\dot V_3 = \nu\mathbb{E}\left[\sum_{i\in\mathcal{R}}\left(\omega_0\|\delta_i\|^2-\int_0^{+\infty}\omega(\eta)\|\delta_i(t-\eta)\|^2d\eta\right)\right], \tag{20}$$

where $\hat\varrho_i = \sum_{k\in\mathbb{S}}\sum_{j\in\mathcal{R}}\alpha_k a_{ji}^{(k)}$ and $\omega_0 = \int_0^{+\infty}\omega(s)ds$. Then, from (18)-(20), one has

$$\begin{aligned}\dot V \le& \mathbb{E}\left[\sum_{i\in\mathcal{R}}(-\varpi\rho_i+\varpi\hat\varrho_i+1/\varpi)\|\delta_i\|^2\right]\\ &+\nu\mathbb{E}\left[\sum_{i\in\mathcal{R}}\left(\omega_0\|\delta_i\|^2-\int_0^{+\infty}\omega(\eta)\|\delta_i(t-\eta)\|^2d\eta\right)\right]\\ &+\varpi^3ce^{-bt}.\end{aligned} \tag{21}$$

We have $-\varpi\rho_i+\varpi\hat\varrho_i = -\varpi\sum_{j\in\mathcal{D}}\alpha_k a_{ij}^{(k)}$. Then, we have that for a sufficiently large constant ϖ, there exists a constant $\psi>0$ such that $-\varpi\sum_{j\in\mathcal{D}}\alpha_k a_{ij}^{(k)}+\frac{1}{\varpi}<-\psi,\ i\in\mathcal{R}$. It then follows from (21) that

$$\begin{aligned}\dot V \le& -\psi\mathbb{E}\left[\sum_{i\in\mathcal{R}}\|\delta_i\|^2\right]+\varpi^3ce^{-bt}\\ &+\nu\mathbb{E}\left[\sum_{i\in\mathcal{R}}\left(\omega_0\|\delta_i\|^2-\int_0^{+\infty}\omega(\eta)\|\delta_i(t-\eta)\|^2d\eta\right)\right]\\ =& -\nu\left(\mathbb{E}\left[\sum_{i\in\mathcal{R}}\|\delta_i\|^2+\sum_{i\in\mathcal{D}}\int_0^{+\infty}\omega(\eta)\|\delta_i(t-\eta)\|^2d\eta\right]\right)\\ &+\varpi^3ce^{-bt},\end{aligned} \tag{22}$$

where $\nu = \frac{\psi}{1+\omega_0}$.

It follows from the definition of $V_1(t,k)$, $V_2(t,k)$, and $V_3(t,k)$ directly that

$$V_1 = \mathbb{E}\left[\sum_{i\in\mathcal{R}}\|\delta_i\|^2\right], \tag{23}$$

$$\begin{aligned}V_2 =& \varpi\mathbb{E}\left[\sum_{i,j\in\mathcal{R}}\rho_{ij}\int_0^t\|\delta_j(t-\eta)\|^2\int_{t-\eta}^{+\infty}\omega_{ij}(s)dsd\eta\right]\\ \le& \varpi\mathbb{E}\left[\sum_{i,j\in\mathcal{R}}\rho_{ij}\int_0^t\|\delta_j(t-\eta)\|^2\int_0^{+\infty}\omega_{ij}(\eta+s)dsd\eta\right]\\ \le& \varpi\omega_0\hat\varrho\mathbb{E}\left[\sum_{i,j\in\mathcal{R}}\int_0^{+\infty}\omega(\eta)\|\delta_i(t-\eta)\|^2d\eta\right],\end{aligned} \tag{24}$$

and

$$V_3 = \nu\mathbb{E}\left[\sum_{i\in\mathcal{R}}\int_{-\infty}^{t}\|\delta_i(\eta)\|^2\int_{t}^{+\infty}\omega(s+\eta)dsd\eta\right]$$
$$\leq \nu\omega_0\mathbb{E}\left[\sum_{i\in\mathcal{R}}\int_{0}^{+\infty}\omega(\eta)\|\delta_i(t-\eta)\|^2d\eta\right], \quad (25)$$

where $\hat{\varrho} = \max_{i\in\mathcal{R}}\{\hat{\varrho}_i\}$. Then, combining (23)-(25), we have

$$V \leq \Theta\mathbb{E}\left[\sum_{i\in\mathcal{R}}\left(\|\delta_i\|^2 + \int_0^{+\infty}\omega(\eta)\|\delta_i(t-\eta)\|^2d\eta\right)\right], \quad (26)$$

where $\Theta = \varpi\omega_0\hat{\varrho} + \nu\omega_0$. Furthermore, from (22), one has $\dot{V}(t) \leq -\Gamma V + \varpi^3 ce^{-bt}$, where $\Gamma = \frac{\nu}{\Theta}$. According to the comparison lemma, it follows that $V \leq V(0)e^{-\Gamma t} + \varpi^3 c\chi(t,b)$, where $\chi(t,b)$ is a function described by $\chi(t,b) = \begin{cases} te^{-\Gamma t}, & \text{if } \Gamma = b, \\ \frac{1}{\Gamma - b}\left(e^{-bt} - e^{-\Gamma t}\right), & \text{if } \Gamma \neq b. \end{cases}$ It can be seen that $\lim_{t\to+\infty}\chi(t,b) = 0$. Moreover, $V \geq \mathbb{E}[\|\delta\|^2]$. Thus, one can get $\lim_{t\to\infty}\mathbb{E}[\|\delta\|^2] = 0$ exponentially. From [16, Lemma 3], for any $0 < \rho_0 < \min\{\Gamma, b\}$, there exists a $\nu_0 > 0$ such that $\|e^{St}\| \leq \nu_0 e^{\frac{\rho_0}{2}t}$. It is noted that the state containment error is: $\zeta = \vartheta + (L_1^{-1}L_2 \otimes I_q)z = (I_N \otimes e^{St})(\tilde{\vartheta} + (L_1^{-1}L_2 \otimes I_q)\tilde{z}) = (I_N \otimes e^{St})\delta$. Then, one has $\lim_{t\to+\infty}\mathbb{E}\left[\|\vartheta + (L_1^{-1}L_2 \otimes I_q)z\|^2\right] = 0$, which implies $\lim_{t\to+\infty}\mathbb{E}[\|\vartheta_i - \sum_{j\in\mathcal{D}}\beta_{ij}z_j\|^2] = 0$ exponentially. ■

Remark 3: [15] investigated containment control problem of MASs with infinite delays by using the frequency domain approach, which is hard to address time-varying delayed systems in this work. Therefore, the time domain Lyapunov method is adopted. Compare with [16], where the case of deterministic switching topologies was studied, this work studies the Markovian randomly switching topologies. Different from [10], where output containment control problem of MASs under Markovian randomly switching topologies and bound distributed delays was considered, this paper extends the case of bounded distributed delays to the more general infinite distributed delays.

B. Controller Design and Convergence Analysis

A novel distributed controller with Markovian randomly switching topologies and infinite communication delays is developed according to the newly proposed distributed observer (8) as follows:

$$u_i = K_{1i}\hat{x}_i + K_{2i}\vartheta_i, \; i\in\mathcal{R} \quad (27a)$$
$$\dot{\hat{x}}_i = A_i\hat{x}_i + B_iu_i + H_i\left(C_ix_i - C_i\hat{x}_i\right), \quad (27b)$$
$$\dot{\vartheta}_i(t) = S\vartheta_i(t) - \varpi\sum_{j\in\mathcal{R}}a_{ij}^{\sigma(t)}\big(\vartheta_i(t) - T_{ij}(S,\vartheta_j,\vartheta_j^0)\big) - \varpi\sum_{j\in\mathcal{D}}c_{ij}^{\sigma(t)}\big(\vartheta_i(t) - T_{ij}(S,z_j,z_j^0)\big), \quad (27c)$$

where $\hat{x}_i \in \mathbb{R}^{p_i}$ is the Luenberger observer's state, $\vartheta_i \in \mathbb{R}^n$ is state of the distributed observer in (8), and K_{1i}, K_{2i}, H_i are matrices to be designed.

The main result of this work is presented in the following.

Theorem 1: Consider MAS (1)-(2) under Markovian randomly switching topologies and infinite delays. Let Assumptions 1-5 hold. Choose K_{1i} and H_i such that $A_i + B_iK_{1i}$ and $A_i - H_iC_i$ are Hurwitz, and $K_{2i} = \Xi_i - K_{1i}X_i, i \in \mathcal{R}$, where $(X_i, \Xi_i), i \in \mathcal{R}$ are solutions to (4). Then the distributed output containment control problem is solved by the proposed controller (27) with sufficiently large ϖ.

Proof: Define $x = (x_1^T, \ldots, x_N^T)^T$, $v = (v_1^T, \ldots, v_N^T)^T$, $y_F = (y_1^T, \ldots, y_N^T)^T$, $y_L = (y_{N+1}^T, \ldots, y_{N+M}^T)^T$, and $e = (e_1^T, \ldots, e_N^T)^T$. In this work, the output containment error is denoted as $e = y_F + \left(\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes I_q\right)y_L$. Denote $A = \text{blockdiag}\{A_1, \ldots, A_N\}$, $B = \text{blockdiag}\{B_1, \ldots, B_N\}$, $C = \text{blockdiag}\{C_1, \ldots, C_N\}$, $H = \text{blockdiag}\{H_1, \ldots, H_N\}$, $K_1 = \text{blockdiag}\{K_{11}, \ldots, K_{1N}\}$, and $K_2 = \text{blockdiag}\{K_{12}, \ldots, K_{2N}\}$. From regulator equations (4) in Assumption 2, we have $AX + B\Xi = X\left(I_N \otimes S\right)$, $CX = I_N \otimes R$, where $X = \text{blockdiag}\left\{\Pi_1, \ldots, \Pi_N\right\}$ and $\Xi = K_1\Pi + K_2$. Define $\varphi = x + \Pi\left(L_1^{-1}L_2 \otimes I_q\right)z$ and $\tilde{x} = x - v$. Then, the heterogeneous MAS (1)-(2) under controller (27) is given as follows:

$$\begin{cases} \dot{\varphi} = (A + BK_1)\varphi + BK_2\zeta - BK_1\tilde{x}, \\ \dot{\tilde{x}} = (A - HC)\tilde{x}, \\ \dot{z} = (I_M \otimes S)z, \\ \dot{\vartheta} = \left(I_N \otimes S\right)\vartheta - \varpi\Bigg(\left(D_{\sigma(t)} \otimes I_q\right)\vartheta \\ \quad + \displaystyle\int_0^t\left(L_1^{\sigma(t)}(\eta) \otimes e^{S\eta}\right)\vartheta(t-\eta)d\eta \\ \quad + \displaystyle\int_0^t\left(L_2^{\sigma(t)}(\eta) \otimes e^{S\eta}\right)z(t-\eta)d\eta \\ \quad + e^{(I_N\otimes S)t}\displaystyle\int_t^{+\infty}\left(L_1^{\sigma(t)}(\eta) \otimes I_q\right)\vartheta^0(t-\eta)d\eta \\ \quad + e^{(I_N\otimes S)t}\displaystyle\int_t^{+\infty}\left(L_2^{\sigma(t)}(\eta) \otimes I_q\right)z^0(t-\eta)d\eta \\ \zeta = \vartheta + \left(L_1^{-1}L_2 \otimes I_q\right)z, \\ e = C\varphi. \end{cases} \quad (28)$$

Noting that $A_i - H_iC_i, i \in \mathcal{R}$ are Hurwitz, it follows that $\lim_{t\to+\infty}\mathbb{E}[\|\tilde{x}\|^2] = 0$ as indicated by [17, Lemma 2]. Moreover, since $\lim_{t\to+\infty}\mathbb{E}[\|\zeta\|^2] = 0$ via Lemma 1. From the fact that $A_i + B_iK_{1i}, i \in \mathcal{R}$ are Hurwitz, one has $\lim_{t\to+\infty}\mathbb{E}[\|\varphi\|^2] = 0$ via [18, Lemma 1]. Because $\lim_{t\to+\infty}\mathbb{E}[\|\varphi\|^2] = 0$, it follows that $\lim_{t\to+\infty}\mathbb{E}[\|e\|^2] = 0$, i.e., $\lim_{t\to+\infty}\mathbb{E}\left[\left\|y_F + (L_1^{-1}L_2 \otimes I_q)y_L\right\|^2\right] = 0$. According to (9), we have $\lim_{t\to+\infty}\mathbb{E}\left[\|y_i - \sum_{j\in\mathcal{R}\cup\mathcal{D}}\beta_{ij}y_j\|^2\right] = 0, i \in \mathcal{R}$. Hence, by Definition 1, the distributed output containment control problem is solved. ■

IV. Simulation Example

Consider the heterogeneous MAS described as in the form of (1) and (2) with $A_i = (0,1,0;0,0,t_i;0,-g_i,-s_i)$, $B_i = (0,0,f_i)^T$, $C_i = (1,0,0)$ and $S = (0,1;-1,0)$,

$R = (1, 0)$, where parameters $\{s_i, f_i, t_i, g_i\}$ are selected as $\{4, 2, 1, 0\}$, $\{10, 2, 1, 0\}$, $\{2, 1, 1, 10\}$, and $\{2, 1, 1, 1\}$ for four followers $i = 5, 6, 7, 8$, respectively. Through the resolution of the regulator equations (6), we can derive $X_i = (1, 0; 0, 1; -\frac{1}{t_i}, 0)$ and $\Xi_i = (-\frac{s_i}{f_i t_i}, -\frac{1}{f_i t_i} + \frac{g_i}{f_i})$. The communication topologies are randomly switched between two digraphs as given in Fig. 1. It can be observed that

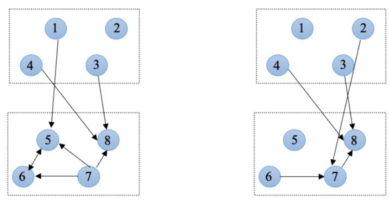

Fig. 1: The communication topologies.

in each digraph the leader set is not globally reachable. However, the leader set attains globally reachable in $\mathcal{G}_{un}$. The switch signal $\{\sigma(t), t \geq 0\}$ is determined by a continuous-time Markov chain. Let $\mathbb{S} = \{1, 2\}$ and $\Omega = (-1, 1; 2, -2)$. The initial distribution of $\{\sigma(t), t \geq 0\}$ is taken as its stationary distribution $\alpha = (2/3, 1/3)^T$. The delay kernel functions are chosen as $\omega_{51}(\eta) = \omega_{56}(\eta) = \omega_{57}(\eta) = \omega_{87}(\eta) = e^{-\eta}$, $\omega_{65}(\eta) = \omega_{67}(\eta) = \omega_{84}(\eta) = 4\eta e^{-2\eta}$, $\omega_{76}(\eta) = \omega_{72}(\eta) = \omega_{83}(\eta) = \frac{6}{5}e^{-\frac{6}{5}\eta}$, and choose $\omega(\eta) = \frac{6}{5}e^{-\eta}, \eta \in [0, +\infty)$, which implies that condition (5) holds. Let $\varpi = 5$ and the initial communication information be chosen as: $z_1^0(t_0) = (2, 6)^T$, $z_2^0(t_0) = (3, 6)^T$, $z_3^0(t_0) = (6, 3)^T$, $z_4^0(t_0) = (3, 1)^T$, $\vartheta_5^0(t_0) = (3, 12)^T$, $\vartheta_6^0(t_0) = (-3, 5)^T$, $\vartheta_7^0(t_0) = (-3, 7)^T$, $\vartheta_8^0(t_0) = (4, 8)^T$, $t_0 \in (-\infty, 0)$, which implies that condition (6) holds. Choose $K_{15} = (-12.5, -12.5, -2.5)$, $K_{16} = (-12.5, -12.5, -0.5)$, $K_{17} = (-25, -15, -7)$, $K_{18} = (-25, -25, -7)$ such that $A_i + B_i K_{1i}$ are Hurwitz for $i = 5, 6, 7, 8$ and K_{2i} can be obtained using $K_{2i} = \Xi_i - K_{1i} X_i$. For $i = 5, 6, 7, 8$, let $\zeta_i = \vartheta_i - \sum_{j \in \mathcal{D}} \beta_{ij} z_j = (\zeta_{i1}, \zeta_{i2})^T$ denote the state containment error. Generate 500 sample paths to approximate $\mathbb{E}[\|\zeta_i\|^2]$ and $\mathbb{E}[\|e_i\|^2]$ as depicted in Figs. 2 and 3, respectively. It is shown in Fig. 3 that the output containment errors converge to 0 as $t \to +\infty$ under the controller (27).

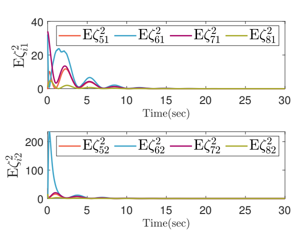

Fig. 2: State errors.

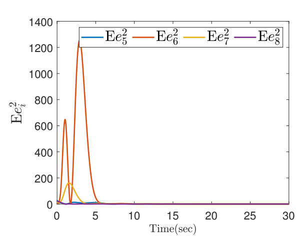

Fig. 3: Output errors.

V. Conclusions

This paper has studied the output containment control problem for heterogeneous MASs over Markovian randomly switching topologies and infinite delays. To address the concerned problem, firstly, a distributed containment observer has been designed for followers to address the uncertainty of the communication topologies and the occurrence of infinite delays. Then, a distributed containment controller has been constructed. An example is given for illustrating the effectiveness of the proposed controller.

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