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Discrete-Time Solution to the Performance Guaranteeing H_∞ Event-Triggered Control Problem

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***Abstract*— We study event-triggered control for discrete systems with the H_∞ (induced ℓ_2) performance measure. We construct event-triggered controllers generating sampling intervals no smaller than those of the optimal time-triggered controller under the same H_∞ performance bound γ. The design philosophy is based on a parametrization of discrete, possibly nonlinear and time varying, γ-suboptimal controllers and triggering events via the Q parameter that renders the parametrization sampled-data. Although this is similar to our previous event-triggered design for continuous-time systems, the lack of continuity of discrete behaviors constitutes non-trivial differences that require special treatment. In particular, dynamic event-triggering is proposed to compensate for premature triggering, which can only be detected a posteriori. As a result of the discrete-time nature of the problem, there appears to be a wider class of signals that causes our event-triggered controllers to generate the optimal time-triggered sampling pattern. We characterize a narrow subclass, the continuous-time counterpart, through the associated difference Riccati equation.**

I. INTRODUCTION

Sampled-data controllers may be thought of as controllers in which the information from sensors to actuators is sent only at sampling instances, regular or intermittent. Classical time-triggered control (TTC) has sampling instances generated in a process independent fashion, associated with either an external clock [1] or process independent communication opportunities [2]. In event-triggered control (ETC), sampling generation depends on the controlled signals. A key motivation behind introducing event triggering is the potential to reduce the communication burden without compromising the attainable performance. This potential was first demonstrated analytically [3] in the LQG setting and was exploited in numerous publications during the last couple of decades, see [4–6] and the references therein. Finding an optimal event triggering may still be an open problem for general linear time-invariant (LTI) systems. Yet for LQG, there are ETC algorithms strictly outperforming TTC [7] in the sense that it guarantees strictly sparser average sampling under the same performance as the optimal TTC.

Choosing an event-triggered controller is justified if it strictly outperforms TTC or, if not strictly, the conditions rendering strict superiority unattainable are understood and their restrictiveness quantified. To the best of our knowledge, the first H_∞ event-triggering solution outperforming the optimal TTC was proposed in the analogue output-feedback setting [8] (the first version with the same title was presented at the 58th CDC in 2019). There, we exhaustively characterized a class of internal *spoilers* signals that prevent the ETC scheme from strictly outperforming TTC. The spoiler class is shown to belong to a finite-dimensional subspace of L_2 and may thus be regarded as atypical. A discrete state-feedback ETC outperforming TTC was independently proposed [9] using game-theoretic techniques, and was extended to the output-feedback setting [10, Ch. 7].

In this paper we also address the discrete version of the problem, but following the the technique of [8]. Apart from a substantially dissimilar solution procedure, our result differs from those in [10] in the following aspects. First, our setup is less restrictive, as we do not impose any limiting assumption on the ranks of the control penalty and measurement noise weights, nor do we require the invertibility of the associated Riccati solutions. Second, we characterize spoiler signals for some situations, whereas [10] does not address this question. Third, our control law is nontrivially different from those in [10]. Specifically, we effectively implement an open-loop worst-case disturbance generating the control signal during the intersample, whereas the results of [9] use the zero-order hold after some steps.

Our previous design philosophy is to start with a preferably complete parametrization of possibly nonlinear, time-varying controllers solving a non-strict version of the H_∞ γ-suboptimal problem, where the closed-loop norm is no larger than γ. Next, all controllers that are sampled-data within this parametrization are extracted by imposing a strict causality constraint. The "central controller" of this sampled-data family has a reset system as its Q parameter. It is thus a convenient choice for outfitting an event-triggering mechanism, where a reset is triggered whenever the input and output signals of Q detect its γ-contractivity on the current sampling interval. Moreover, this approach affords a complete characterization of the spoiler class, barring which we have a stronger claim of increased sampling sparsity per sampling interval rather than on average.

The steps described above can in principle be carried over to the discrete setting, but not without a few tweaks to adjust to the discontinuous behavior of the signals. One implication of discrete signals is that one cannot expect a system to exactly reach a feasible norm bound. Another implication is that the obstruction to a strictly larger sampling interval includes not just the norm bound achieving inputs to Q like in the L_2 case. This means that a favorable conclusion for

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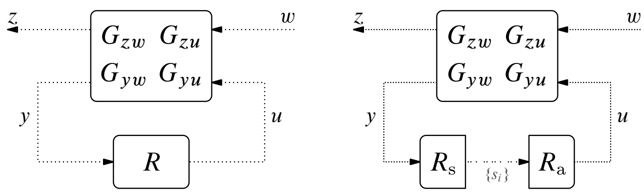


(a) general discrete controller (b) intermittent controller

Fig. 1: Standard problems with generalized plants

ETC might not be as strong in the ℓ_2 case. Additionally, to characterize even just the norm bound achieving inputs of Q requires an alternative approach, as the previous method is not readily portable.

It is therefore the objective of this note to highlight our solutions to the technical problems unique to the ℓ_2 setting, besides the main program of designing performance guaranteeing event-triggered controllers that are provably more efficient in spending the sampling budget.

Notation: The set of all non-negative integers is denoted as $\mathbb{Z}_+$ and $\mathbb{Z}_{i_1..i_2} := \{i \in \mathbb{Z} \mid i_1 \leq i \leq i_2\}$. The dimension of a signal x is denoted as n_x. By $\ell_2(\mathbb{I})$ we denote the space of square integrable signals $x : \mathbb{I} \to \mathbb{R}^{n_x}$ for some $\mathbb{I} \subset \mathbb{Z}_+$ and $\|\cdot\|_2$ denotes the $\ell_2(\mathbb{I})$ signal norm. When a system G is considered on the interval $\mathbb{Z}_{0..h-1}$ for some $h \in \mathbb{Z}_+$, we write G_h. The $\ell_2(\mathbb{Z}_{0..h-1})$-induced norm of G_h is also denoted $\|G_h\|_\infty$. It is the minimal $\gamma \geq 0$ such that $\|y\|_2^2 \leq \gamma^2\|u\|_2^2$ for all $u \in \ell_2(\mathbb{Z}_{0..h-1})$. Given a $\tau \in \mathbb{Z}_+$, the truncation operator P_τ acts on a discrete signal x as

$$(P_\tau x)[t] = \begin{cases} x[t] & t < \tau \\ 0 & \text{otherwise.} \end{cases}$$

The lower linear-fractional transformation (LFT) of Ω by the generator Φ is defined as

$$\mathcal{F}_l(\Phi, \Omega) := \Phi_{11} + \Phi_{12}\Omega(I - \Phi_{22}\Omega)^{-1}\Phi_{21}.$$

II. Problem setup

The setups in Fig. 1 represent the standard problem of designing a controller for a generalized plant in discrete time. The generalized plant is a linear shift-invariant (LSI) system, whose transfer function is given by its state-space realization

$$G(z) = \begin{bmatrix} G_{zw}(z) & G_{zu}(z) \\ G_{yw}(z) & G_{yu}(z) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array}\right],$$

satisfying standard assumptions on stabilizability and well-posedness, viz.

$\boldsymbol{\mathcal{A}_1}$: (C_y, A, B_u) is stabilizable and detectable,

$\boldsymbol{\mathcal{A}_2}$: $\left[\begin{smallmatrix} A-\mathrm{e}^{\mathrm{j}\theta}I & B_u \\ C_z & D_{zu} \end{smallmatrix}\right]$ is left invertible $\forall\theta \in [-\pi, \pi]$,

$\boldsymbol{\mathcal{A}_3}$: $\left[\begin{smallmatrix} A-\mathrm{e}^{\mathrm{j}\theta}I & B_w \\ C_y & D_{yw} \end{smallmatrix}\right]$ is right invertible $\forall\theta \in [-\pi, \pi]$.

What distinguishes the setup in Fig. 1(b) from that in 1(a) is the sampling instances at which the information flows from the sensor R_s to the actuator R_a. A sampling sequence $\{s_i\}$ is composed of strictly increasing sampling instances $s_i \in \mathbb{Z}_+$; the differences between consecutive sampling instances are sampling intervals $h_i := s_{i+1} - s_i$. The restriction of a controller's information transfer to a sampling sequence $\{s_i\}$ can be formalized by a causality constraint

$$\mathcal{C}_{\{s_i\}} := \{R \mid P_\tau R = P_\tau R P_{s_i},\ \tau \in \mathbb{Z}_{s_i+1..s_{i+1}},\ i \in \mathbb{Z}_+\}.$$

Controllers belonging to $\mathcal{C}_{\{s_i\}}$ are termed sampled-data and are strictly causal, that is, the control signal u in the current interval depends only on measurement y received during previous intervals. (Note that there is no loss of generality in dealing with strictly causal controllers, as we do not assume $D_{yu} = 0$ as is conventionally done.) The sampling sequence in Fig. 1(a) is the baseline periodic sampling $\{i\} = \mathbb{Z}_+$, whereas the one in Fig. 1(b) is a possibly intermittent and sparser $\{s_i\}$. The latter may be event or time-triggered; in other words, it may be process dependent or independent. In the case of event-triggered controllers, we note that the sampling instances s_i generated by the controller depends only on signals in previous intervals as well.

The goal of the γ-suboptimal H_∞ problem associated with $\{s_i\}$, denoted $\mathbf{OP}_{\{s_i\}}$, is to find sampled-data controllers $R \in \mathcal{C}_{\{s_i\}}$ that internally stabilize G while keeping the ℓ_2 gain of the closed-loop system T_{zw} from the exogenous input w to the regulated output z under a specified performance level γ. It is more restrictive than $\mathbf{OP}_{\{i\}}$ associated with the sampling sequence $\{i\}$ in the sense that the sampled-data controller $R \in \mathcal{C}_{\{s_i\}}$ needs to wait longer to send information from the sensor to the actuator.

Let the minimal performance measure attainable by $\mathbf{OP}_{\{i\}}$ be denoted γ_1. If $\mathbf{OP}_{\{i\}}$ is solvable, that is $\gamma \geq \gamma_1$, then there is a complete parametrization of LSI strictly causal controllers. The solution [11] is in terms of the parameters of $G(z)$, the stable and γ-contractive system Q, and the solutions to the standard H_∞ discrete algebraic Riccati equations [12] associated with the necessary and sufficient solvability conditions of $\mathbf{OP}_{\{i\}}$

$$X = \Phi_{xx} - \begin{bmatrix} \Phi_{xw} & \Phi_{xu} \end{bmatrix} \begin{bmatrix} \Phi_{ww} & \Phi_{wu} \\ \Phi_{uw} & \Phi_{uu} \end{bmatrix}^{-1} \begin{bmatrix} \Phi_{wx} \\ \Phi_{ux} \end{bmatrix}, \tag{1a}$$

$$Y = \Psi_{xx} - \begin{bmatrix} \Psi_{xz} & \Psi_{xy} \end{bmatrix} \begin{bmatrix} \Psi_{zz} & \Psi_{zy} \\ \Psi_{yz} & \Psi_{yy} \end{bmatrix}^{-1} \begin{bmatrix} \Psi_{zx} \\ \Psi_{yx} \end{bmatrix}, \tag{1b}$$

where

$$\begin{bmatrix} \Phi_{xx} & \Phi_{xw} & \Phi_{xu} \\ \Phi_{wx} & \Phi_{ww} & \Phi_{wu} \\ \Phi_{ux} & \Phi_{uw} & \Phi_{uu} \end{bmatrix} := -\begin{bmatrix} 0 & 0 & 0 \\ 0 & \gamma^2 I & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} A' & C_z' \\ B_w' & D_{zw}' \\ B_u' & D_{zu}' \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \end{bmatrix}$$

and

$$\begin{bmatrix} \Psi_{xx} & \Psi_{xz} & \Psi_{xy} \\ \Psi_{zx} & \Psi_{zz} & \Psi_{zy} \\ \Psi_{yx} & \Psi_{yz} & \Psi_{yy} \end{bmatrix} := -\begin{bmatrix} 0 & 0 & 0 \\ 0 & \gamma^2 I & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} A & B_w \\ C_z & D_{zw} \\ C_y & D_{yw} \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A' & C_z' & C_y' \\ B_w' & D_{zw}' & D_{yw}' \end{bmatrix}.$$

For the more restrictive $\mathbf{OP}_{\{s_i\}}$ to be solvable, γ must be strictly larger than γ_1. As such, the Riccati equations (1) have stabilizing solutions, namely, $X \geq 0$ and $Y \geq 0$ such that the matrices $A_K := A + B_w K_w + B_u K_u$ and $A_L := A + L_z C_z + L_y C_y$ are Schur, where

$$\begin{bmatrix} K_w \\ K_u \end{bmatrix} := -\begin{bmatrix} \Phi_{ww} & \Phi_{wu} \\ \Phi_{uw} & \Phi_{uu} \end{bmatrix}^{-1} \begin{bmatrix} \Phi_{wx} \\ \Phi_{ux} \end{bmatrix},$$

$$\begin{bmatrix} L_z & L_y \end{bmatrix} := -\begin{bmatrix} \Psi_{xz} & \Psi_{xy} \end{bmatrix} \begin{bmatrix} \Psi_{zz} & \Psi_{zy} \\ \Psi_{yz} & \Psi_{yy} \end{bmatrix}^{-1}.$$

Moreover, $Z := (I - \gamma^{-2} YX)^{-1}$ is nonsingular and

$$\overline{\sigma}\left(\begin{bmatrix} X^{1/2} A Y^{1/2} & X^{1/2} B_w \\ C_z Y^{1/2} & D_{zw} \end{bmatrix}\right) < \gamma.$$

where $\overline{\sigma}(\cdot)$ denotes the largest singular value of a matrix.

III. H_∞ Event-Triggered Control

The ETC designs proposed in this note shares the basic controller architecture with the TTC solution to $\mathbf{OP}_{\{s_i\}}$. The solvability of the TTC problem both serves as a prerequisite for that of the ETC problem and provides a basis for comparing sampling sparsity of the two designs. The existing time-triggered intermittent controllers solving $\mathbf{OP}_{\{s_i\}}$ given in [13, (8)] are extracted from the complete parametrization of linear time-varying controllers solving $\mathbf{OP}_{\{i\}}$. It is not directly usable, as event-triggered controllers are nonlinear time-varying, which necessitates a dedicated solution. We therefore first enlarge the search space of $\mathbf{OP}_{\{i\}}$ to account for general nonlinear controllers before arriving at a sampled-data one in §III-A. We then propose two ETC strategies that can outperform the optimal TTC in §III-B, and comment on characteristics unique to the discrete-time problem. In §III-C, we identify a class of signals reducing the ETC sampling pattern to that generated by the optimal TTC.

A. Controller architecture

We adopt our previous approach of extracting sampled-data controllers from the parametrization for $\mathbf{OP}_{\{i\}}$ by imposing the causality constraint $\mathcal{C}_{\{s_i\}}$ on the γ-contractive Q parameter [14]. In order to end up with a controller architecture friendly to our previous ETC design idea of triggering events by resetting the Q parameter, we derive the following parametrization for $\mathbf{OP}_{\{i\}}$. Define the LSI system $J\colon (y, \eta) \mapsto (u, \epsilon)$

$$J(z) = \left[\begin{array}{c|cc} A_K + Z L_y \tilde{C}_y & -Z L_y & Z \tilde{B}_u S_u^{-1} \\ \hline K_u & 0 & S_u^{-1} \\ -S_y^{-1} \tilde{C}_y & S_y^{-1} & -S_y^{-1} \tilde{D}_{yu} S_u^{-1} \end{array}\right] \quad (2)$$

where $\tilde{B}_u := B_u + L_z D_{zu} + L_y D_{yu}$, $\tilde{C}_y := C_y + D_{yw} K_w + D_{yu} K_u$, and S_u and S_y are any square matrices satisfying

$$\begin{bmatrix} S_u' S_u & \tilde{D}_{yu}' \\ \tilde{D}_{yu} & S_y S_y' \end{bmatrix} := \begin{bmatrix} \Phi_{uu} & D_{yu}' \\ D_{yu} & 0 \end{bmatrix} + \begin{bmatrix} \Phi_{ux} Y & \Phi_{uw} \\ C_y Y & D_{yw} \end{bmatrix} \times \begin{bmatrix} \gamma^2 I - \Phi_{xx} Y & -\Phi_{xw} \\ -\Phi_{wx} Y & -\Phi_{ww} \end{bmatrix}^{-1} \begin{bmatrix} \Phi_{xu} & C_y' \\ \Phi_{wu} & D_{yw}' \end{bmatrix}.$$

Lemma 3.1: Given $\boldsymbol{\mathcal{A}}_{\mathbf{1\text{–}3}}$ and $\gamma > \gamma_1$, then all parameters of J in (2) are well defined and a class of strictly causal stabilizing controllers that ensures $\|T_{zw}\|_\infty \leq \gamma$ is parametrized as $R = \mathcal{F}_l(J, Q)$ for all strictly causal Q such that $\|Q\|_\infty \leq \gamma$.

The parametrization in Lemma 3.1 admits nonlinearity in the Q parameter and is extended to non-strict norm bound, as is better suited to the ETC design later on. If Q is restricted to be linear, the result is still both necessary and sufficient with the non-strict norm bound. The proof steps of Lemma 3.1 parallels that of the continuous time counterpart [8, Prop. A.3], and is omitted because of space limitation.

Compared to the parametrization in [12], Lemma 3.1 parametrizes strictly causal rather than causal controllers; compared to the one used in [13], the present parametrization yields a sampled-data controller in the next proposition that allows us to reset the Q parameter directly rather than an augmented version of it.

Proposition 3.1: The controller $\mathcal{F}_l(J, Q_{\text{stat}})$ satisfies the sampled-data causality constraint $\mathcal{C}_{\{s_i\}}$, and is γ-suboptimal if and only if $\|Q_{\text{stat}}\| \leq \gamma$, where $Q_{\text{stat}}\colon \epsilon \mapsto \eta$ is

$$Q_{\text{stat}}\colon \begin{cases} x_q[t+1] = A_q x_q[t] + Z L_y S_y \epsilon[t], \quad x_q[s_i] = 0 \\ \qquad \eta[t] = S_u K_u x_q[t], \end{cases} \quad (3)$$

with $A_q = A_K - Z(\tilde{B}_u - L_y \tilde{D}_{yu}) K_u$.

With this choice, we can directly control the norm bound on Q_{stat} by resetting its state x_q. The proof follows the same logic as that of [15, Lem. 5] and is omitted due to space limitation.

Remark 3.1: In the time-triggered case, $\mathcal{F}_l(J, Q_{\text{stat}})$ is linear, and by the linear version of Lemma 3.1, the parametrization $\mathcal{F}_l(J, Q_{\text{stat}})$ for $\|Q_{\text{stat}}\| \leq \gamma$ in Prop. 3.1 is complete. The optimal TTC sampling period h_γ is then the maximum h_i such that $\|Q_{\text{stat}}\| \leq \gamma$ (or equivalently, that the condition expressed using a difference Riccati solution in [13, Thm. 1] is satisfied). Consequently, the optimal TTC controller is the optimal periodic controller with sampling period h_γ, and we need only compare the ETC sampling sequence with $\{i h_\gamma\}$.

For implementation, substituting (3) into (2) yields the controller with the sensor-side part R_s given by

$$\begin{aligned} x_s[t+1] = (A_K + Z L_y \tilde{C}_y - Z \tilde{B}_u K_u) x_s[t] \\ - Z L_y y[t] + Z \tilde{B}_u u[t] \end{aligned} \quad (4a)$$

and the actuator-side part R_a given by

$$x_a[t+1] = A_K x_a[t], \quad x_a[s_i] = x_s[s_i] \quad (4b)$$

$$u[t] = K_u x_a[t]. \quad (4c)$$

where $x_s = x_j$, $x_a = x_j + x_q$, and x_j is the state of J in (2).

B. Event-triggering algorithm design

In the TTC setting, given an intermittent sampling sequence, Q_{stat} resets at every s_i. Consequently, $\|Q_{\text{stat}}\|$ is determined by the largest sampling interval of the given sequence, which must respect h_γ. Otherwise if $h > h_\gamma$, there exists an input $\epsilon \in \ell_2(\mathbb{Z}_{0..h-1})$ such that $\|Q_{\text{stat},h}\|$ exceeds γ, and we can no longer guarantee closed-loop performance.

However, there is no need to guard against such an input by limiting the sampling period to h_γ if the controller is capable of inferring its absence. In fact, the controller can do so by monitoring whether $\|\eta\|_2/\|\epsilon\|_2 \leq \gamma$ for given $\eta, \epsilon \in \ell_2(\mathbb{Z}_{s_i..t})$ at the current time t, or equivalently $e[t] \leq 0$ where

$$e[t] := \sum_{i=s_i}^{t} \|\eta[i]\|^2 - \gamma^2 \sum_{i=s_i}^{t} \|\epsilon[i]\|_2^2. \tag{5}$$

Better yet, it is possible to detect whether $e[t+1]$ might exceed 0 with currently available information and to trigger a warning to reset Q_{stat}. A mechanism for doing that is

$$s_{i+1} = 1 + \min\left\{t \geq s_i \mid e^+[t] > 0\right\}, \quad \forall i, \tag{6}$$

where

$$e^+[t] := \sum_{i=s_i}^{t+1} \|\eta[i]\|^2 - \gamma^2 \sum_{i=s_i}^{t} \|\epsilon[i]\|_2^2. \tag{7}$$

The computation of $\eta[t+1]$ is what enables the controller to predict at the current time step if the finite-horizon ℓ_2 gain on the current interval *might* exceed γ at the next time step. The information needed for computing $\eta[t+1]$ is made available by the strict causality of Q_{stat}, see (3). At time t, $x_q[t+1]$ is known, so $\eta[t+1]$ can be obtained from the output equation. Although we do not have knowledge about $\epsilon[t+1]$ at time t, the worst possible increase of the ℓ_2 gain happens for $\epsilon[t+1] = 0$. To see this, note that

$$e[t+1] = e^+[t] - \gamma^2 \|\epsilon[t+1]\|_2^2 \leq e^+[t].$$

These considerations reflect the first distinct features of our discrete-time ETC design, which are absent in the continuous-time case. When the signals are in continuous-time, there are no jump from $e[t]$ to $e[t+1]$ before any reset occurs, so the continuous-time triggering only needs to be conditioned on $e(t)$ reaching exactly 0.

A further observation reveals why the strategy in (6) may be naive. Enforcing (6) likely does not result in $\|\eta\|_2/\|\epsilon\|_2$ equal precisely to γ (in this paragraph the signals are considered on $\mathbb{Z}_{s_i..t}$). One step before triggering due to positive $e^+[s_{i+1}-1]$, the actual norm difference $e[s_{i+1}-1]$ may fall short of 0. From the viewpoint of prolonging the sampling intervals as much as possible until $\|\eta\|_2/\|\epsilon\|_2$ reaches γ, one might wish to reset at s_{i+1} only when $e[s_{i+1}] > 0$. But this scheme is not causally implementable, and on top of that the possibility of $e[s_{i+1}-1] < 0$ remains unchanged. The conservatism introduced by the undershoot is thus unavoidable on any given sampling interval. Fortunately, with not just one but a sequence of sampling intervals, we can improve (6) by relaxing the triggering threshold on h_{i+1} to compensate for the undershoot incurred on h_i:

$$s_{i+1} = 1 + \min\left\{t \geq s_i \mid e^+[t] > \beta_i\right\}, \quad \forall i \tag{8}$$

with

$$\begin{aligned} \beta_0 &= 0 \\ \beta_{i+1} &= \beta_i - e[s_{i+1}-1]. \end{aligned} \tag{9}$$

The phenomenon of undershooting a specified norm bound, and thus the remedy applied to reduce the unrealized potential, are another aspect unique to the discrete-time ETC design. In contrast, the continuous-time event-triggered controller devised in [8, (12)] reaches the γ-bound on $\|Q_{\text{stat}}\|$ on every sampling interval. As a small side effect, a safety check must be in place to ensure that the equality there is not caused by zero ϵ to avoid needless (potentially Zeno) triggering, which need not be considered here.

With the ETC strategies, whether static (6) or dynamic (8), $\|Q_{\text{stat}}\| \leq \gamma$ is a consequence of $\|Q_{\text{stat},h_i}\| \leq \gamma$ for every i. As a matter of fact, in both (6) and (8), the search space can be equivalently restricted to $t \geq s_i + h_\gamma - 1$. This means that the sampling sequences generated by both algorithms cannot have sampling intervals smaller than the optimal TTC sampling period.

On the implementation side, the controller architecture is based on the sensor-actuator separation structure (4). The triggering signals that prevents Q_{stat} from overstepping the γ-bound shall be computed using signals available to (4).

$$\begin{aligned} \epsilon[t] &= S_y^{-1}\left(y[t] - (\tilde{C}_y - \tilde{D}_{yu}K_u)x_s[t] - \tilde{D}_{yu}u[t]\right) \\ \eta[t] &= S_u\left(u[t] - K_u x_s[t]\right). \end{aligned} \tag{10}$$

The main result of this subsection is then summarized.

Theorem 3.2: Let assumptions $\boldsymbol{\mathcal{A}}_{\mathbf{1\text{–}3}}$ hold and $\gamma > \gamma_1$. The ETC controller comprising the control architecture (4) and the event-triggering rule (6) or (8) with the triggering signal (7) realized as (10), internally stabilizes the system in Fig. 1(b), guarantees that $\|T_{zw}\|_\infty \leq \gamma$ and that $h_i \geq h_\gamma$ for every i.

Proof: As both (6) and (8) ensures γ-contractive Q_{stat} by construction, the part about solving the γ-suboptimal sampled-data H_∞ problem follows from Prop. 3.1. It remains to be shown that $h_i \geq h_\gamma$, which we prove only for the static triggering condition (6), as the case for the dynamic one (8) is implied by the former. This is because the dynamic thresholding of (8) allows larger sampling periods, since it is harder (takes more time) for $e^+[t]$ to be greater than $\beta \geq 0$ than the 0 threshold.

Recall that Prop. 3.1 specialized to the TTC case means that $\|Q_{\text{stat}}\| \leq \gamma$ is also necessary for the γ-suboptimal H_∞ problem to be solvable. It follows (restricting to the first sampling interval without loss of generality) that $\|Q_{\text{stat},h_\gamma}\| \leq \gamma$, or alternatively, $e[h_\gamma - 1] \leq 0$. Suppose that there exists an $h < h_\gamma$ generated by the ETC controller with the static triggering rule (6). This can only happen when $e^+[h-1] > 0$, which means that $\|\eta\|_2^2 - \gamma^2\|\epsilon\|_2^2 > 0$ (these signals are over $\mathbb{Z}_{0..h-1}$). Construct an input $(\epsilon[0], \ldots, \epsilon[h-1], 0, \ldots)$ which is padded by $h_\gamma - h$ zeros, the output of Q_{stat} must be $(\eta[0], \ldots, \eta[h], \eta(h+1), \ldots)$. Thus, $e[h_\gamma - 1] \geq e^+[h-1] > 0$, which is a contradiction. ■

C. How to spoil the ETC advantage

We have shown that on every sampling interval, both static and dynamic ETC designs generate sampling intervals no smaller than the optimal TTC sampling period. The question is then what adversarial inputs to Q_{stat} will force $h_i = h_\gamma, \forall i$ to render the ETC designs less attractive.

In the L_2 case, the spoiler signals are a narrow class of inputs to Q_{stat} defined on the interval $[0, h_\gamma)$, which exactly achieve the norm bound on Q_{stat,h_γ}. This would correspond to $e[h-1] = 0$ in the ℓ_2 case, which however, does not determine the next sampling instance in either (6) or (8). Unlike the L_2 design where events are generated upon an equality, the deciding factor in the static algorithm (6) is a strictly positive e^+. The class of ℓ_2 spoiler signals analogous to the L_2 counterpart is thus defined as

$$\{\epsilon : \mathbb{Z}_{0..h_\gamma-1} \to \mathbb{R}^{n_y} \mid e[h_\gamma - 1] = 0 \;\&\; e^+[h_\gamma - 1] > 0\}. \quad (11)$$

This class works against the dynamic design (8) too, as $e[h-1] = 0$ on every sampling interval leaves $\beta_i = 0$ for all i, see (9).

To characterize (11), we focus on (h, γ_h) pairs, where γ_h is the minimally attainable performance measure for a given sampling period h. This is done without loss of generality from the TTC perspective, as there is no reason to choose $\gamma > \gamma_h$ when both provide the same optimal TTC period. Let (A_q, B_q, C_q) denote the parameters of the strictly causal reset system Q_{stat} in (3). The associated difference Riccati equation is

$$\begin{aligned} P[t+1] &= A_q' P[t] A_q + C_q' C_q \\ &\quad - A_q' P[t] B_q (B_q' P[t] B_q - \gamma_h^2 I)^{-1} B_q' P[t] A_q \end{aligned} \quad (12)$$

with initial condition $P[0] = 0$.

Proposition 3.3: Given $h = h_\gamma$, the class of spoilers (11) can be computed as

$$\tilde{\epsilon}[t] = (\gamma_h^2 I - B_q' P[h-1-t] B_q)^{-1} B_q' P[h-1-t] A_q x_q[t] \quad (13)$$

for all $t \in \mathbb{Z}_{1..h-1}$, with any $\tilde{\epsilon}[0] \in \ker(\gamma_h^2 I - B_q' P[h-1] B_q)$, where $P[t]$ is the solution to (12), iff

$$\tilde{\eta}_h := C_q A_r[h-1] \cdots A_r[1] B_q \tilde{\epsilon}[0] \neq 0,$$

where $A_r[t] = (I - \gamma_h^{-2} B_q B_q' P[h-1-t])^{-1} A_q$.

Proof: It is based on the finite-horizon Bounded Real Lemma A.1. Details are omitted due to space limitation. ■

The condition $\tilde{\eta}_h \neq 0$ in Proposition 3.3 does not have a counterpart in the continuous-time spoiler class characterization. This is a consequence of triggering based on a next-step estimate in the discrete-time mechanism rather than on the current-time status in the continuous-time mechanism.

The derivation of the spoiler class (11) goes through characterizing a finite horizon operator Q, whose ℓ_2-induced norm is γ. In the continuous-time case, this was done by identifying the singularity condition of the system with two point boundary conditions $I - \gamma^{-2} Q'Q$, where Q' stands for the adjoint operator, see [8, Lem. B.1] for details. However, the adjoint of a discrete system is not representable by standard state-space model, unless one assumes a nonsingular "A" matrix. This could be circumvented by the use of the descriptor systems formalism. Yet solving state equations in the descriptor form would require a special state transformation, which would destroy the structure of the problem. This motivated us to take an alternative approach, where we use the difference Riccati equation.

Remark 3.2: Note that the class $\tilde{\epsilon}$ exists only for $h > 1$, as $\ker(\gamma_h^2 I - B_q' P[h-1] B_q)$ is trivial for $h = 1$. This is also consistent with the fact that resetting Q_{stat} at every time step results in $\eta[t] \equiv 0$. Consequently, $e[h-1] < 0$, which falls outside the set (11).

Remark 3.3: The behavior of $\tilde{\epsilon}$ at the end of the sampling interval is $\tilde{\epsilon}[h-1] = 0$, which also features in the continuous time spoiler signals.

To summarize this section, one way to prevent our ETC controllers from strictly outperforming the optimal TTC is the following.

Corollary 3.4: For $h = h_\gamma > 1$, if $\epsilon[t] = \tilde{\epsilon}[t]$ defined in (13) for all $t \in \mathbb{Z}_{s_i..s_i+h-1}$ on every sampling interval i and $\tilde{\eta}_h \neq 0$, then $\{s_i\} = \{i h_\gamma\}$ for both static (6) and dynamic (8) ETC controllers.

There are all sorts of ways to force the optimal TTC sampling pattern other than having ϵ belonging to the class (11) on every sampling interval. One obvious choice for the static triggering strategy is

$$\{\epsilon : \mathbb{Z}_{0..h_\gamma-1} \to \mathbb{R}^{n_y} \mid e[h_\gamma - 1] < 0 \;\&\; e^+[h_\gamma - 1] > 0\}.$$

Although the dynamic triggering strategy is expressly designed to mitigate the effect of this type of signals, one can similarly define the above class with varying threshold β_i for each sampling interval. Therefore, considered on the entire horizon, there appears to be a wider class of input signals $\epsilon(\mathbb{Z}_+)$ that forces $\{s_i\} = \{i h_\gamma\}$ for both (6) and (8).

IV. Illustrative Example

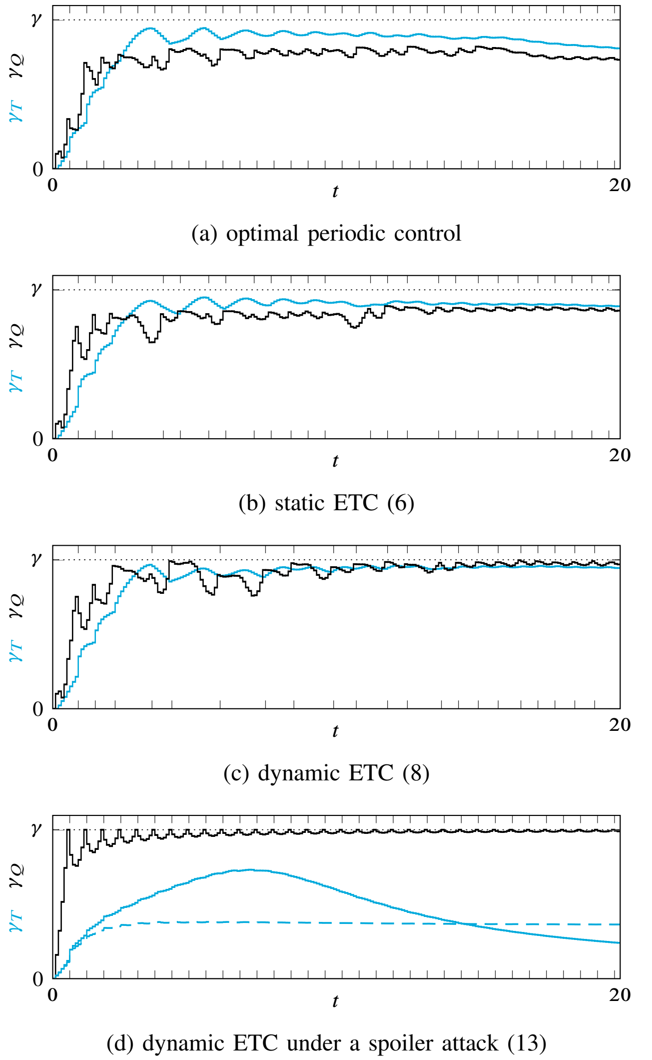
We consider a discrete-time single integrator with a baseline sampling period $h_0 = 0.1$

$$G(z) = \left[\begin{array}{c|cc:c} 1 & h_0 & 0 & h_0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\varrho} \\ \hdashline 1 & 0 & \sqrt{\varsigma} & 0 \end{array}\right].$$

It is obtained by the standard step-invariant discretization of the continuous-time single integrator example in [8, §V] with $\varrho = 1$ and $\varsigma = 0.0025$. For a given sampling period of $h = 6$ steps, the optimally attainable performance γ_h is found to be $\gamma_6 = 1.209$.

Inject into the closed-loop system $w[t] = (d[t], n[t])$, where the input disturbance $d[t]$ is sinusoidal with an increasing frequency in the range of $[0.2\pi, \pi]$ and the measurement noise $n[t]$ is zero-mean wide-band with the ℓ_2 norm equal to $\sqrt{\varsigma}$. The moving horizon ℓ_2 gains of Q_{stat} (black) and T_{zw} (blue) on the time interval $[0, 20]$, defined respectively as $\gamma_Q[t] = \|P_{t+1}\eta\|_2 / \|P_{t+1}\epsilon\|_2$ and $\gamma_T[t] = \|P_{t+1}z\|_2 / \|P_{t+1}w\|_2$, are depicted in Fig. 2.

Compared to the optimal uniform sampling period in Fig. 2(a), the average sampling periods of the ETC controllers are increased by 25% for the static design in Fig. 2(b) and 37% for the dynamic design in Fig. 2(c). Not pictured here is another experiment with a measurement noise from a different seed for the random number generator, where we



(a) optimal periodic control

(b) static ETC (6)

(c) dynamic ETC (8)

(d) dynamic ETC under a spoiler attack (13)

Fig. 2: Moving horizon ℓ_2 energy gains of Q_{stat} and T_{zw} for $\gamma = 1.209$; time axis ticks indicate sampling instances $\{s_i\}$.

see a more significant 73% reduction of sampling instances, demonstrating the potential of our ETC designs.

The periodic sampling sequence in Fig. 2(d) identical to the one in Fig. 2(a) is obtained by introducing exogenous signals rendering $\epsilon = \tilde{\epsilon}$ according to the spoiler formula (13) on every sampling interval. The two blue curves represent the closed-loop responses to two choices of $w[t]$ such that at each sampling instance $s_i = ih_\gamma$, $\tilde{\epsilon}[s_i] = 1$ which belongs to $\ker \gamma_h^2 I - B_q' P[h-1] B_q = \mathbb{R}$.

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Appendix

We present a discrete-time finite-horizon Bounded Real Lemma. A similar result appears in [16, Lem. 4] for time-varying systems, though the difference Riccati equation therein contains a pseudo-inverse, and no characterization of the minimal horizon achieving the norm bound is provided.

Consider the system

$$G\colon \begin{cases} x[t+1] = Ax[t] + Bu[t], \quad x[0] = 0 \\ \quad\; y[t] = Cx[t] + Du[t] \end{cases} \tag{14}$$

operating over the horizon $\mathbb{Z}_{0..h-1}$ and introduce the difference Riccati equation

$$P[t+1] = Q[h-t] - S'[h-t]R^{-1}[h-t]S[h-t]$$

for $P[0] = 0$, where

$$\begin{bmatrix} Q[t] & S'[t] \\ S[t] & R[t] \end{bmatrix} := \begin{bmatrix} A & B \\ C & D \end{bmatrix}' \begin{bmatrix} P[h-t] & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix}$$

for all $t \in \mathbb{Z}_{1..h}$.

Lemma A.1: The finite-horizon ℓ_2 gain of G in (14) satisfies $\|G_h\|_\infty < \gamma$ iff $R[h-t] < 0$ for all $t \in \mathbb{Z}_{0..h-1}$. Moreover, if $\|G_h\|_\infty < \gamma$, then $\|G_{h+1}\|_\infty = \gamma$ iff $\lambda_{\max}(R[0]) = 0$ and every input u for which $\|y\|_2^2 = \gamma^2\|u\|_2^2$ is of the form

$$u[t] = \begin{cases} u_0 & \text{if } t = 0 \\ -R^{-1}[t]S[t]x[t] & \text{otherwise} \end{cases}$$

for an arbitrary $u_0 \in \ker R[0]$.