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A New Optimal Design of Set-Theoretic Unknown Input Observer for Robust State Estimation

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Abstract— **Set-theoretic unknown input observer (SUIO) and set-valued observer (SVO) are two different kinds of robust state estimation methods, either with respective advantages in terms of state estimation conservatism. In this paper, we propose a new optimal design method for set-theoretic unknown input observer based on zonotopes and F-radius metric. We prove that the proposed method combines the advantages of both SUIO and SVO in conservatism without introducing extra computational complexity. Specifically, under the corresponding F-radius optimal designs, the worst state estimation outcome of the proposed method is as precise as the best outcome of both SUIO and SVO. To further reduce the computational cost, we establish the existence condition and proposed a computing method for the constant optimal observer parameters as time tends to infinity. Finally, we use a numerical example to illustrate the effectiveness of the proposed methods.**

I. INTRODUCTION

The precise estimation of a system state in the presence of uncertainty, such as disturbance and noise, is a fundamental problem in many engineering applications. The stochastic state estimation approaches, such as Kalman filter, have been extensively studied and applied on various systems. However, when dealing with systems with unknown but deterministic behaviors, it is not well-suited to model uncertainties as the probability distributions required by these filters [1]. To address this problem, deterministic approaches that do not rely on probability distributions have been proposed, where geometric sets such as intervals, polytopes, ellipsoids, zonotopes, and constrained zonotopes are used to characterize the boundary of uncertainties.

From a practical point of view, the conservatism (i.e., the precision of estimation) and computational complexity are two essential aspects for deterministic approaches. Theoretically, the set-membership estimation (SME) [2] has the lowest conservatism among deterministic approaches, i.e., such method can provide the most compact set that contains the real state [3]. The basic idea of SME is to calculate a prediction state set based on the system dynamics (through set mapping and Minkowski sum) and then correct this set using another state set that is consistent with the measurement (through intersection). However, since some set operations (e.g., the nonlinear mapping, the Minkowski sum of ellipsoids, and the intersection of ellipsoids and zonotopes) are not closed, the SME typically needs to be implemented with overapproximation, which prevents it from achieving the theoretically lowest conservatism [4]. Although this overapproximation can be avoided on linear systems using constrained zonotopes and polytopes, the computational complexity is still relatively high or even unacceptable [4], [5]. Contrary to the SME, the interval observer (IO) [6], [7] is a deterministic approach with extremely low computational complexity. Based on the monotone system theory [8], the IO provides an interval (i.e., the lower and upper bounds) estimation of the real state. However, compared with other geometric sets, the interval is naturally with considerable conservatism [9]. In order to satisfy the cooperativity property, the IO needs to sacrifice part of the parametric design freedom or additionally introduce interval hull approximations [7], [10], resulting in relatively conservative estimation.

Different from the SME and IO, the set-based observer [11], [12], [13] effectively balances the computational complexity and the conservatism [4], which constructs a parametric set-version observer to calculate a state estimation set using the system dynamics and the measurement. Except for intersection, the common set operations (e.g., linear mapping and Minkowski sum) can be efficiently and exactly computed using zonotopes [14]. Since the set-based observers do not require intersection, it is quite suitable for linear systems to implement such methods using zonotopes without requiring extra approximations. Based on the Luenberger observer structure, [11] proposed the SVO and corresponding F-radius optimal design for linear systems. Following [11], the SVO was further extended to other areas, including fault detection [15], human-robot interaction [16], and nonlinear reachability analysis [17]. As another kind of set-based observer, the SUIO proposed in [12] adopts set-valued approaches to extend classical unknown input observers, which relaxes the severe designing conditions of classical unknown input observers [18]. Furthermore, [13] proposed an F-radius optimal design for the SUIO and compared its conservatism with the SVO under the corresponding optimal designs. As the conclusion of [13], the SUIO and SVO have respective advantages in conservatism. In detail, when the disturbance level of the system is higher than the measurement noise, the SUIO can obtain a tighter estimation set than the SVO. But if the disturbance level of the system is lower than the measurement noise, the SVO can achieve a more precise estimation result.

This paper aims at the optimal design of SUIO. On this topic, a pending issue is the robust convergence of the SUIO under the optimal design. Moreover, there still exist unexploited parametric freedoms to further improve the conservatism of SUIO. Due to these points, this paper proposes a new F-radius optimal design method for SUIO

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and addresses the following issues:

- Propositions 1 and 2 indicate that compared with existing SUIO [12] and [13], the proposed method has a milder condition to design a stable observer;
- To transfer the online computational burden, Theorem 1 establishes the solution of optimal parameters under the metric of F-radius as time tends to infinity, by which the constant optimal parameters of SUIO could be conveniently designed offline;
- Theorem 2 proves that the size of state estimation set obtained by the proposed method is at most as large as that of both the SUIO [12], [13] and SVO under the measure of F-radius. Also, Remark 5 shows such improvement of conservatism is not at the cost of increased computational complexity. The convergence of the observer under the proposed optimal design is a natural corollary of Theorem 2.

The remainder of this paper is organized as follows. Section II gives some preliminaries. Section III introduces system model, the SUIO and the SVO. Section IV presents the main results of this paper. Section V uses a numerical example to show the effectiveness of the proposed method. Finally, some conclusions are drawn in Section VI.

II. PRELIMINARIES

In this paper, the notations $\mathbf{0}_n$, $\mathbf{0}_{m\times n}$ and I_n are denoted as the n-dimensional null vector, the $m\times n$ null matrix and the n-dimensional identity matrix, respectively. Given a matrix $\mathcal{M}$, $\mathrm{tr}(\mathcal{M})$, $\mathrm{rank}(\mathcal{M})$ and $||\mathcal{M}||_F$ are denoted as its trace, rank and Frobenius norm, and $\mathcal{M}>0$ ($\mathcal{M}\geq 0$) denotes that $\mathcal{M}$ is a positive definite (semidefinite) matrix. Given a set $X\subset\mathbb{R}^n$, X is called full-dimensional if the affine dimension of X is n. Given two sets X and Y, the Minkowski sum of X and Y is defined as $X\oplus Y=\{x+y\,|\,x\in X,\,y\in Y\}$. A zonotope $Z\subset\mathbb{R}^n$ of order m/n is defined as $Z=\{z\mid z=p+G\xi,\ \|\xi\|_\infty\leq 1\}$ and abbreviated as $Z=\langle p,G\rangle$, where $p\in\mathbb{R}^n$ is the center vector of Z, $g_1,g_2,...,g_m\in\mathbb{R}^n$ are the generators of Z, and $G=[g_1,g_2,...,g_m]$ is called the generator matrix. A zonotope $Z=\langle p,G\rangle$ is a full-dimensional set if and only if the matrix G is full row rank. For two zonotopes $Z_1=\langle p_1,G_1\rangle$ and $Z_2=\langle p_2,G_2\rangle$, the linear transformation of Z_1 and the Minkowski sum of Z_1 and Z_2 are given by $QZ_1=\langle Qp_1,\,QG_1\rangle$ and $Z_1\oplus Z_2=\langle p_1+p_2,\,[G_1\ G_2]\rangle$, where Q is a matrix with proper dimensions. Definition 1 introduces a common size metric of zonotopes used in this paper.

Definition 1 ([2], [11], [13])**.** Given a zonotope $Z=\langle p,G\rangle$, the *F-radius* of Z is defined as $\phi(Z)=||G||_F=\sqrt{\mathrm{tr}(GG^T)}$.

III. TWO SET-BASED OBSERVERS

A. System Model

The discrete linear time-invariant (LTI) system with process disturbances and measurement noises is modeled as

$$x_{k+1}=Ax_k+Bu_k+E\omega_k, \tag{1a}$$
$$y_k=Cx_k+F\eta_k, \tag{1b}$$

where $A\in\mathbb{R}^{n_x\times n_x}$, $B\in\mathbb{R}^{n_x\times n_u}$, $C\in\mathbb{R}^{n_y\times n_x}$, $E\in\mathbb{R}^{n_x\times n_\omega}$ and $F\in\mathbb{R}^{n_y\times n_\eta}$ are the parametric matrices of the system. $x_k\in\mathbb{R}^{n_x}$, $u_k\in\mathbb{R}^{n_u}$, $y_k\in\mathbb{R}^{n_y}$, $\omega_k\in\mathbb{R}^{n_\omega}$, and $\eta_k\in\mathbb{R}^{n_\eta}$ denote the state, input, output, process disturbances and measurement noises at the k-th time instant, respectively. Moreover, the system (1) satisfies the following two assumptions.

Assumption 1. The initial state x_0, process disturbances ω_k and measurement noise η_k are bounded by zonotopes $X_0=\left\langle p_0^x,G_0^x\right\rangle\subset\mathbb{R}^{n_x}$, $W=\left\langle\mathbf{0}_{n_\omega},G^w\right\rangle\subset\mathbb{R}^{n_\omega}$ and $V=\left\langle\mathbf{0}_{n_\eta},G^v\right\rangle\subset\mathbb{R}^{n_\eta}$, respectively.

Assumption 2. The pair (A,C) is detectable.

B. Set-Theoretic Unknown Input Observer

According to [12], the SUIO is designed based on the classical UIO structure:

$$z_{k+1}^{suio}=Nz_k^{suio}+Tu_k+Ky_k, \tag{2a}$$
$$\hat{x}_k^{suio}=z_k^{suio}+Hy_k, \tag{2b}$$

where $z_k^{suio}\in\mathbb{R}^{n_x}$ is the state of the observer, $\hat{x}_k^{suio}\in\mathbb{R}^{n_x}$ is the estimated state of the system (1), and N, T, K, H are the parameters of the observer to be designed.

According to the design of SUIO proposed in [12], the unknown input ω_k and corresponding distribution matrix E in the system (1) should be divided into

$$\omega_k=[\omega_{1,k}^T\ \ \omega_{2,k}^T]^T,\ \ E=[E_1\ \ E_2],$$

where $\omega_{1,k}\in\mathbb{R}^{n_{\omega_1}}$ includes all unknown inputs that can be actively decoupled, $\omega_{2,k}\in\mathbb{R}^{n_{\omega_2}}$ denotes the remaining unknown inputs, and E_1 and E_2 are the distribution matrices of $\omega_{1,k}$ and $\omega_{2,k}$, respectively. Moreover, $\omega_{2,k}$ is bounded by a known set W_2. In order to actively decouple the influence of $\omega_{1,k}$ and some other terms, the parameters of SUIO should satisfy the following constraints (see [12] for the details):

$$A-HCA-K_1C-N=\mathbf{0}_{n_x\times n_x}, \tag{3a}$$
$$K-K_1-K_2=\mathbf{0}_{n_x\times n_y}, \tag{3b}$$
$$E_1-HCE_1=\mathbf{0}_{n_x\times n_\omega}, \tag{3c}$$
$$B-T-HCB=\mathbf{0}_{n_x\times n_u}, \tag{3d}$$
$$(A-HCA-K_1C)H-K_2=\mathbf{0}_{n_x\times n_y}. \tag{3e}$$

Remark 1. According to the existing design of SUIO [12], [13], the parameter H is determined by (3c). Then once K_1 is further designed, the others (i.e., N, T, K_2 and K) are uniquely determined by (3). Furthermore, Lemma 1 introduces conditions for designing an SUIO.

Lemma 1 ([18])**.** The necessary and sufficient conditions to design a stable SUIO for the system (1) are

1) $\mathrm{rank}(E_1)=\mathrm{rank}(CE_1)$;
2) The pair $(A_1,\ C)$ is detectable, where $A_1=A-E_1[(CE_1)^TCE_1]^{-1}(CE_1)^TCA$.

Remark 2. In Lemma 1, the condition 1) ensures the solvability of the constraint (3c), and the condition 2) is necessary and sufficient for the existence of a stable SUIO.

In this paper, the state estimation error for a given observer is defined as

$$e_k = x_k - \hat{x}_k, \tag{4}$$

where $\hat{x}_k$ and e_k are the estimated state and state estimation error of this observer, respectively. Then let e_k^{suio} denote the state estimation error of SUIO. Based on (1), (2), (3) and (4), the dynamics of e_k^{suio} is derived as

$$\begin{aligned} e_{k+1}^{suio} = &(A - HCA - K_1C)e_k^{suio} + (E_2 - HCE_2)\omega_{2,k} \\ &- HF\eta_{k+1} - K_1F\eta_k, \end{aligned} \tag{5}$$

where $e_0^{suio} \in E_0^{suio}$ and E_0^{suio} is the initial state estimation error set. Based on (5), the state estimation error set E_k^{suio} can be computed recursively by

$$\begin{aligned} E_{k+1}^{suio} = &(A - HCA - K_1C)E_k^{suio} \oplus (E_2 - HCE_2)W_2 \\ &\oplus (-HF)V \oplus (-K_1F)V \end{aligned} \tag{6}$$

such that $e_k^{suio} \in E_k^{suio}, \forall k \geq 1$ hold. After obtaining the state estimation error set, the set version of (4) can be used to compute a state estimation set for a given set-based observer (e.g., SVO and SUIO):

$$X_k = E_k \oplus \hat{x}_k, \tag{7}$$

where E_k and X_k are the state estimation error set and state estimation set of this observer, respectively.

C. Set-Valued Observer

According to [11], the SVO is proposed based on the following set-propagated dynamics:

$$X_{k+1}^{svo} = (A - LC)X_k^{svo} \oplus EW \oplus (-LF)V \oplus Bu_k \oplus Ly_k, \tag{8}$$

where L and X_k^{svo} are the gain matrix to be designed and the state estimation set of SVO at time instant k, respectively. Note that, under Assumption 2, there exists a gain matrix L which can guarantee the stability of the SVO.

Unlike the SUIO, the SVO does not provide a point estimate of the state (i.e., the estimated state). To facilitate the analysis in the following section, we first translate the SVO into a similar form of the SUIO. Specifically, it is natural to choose the center of the state estimation set X_k^{svo} as the the estimated state of SVO. Since $W = \left\langle \mathbf{0}_{n_\omega}, G^w \right\rangle$ and $V = \left\langle \mathbf{0}_{n_\eta}, G^v \right\rangle$ (as assumed in [11]), based on (8), the estimated state of SVO can be recursively computed by

$$\hat{x}_{k+1}^{svo} = (A - LC)\hat{x}_k^{svo} + Bu_k + Ly_k, \tag{9}$$

where $\hat{x}_k^{svo}$ is both the estimated state of SVO and the center of X_k^{svo}. Similar to E_k^{suio}, E_k^{svo} is denoted as the state estimation error set of SVO. Based on (7) and (9), the dynamics of E_k^{svo} is derived as

$$\begin{aligned} E_{k+1}^{svo} &= X_{k+1}^{svo} \oplus (-\hat{x}_{k+1}^{svo}) \\ &= (A - LC)E_k^{svo} \oplus EW \oplus (-LF)V. \end{aligned}$$

IV. MAIN RESULTS

A. Generalized Set-Theoretic Unknown Input Observer

Motivated by [13], this paper relaxes the constraints of SUIO to further reduce its conservatism. In order to distinguish from the existing SUIO in [12] and [13] for later analysis, we use the following Definition 2 to introduce the proposed design of this paper.

Definition 2. An observer is a generalized set-theoretic unknown input observer (GSUIO) for the system (1) if this observer has the following structure

$$z_{k+1}^{gsuio} = Nz_k^{gsuio} + Tu_k + Ky_k, \tag{10a}$$

$$\hat{x}_k^{gsuio} = z_k^{gsuio} + Hy_k, \tag{10b}$$

with the parameters satisfying

$$A - HCA - K_1C - N = \mathbf{0}_{n_x \times n_x}, \tag{11a}$$

$$K - K_1 - K_2 = \mathbf{0}_{n_x \times n_y}, \tag{11b}$$

$$B - T - HCB = \mathbf{0}_{n_x \times n_u}, \tag{11c}$$

$$(A - HCA - K_1C)H - K_2 = \mathbf{0}_{n_x \times n_y}. \tag{11d}$$

Remark 3. Reminded by the reviewers, we have noticed that a so-called TNL observer has the equivalent structure with (2) and (10), which was first proposed in [7] and then developed in [10]. However, the TNL observer is an IO based on the monotone system theory and optimized by H_∞ technique. Except for the observer structure, the GSUIO presented in this paper is different from the TNL observer in terms of focus, usage, and optimization methods.

Let e_k^{gsuio} and E_k^{gsuio} denote the state estimation error and state estimation error set of GSUIO. Based on Definition 2 and (4), the dynamics of e_k^{gsuio} is derived as

$$\begin{aligned} e_{k+1}^{gsuio} = &(A - HCA - K_1C)e_k^{gsuio} + (E - HCE)\omega_k \\ &- HF\eta_{k+1} - K_1F\eta_k. \end{aligned} \tag{12}$$

Then based on (12), E_k^{gsuio} can be computed by

$$\begin{aligned} E_{k+1}^{gsuio} = &(A - HCA - K_1C)E_k^{gsuio} \oplus (E - HCE)W \\ &\oplus (-HF)V \oplus (-K_1F)V. \end{aligned} \tag{13}$$

Furthermore, the following proposition presents the necessary and sufficient condition to design a stable GSUIO.

Lemma 2 ([18])**.** Let $\tilde{C} = [(CA)^T \ C^T]^T$, the detectability for the pair $(A, \tilde{C})$ is equivalent to that for the pair (A, C).

Proposition 1. The GSUIO is stabilizable if and only if the pair (A, C) is detectable.

Proof. To prove the necessity, let $\tilde{L} = [H \ \ K_1]$, the parametric matrix N can be reformulated as $N = A - \tilde{L}\tilde{C}$ (see (11a)). If the dynamics of GSUIO (i.e., (10a)) is stable, the matrix N needs to be a Schur matrix, i.e., the pair $(A, \tilde{C})$ is detectable. Hence, according to Lemma 2, the pair (A, C) is also detectable. Since the reverse process of proof also holds, the sufficiency is proved as well. □

Then Proposition 2 shows that the existence condition of the GSUIO (i.e., Proposition 1) is milder than that of the SUIO (i.e., Lemma 1), as Proposition 1 is a necessary but not sufficient condition of Lemma 1.

Proposition 2. The pair (A, C) is detectable if the pair (A_1, C) is detectable, where A_1 has been defined in Lemma 1.

Proof. Proof by contradiction. According to Hautus lemma, the pair (A, C) is not detectable if and only if there exists a complex value λ satisfying $|\lambda| \geq 1$ such that

$$\text{rank}\begin{bmatrix} A-\lambda I_{n_x} \\ C \end{bmatrix} < n_x. \tag{14}$$

Then the inequality (14) holds if and only if there exists a vector $\zeta \neq \mathbf{0}_{n_x}$ such that

$$\begin{bmatrix} A-\lambda I_{n_x} \\ C \end{bmatrix}\zeta = \mathbf{0}_{n_x+n_y}. \tag{15}$$

From (15), we have $A\zeta = \lambda\zeta$ and $C\zeta = \mathbf{0}_{n_y}$, and hence

$$\begin{aligned} \lambda\zeta &= A\zeta - \lambda E_1[(CE_1)^T CE_1]^{-1}(CE_1)^T\mathbf{0}_{n_y} \\ &= A\zeta - \lambda E_1[(CE_1)^T CE_1]^{-1}(CE_1)^T(C\zeta) \\ &= A\zeta - E_1[(CE_1)^T CE_1]^{-1}(CE_1)^T CA\zeta = A_1\zeta. \end{aligned} \tag{16}$$

Similar to the above proof, due to $\lambda\zeta = A_1\zeta$ and $C\zeta = \mathbf{0}_{n_y}$, we have

$$\begin{bmatrix} A_1-\lambda I_{n_x} \\ C \end{bmatrix}\zeta = \mathbf{0}_{n_x+n_y} \text{ and } \text{rank}\begin{bmatrix} A_1-\lambda I_{n_x} \\ C \end{bmatrix} < n_x.$$

Based on Hautus lemma, the pair (A_1, C) is also not detectable. Since the reverse process of the above proof does not hold (i.e., (16) can not reverse), the detectablility of the pair (A, C) is only a necessary condition for the detectablility of the pair (A_1, C). □

Then Proposition 3 shows that the existing SUIO and SVO can be unified under the GSUIO.

Proposition 3. The SUIO and SVO are two special cases of the GSUIO.

Proof. According to the introduction in Section III. B and Definition 2, any SUIO can be represented by a GSUIO with the parameter H satisfying $E_1 - HCE_1 = \mathbf{0}_{n_x \times n_{\omega_1}}$. Similarly, any SVO can be represented by a GSUIO with $K = L$, $N = A - LC$, $T = B$ and $H = \mathbf{0}_{n_x \times n_y}$. □

B. Optimal Design of GSUIO

In order to obtain a tighter state estimation set for the GSUIO, the following optimization problem is formulated to minimize the F-radius of state estimation error set:

$$\min_{H_k, K_{1,k}} \quad \phi(E_{k+1}^{gsuio})^2, \tag{17}$$

where H_k and $K_{1,k}$ are the values of H and K_1 of GSUIO at the k-th time instant, respectively.

According to Assumption 1 and (7), the initial state estimation error set of GSUIO is formulated as $E_0^{gsuio} = \langle p_0^x - \hat{x}_0, G_0^x \rangle$. Based on (13), the state estimation error set at each time instant is also a zonotope, which can be denoted as $E_k^{gsuio} = \langle p_k^E, G_k^E \rangle$. Using (13) and the properties of zonotopes, G_k^E ($k \geq 1$) can be recursively computed by

$$G_{k+1}^E = \left[(A - \tilde{L}_k\tilde{C})G_k^E \quad (E - \tilde{L}_k\Theta)G^w \quad \tilde{L}_k\Lambda \quad \tilde{L}_k\Gamma \right], \tag{18}$$

where

$$\Lambda = -\begin{bmatrix} \mathbf{0}_{n_y \times n_y} \\ I_{n_y} \end{bmatrix} FG^v, \quad \Theta = \begin{bmatrix} I_{n_y} \\ \mathbf{0}_{n_y \times n_y} \end{bmatrix} CE,$$

$$\Gamma = -\begin{bmatrix} I_{n_y} \\ \mathbf{0}_{n_y \times n_y} \end{bmatrix} FG^v, \quad \tilde{C} = \begin{bmatrix} CA \\ C \end{bmatrix}, \quad \tilde{L}_k = [H_k \ K_{1,k}].$$

To solve (17), let $S_k = G_k^E(G_k^E)^T$ and $J = G^w(G^w)^T$. Using the above expressions and Definition 1, $\phi(E_{k+1}^{gsuio})^2$ can be formulated as

$$\phi(E_{k+1}^{gsuio})^2 = \left\| G_E^{k+1} \right\|_F^2 = \text{tr}(S_{k+1}), \tag{19}$$

with

$$\begin{aligned} S_{k+1} =& (A - \tilde{L}_k\tilde{C})S_k(A - \tilde{L}_k\tilde{C})^T + \tilde{L}_k\Lambda\Lambda^T\tilde{L}_k^T \\ &+ (E - \tilde{L}_k\Theta)J(E - \tilde{L}_k\Theta)^T + \tilde{L}_k\Gamma\Gamma^T\tilde{L}_k^T. \end{aligned} \tag{20}$$

It is observed that the problem (17) is a convex and unconstrained optimization problem. Hence, based on (19) and (20), the time-varying optimal parameter $\tilde{L}_k^*$ can be obtained by solving $\frac{\partial S_{k+1}}{\partial \tilde{L}_k} = \mathbf{0}_{n_x \times 2n_y}$ The solution of the above differential equation is

$$\tilde{L}_k^* = (AS_k\tilde{C}^T + EJ\Theta^T)(\tilde{C}S_k\tilde{C}^T + U)^{-1}, \tag{21}$$

where $U = \Theta J\Theta^T + \Lambda\Lambda^T + \Gamma\Gamma^T$.

Inspired by Theorem 4.2 in [13], Theorem 1 further gives the solution of (17) as time tends to infinity.

Theorem 1. Under Assumption 2, if the matrix U is nonsingular, the constant optimal parameters of GSUIO as $k \to \infty$ can be obtained by solving

$$\begin{aligned} S_\infty =& \tilde{A}S_\infty\tilde{A}^T - \tilde{A}S_\infty\tilde{C}^T(\tilde{C}S_\infty\tilde{C}^T + U)^{-1}\tilde{C}S_\infty\tilde{A}^T \\ &+ EJE^T - PU^{-1}P^T, \end{aligned} \tag{22a}$$

$$\tilde{L}_\infty^* = (AS_\infty\tilde{C}^T + EJ\Theta^T)(\tilde{C}S_\infty\tilde{C}^T + U)^{-1}, \tag{22b}$$

where $P = EJ\Theta^T$, $\tilde{A} = A - PU^{-1}\tilde{C}$, $\tilde{L}_\infty^* = [H_\infty^* \ K_{1,\infty}^*]$ is the optimal parameter as $k \to \infty$, and the other parameters (i.e., N_∞^*, T_∞^* and K_∞^*) are obtained by solving the equations (11).

Proof. Substituting (21) and $U = \Theta J\Theta^T + \Lambda\Lambda^T + \Gamma\Gamma^T$ into (20) and eliminating $\tilde{L}_k$, S_{k+1} is derived as

$$\begin{aligned} S_{k+1} =& AS_kA^T + EJE^T - (AS_k\tilde{C}^T + EJ\Theta^T) \\ &\times (\tilde{C}S_k\tilde{C}^T + U)^{-1}(AS_k\tilde{C}^T + EJ\Theta^T)^T. \end{aligned} \tag{23}$$

Note that there is no absolute error-free estimation for real systems, i.e., the state estimation error set $E_k^{gsuio} = \langle p_k^E, G_k^E \rangle$ is a full-dimensional zonotope. Hence, G_k^E is a full row rank matrix and $S_k = G_k^E(G_k^E)^T \succ 0$. Then using Woodbury matrix identity, if the matrix U is nonsingular, we have the following identities:

$$S_k - S_k\tilde{C}^T(\tilde{C}S_k\tilde{C}^T + U)^{-1}\tilde{C}S_k = (S_k^{-1} + \tilde{C}^TU^{-1}\tilde{C})^{-1}, \tag{24a}$$

$$(S_k^{-1} + \tilde{C}^TU^{-1}\tilde{C})^{-1}\tilde{C}^TU^{-1} = S_k\tilde{C}^T(\tilde{C}S_k\tilde{C}^T + U)^{-1}. \tag{24b}$$

Substituting $P = EJ\Theta^T$ and $A = \tilde{A} + PU^{-1}\tilde{C}$ into (23) and using the identities (24), after some lengthy algebraic operations, it yields

$$\begin{aligned} S_{k+1} =& \tilde{A}S_k\tilde{A}^T - \tilde{A}S_k\tilde{C}^T(\tilde{C}S_k\tilde{C}^T + U)^{-1}\tilde{C}S_k\tilde{A}^T \\ &+ EJE^T - PU^{-1}P^T, \end{aligned} \tag{25}$$

which is a Riccati Difference Equation (RDE).

According to Theorem 4.2 in [19], if the pair $(\tilde{A}, \tilde{C})$ is detectable, $EJE^T - PU^{-1}P^T \geq 0$ and $S_0 \geq S_\infty$, the sequence $\{S_k\}$ obtained by RDE (25) uniquely converges to a constant matrix S_∞ satisfying (22a), where (22a) is an ARE.

The remaining proof will show that the above convergence conditions can all be satisfied. Based on Assumption 2 and Lemma 2, the pair $(A, \tilde{C})$ is detectable. By applying the same proving technique as in Proposition 2, it can be proved that the detectability of the pairs $(\tilde{A}, \tilde{C})$ and $(A, \tilde{C})$ are equivalent. Hence, the condition of detectability for the pair $(\tilde{A}, \tilde{C})$ is satisfied. Furthermore, due to $P = EJ\Theta^T$, $U = \Theta J \Theta^T + \Lambda\Lambda^T + \Gamma\Gamma^T$, $\Lambda\Lambda^T + \Gamma\Gamma^T \geq 0$ and $J = G^w(G^w)^T$, we have

$$\begin{bmatrix} U & P^T \\ P & EJE^T \end{bmatrix} = \underbrace{\begin{bmatrix} \Theta G^w \\ EG^w \end{bmatrix} \begin{bmatrix} \Theta G^w \\ EG^w \end{bmatrix}^T}_{\geq 0} + \underbrace{\begin{bmatrix} \Lambda\Lambda^T + \Gamma\Gamma^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\geq 0} \geq 0.$$

Since the matrix U is nonsingular, using Schur complement lemma, it yields

$$\begin{bmatrix} U & P^T \\ P & EJE^T \end{bmatrix} \geq 0 \Leftrightarrow EJE^T - PU^{-1}P^T \geq 0.$$

Hence, the condition $EJE^T - PU^{-1}P^T \geq 0$ is satisfied. Since the initial state set X_0 is specified by designers, one can always set a big enough zonotope $X_0 = \langle p_0^X, G_0^X \rangle$ such that

$$S_0 = G_0^E (G_0^E)^T = G_0^X (G_0^X)^T > S_\infty.$$

Then, the convergence conditions are all satisfied. □

Remark 4. Note that U is the sum of three positive semidefinite matrices (i.e., $\Theta J \Theta^T$, $\Lambda\Lambda^T$ and $\Gamma\Gamma^T$), which is singular if and only if the intersection of null space of the three matrices is not empty except for the origin. Moreover, since both W and V are full dimensional zonotopes, U is a singular matrix as long as the matrices E or F is full row rank. Hence, U is typically a nonsingular matrix for real systems. Moreover, the ARE (22a) generally can not be solved analytically, but can be solved numerically with a given precision. In practice, since we have proven in Theorem 1 that the RDE (25) converges to the ARE (22a), the solution can be obtained through iterating the RDE (25) with proper initial values.

C. Conservatism and Convergence Analysis

This subsection will analyze the conservatism of GSUIO, SVO and SUIO under the corresponding F-radius optimal design. Before introducing Theorem 2, Proposition 4 is first presented as the auxiliary for proving Theorem 2.

Proposition 4. Given an arbitrary matrix $O \geq 0$, (21) is an optimal solution of the following convex optimization problem:

$$\min_{\tilde{L}_k} \quad \mathrm{tr}(S_{k+1} O).$$

Proof. Similar to the solving process of (17), the general solution of the above problem is

$$\tilde{L}_k^\dagger = (AS_k\tilde{C}^T + EJ\Theta^T + P_b Q_a)(\tilde{C}S_k\tilde{C}^T + U)^{-1}, \quad (26)$$

where $P_b = [p_1, p_2, ..., p_k]$, the group of vectors $p_1, p_2, ..., p_k$ is the basis of the null space of O, and the matrix $Q_a \in \mathbb{R}^{n_y \times 2n_y}$ is arbitrary. Then it is straightforward that (21) is also one case of (26) with $Q_a = \mathbf{0}_{n_y \times 2n_y}$. □

Theorem 2. Denote $X_k^{*,gsuio}$, $X_k^{*,suio}$ and $X_k^{*,svo}$ as the state estimation sets of GSUIO, SUIO and SVO under the corresponding F-radius optimal designs, respectively. Given $E_0^{gsuio} = E_0^{suio} = E_0^{svo}$, the F-radius of $X_k^{*,gsuio}$ is not larger than those of both $X_k^{*,suio}$ and $X_k^{*,svo}$ for arbitrary $k \geq 1$, i.e.,

$$\phi(X_k^{*,gsuio}) \leq \phi(X_k^{*,suio})$$
$$\text{and} \quad \phi(X_k^{*,gsuio}) \leq \phi(X_k^{*,svo}), \ \forall k \geq 1.$$

Proof. Due to space limits, here we only prove $\phi(X_k^{*,gsuio}) \leq \phi(X_k^{*,suio})$, $\forall k \geq 1$. As for $\phi(X_k^{*,gsuio}) \leq \phi(X_k^{*,svo})$, $\forall k \geq 1$, the proof can be completed in the same way.

According to Proposition 3, the SUIO is reformulated as the form of GSUIO with an extra constraint $E_1 - H_k C E_1 = \mathbf{0}_{n_x \times n_{\omega_1}}$. To distinguish the GSUIO and SUIO under the same form, the notations S_k^{gsuio} and S_k^{suio} are used to indicate S_k corresponding to GSUIO and SUIO, respectively. Similarly, $\tilde{L}_k^{*,suio}$ and $\tilde{L}_k^{*,gsuio}$ are used to denote the corresponding $\tilde{L}_k^*$ of GSUIO and SUIO, respectively. Next, due to (19) and $\phi(X_k) = \phi(E_k)$ (see (7)), we will complete the proof of $\phi(X_k^{*,gsuio}) \leq \phi(X_k^{*,suio})$, $\forall k \geq 1$ through proving $\mathrm{tr}(S_k^{gsuio}) \leq \mathrm{tr}(S_k^{suio})$, $\forall k \geq 1$ by mathematical induction.

According to Proposition 4, we have

$$\tilde{L}_k^{*,\,gsuio} = \arg\min_{\tilde{L}_k} \ \mathrm{tr}(S_{k+1}^{gsuio} O) \quad (27)$$

holds for arbitrary $O \geq 0$. Using the same proof techniques shown in Proposition 4, we have the similar conclusion for SUIO, i.e.,

$$\tilde{L}_k^{*,suio} = \arg\min_{\tilde{L}_k} \ \mathrm{tr}(S_{k+1}^{suio} O). \quad (28)$$
$$s.t. \ E_1 - H_k C E_1 = \mathbf{0}.$$

At the time instant $k = 0$, based on $S_0 = G_0^E (G_0^E)^T$ and $E_0^{gsuio} = E_0^{suio}$, we have $S_0^{gsuio} = S_0^{suio}$. Since both S_k^{gsuio} and S_k^{suio} are calculated recursively by (20), the optimization problem (28) is exactly (27) with an extra constraint when $k = 0$, which implies that in this case the problem (28) has the same objective function but a smaller feasible domain than (27). Thus, we have $\mathrm{tr}(S_1^{gsuio} O) \leq \mathrm{tr}(S_1^{suio} O)$, $\forall O \geq 0$.

At the time instant k, assume that $\mathrm{tr}(S_k^{gsuio} O) \leq \mathrm{tr}(S_k^{suio} O)$ holds for arbitrary $O \geq 0$. According to (20), we have

$$\begin{aligned}\mathrm{tr}(S_{k+1}^{gsuio} O) = \mathrm{tr}\{[(A - \tilde{L}_k^{*,gsuio}\tilde{C}) S_k^{gsuio} (A - \tilde{L}_k^{*,gsuio}\tilde{C})^T \\ + \tilde{L}_k^{*,gsuio} \Lambda (\tilde{L}_k^{*,gsuio} \Lambda)^T + \tilde{L}_k^{*,gsuio} \Gamma (\tilde{L}_k^{*,gsuio} \Gamma)^T \\ + (E - \tilde{L}_k^{*,gsuio} \Theta) J (E - \tilde{L}_k^{*,gsuio} \Theta)^T] O\}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathrm{tr}(S_{k+1}^{suio} O) = \mathrm{tr}\{[(A - \tilde{L}_k^{*,suio}\tilde{C}) S_k^{suio} (A - \tilde{L}_k^{*,suio}\tilde{C})^T \\ + \tilde{L}_k^{*,suio} \Lambda (\tilde{L}_k^{*,suio} \Lambda)^T + \tilde{L}_k^{*,suio} \Gamma (\tilde{L}_k^{*,suio} \Gamma)^T \\ + (E - \tilde{L}_k^{*,suio} \Theta) J (E - \tilde{L}_k^{*,suio} \Theta)^T] O\}.\end{aligned}$$

Note that $\tilde{L}_k^{*,suio}$ is only a feasible solution for the optimization problem (27). Thus, given a function $f(S_k^{gsuio}, \tilde{L}_k^{*,suio})$

defined as

$$f(S_k^{gsuio}, \tilde{L}_k^{*,suio}) = \mathrm{tr}\{[(A-\tilde{L}_k^{*,suio}\tilde{C})S_k^{gsuio}(A-\tilde{L}_k^{*,suio}\tilde{C})^T + \tilde{L}_k^{*,suio}\Lambda(\tilde{L}_k^{*,suio}\Lambda)^T + \tilde{L}_k^{*,suio}\Gamma(\tilde{L}_k^{*,suio}\Gamma)^T + (E-\tilde{L}_k^{*,suio}\Theta)J(E-\tilde{L}_k^{*,suio}\Theta)^T]O\},$$

we have $\mathrm{tr}(S_{k+1}^{gsuio}O) \leq f(S_k^{gsuio}, \tilde{L}_k^{*,suio})$. On the other hand, we have

$$\begin{aligned} & f(S_k^{gsuio}, \tilde{L}_k^{*,suio}) - \mathrm{tr}(S_{k+1}^{suio}O) \\ =& \mathrm{tr}[(A-\tilde{L}_k^{*,suio}\tilde{C})(S_k^{gsuio}-S_k^{suio})(A-\tilde{L}_k^{*,suio}\tilde{C})^T O] \\ =& \mathrm{tr}[(S_k^{gsuio}-S_k^{suio})(A-\tilde{L}_k^{*,suio}\tilde{C})^T O(A-\tilde{L}_k^{*,suio}\tilde{C})]. \end{aligned}$$

Let $O_{new} = (A-\tilde{L}_k^{*,suio}\tilde{C})^T O(A-\tilde{L}_k^{*,suio}\tilde{C})$. Then it is derived from $O \geq 0$ that $O_{new} \geq 0$. Since $\mathrm{tr}(S_k^{gsuio}O) \leq \mathrm{tr}(S_k^{suio}O)$ holds for arbitrary $O \geq 0$, the above formula is further derived as

$$\begin{aligned} & f(S_k^{gsuio}, \tilde{L}_k^{*,suio}) - \mathrm{tr}(S_{k+1}^{suio}O) \\ =& \mathrm{tr}[(S_k^{gsuio}-S_k^{suio})O_{new}] \\ =& \mathrm{tr}(S_k^{gsuio}O_{new}) - \mathrm{tr}(S_k^{suio}O_{new}) \leq 0. \end{aligned}$$

Thus we have

$$\mathrm{tr}(S_{k+1}^{gsuio}O) \leq f(S_k^{gsuio}, \tilde{L}_k^{*,suio}) \leq \mathrm{tr}(S_{k+1}^{suio}O).$$

Through mathematical induction, we have completed the proof of $\mathrm{tr}(S_k^{gsuio}O) \leq \mathrm{tr}(S_k^{suio}O), \forall k \geq 1$ for arbitrary $O \geq 0$. Since I_{n_x} is one case of $O \geq 0$, we have $\mathrm{tr}(S_k^{gsuio}) \leq \mathrm{tr}(S_k^{suio}), \forall k \geq 1$. □

Remark 5. Since the F-radius optimal designs of GSUIO, SUIO and SVO are all analytical (see [11], [13] for the details about SVO and SUIO), the computational complexity of these methods are commensurate. Moreover, [11, Theorem 12] has indicated the convergence of SVO under the F-radius optimal design, i.e., $\phi(X_k^{*,svo})$ is bounded as $k \to \infty$. Since it is shown in Theorem 2 that $\phi(X_k^{*,gsuio}) \leq \phi(X_k^{*,svo})$, $\forall k \geq 1$, the convergence of the proposed optimal design emerges as a natural corollary from Theorem 2.

V. ILLUSTRATIVE EXAMPLE

In this example, we will use a numerical example to illustrate the effectiveness of the proposed method. For comparison, the SVO from [11] and the SUIO from [13] are also used to demonstrate the advantage of the GSUIO in conservatism. Specifically, the parametric matrices of the LTI system (1) are given by

$$A = \begin{bmatrix} 0.6887 & -0.3293 \\ 0.0095 & 0.6969 \end{bmatrix}, \quad B = \begin{bmatrix} 1.3905 \\ 0.9064 \end{bmatrix}, \quad C = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.6 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.67 & 0.22 & 0.31 & 0.27 \\ 0.57 & 0.45 & 0.53 & 0.33 \end{bmatrix}, \quad F = \begin{bmatrix} 0.97 & 1.28 \\ 0.49 & 0.91 \end{bmatrix}.$$

Since the SUIO requires to decouple the process disturbances, the 1-st component of w_k is chosen to be actively decoupled. Hence, the distribution matrix E is divided into

$$E_1 = \begin{bmatrix} 0.67 \\ 0.57 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.22 & 0.31 & 0.27 \\ 0.45 & 0.53 & 0.33 \end{bmatrix}.$$

Moreover, we assign the initial state $x_0 = [0, 0]^T$ and input $u_k = \pi sin(0.5k)/3$. The initial state estimation set, disturbance set and noise set are considered as $X_0 = \langle \mathbf{0}, I_{2\times 2} \rangle$, $W = \langle \mathbf{0}, 0.5I_{4\times 4} \rangle$ and $V = \langle \mathbf{0}, 0.5I_{2\times 2} \rangle$, respectively.

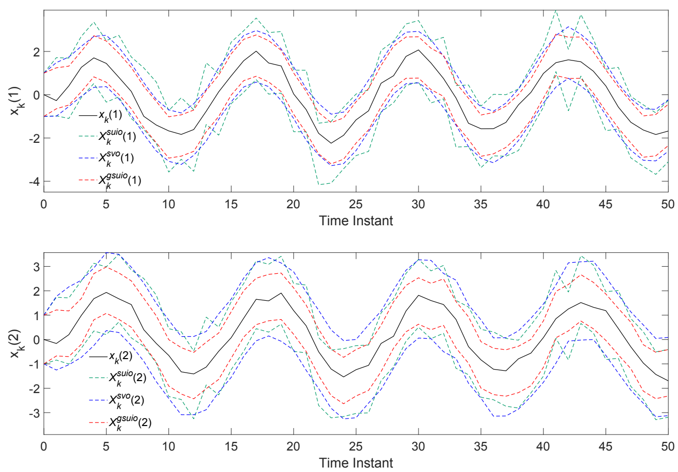
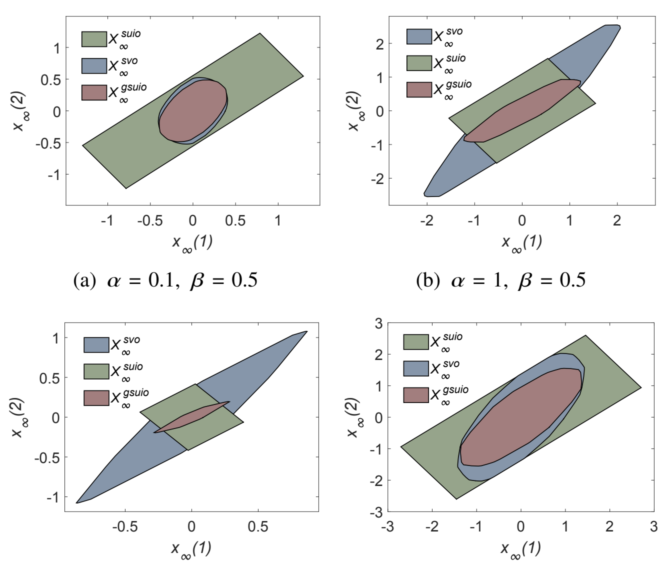


Fig. 1: The state estimation results of the SUIO, the SVO and the GSUIO from $k = 0$ to $k = 50$.



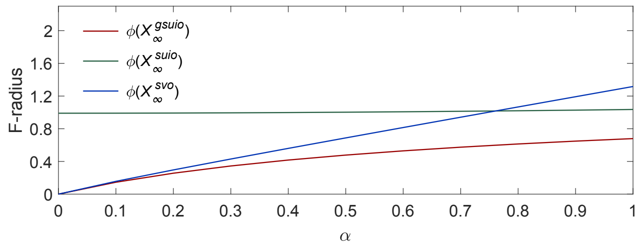
(a) $\alpha = 0.1$, $\beta = 0.5$ (b) $\alpha = 1$, $\beta = 0.5$

(c) $\alpha = 0.5$, $\beta = 0.1$ (d) $\alpha = 0.5$, $\beta = 1$

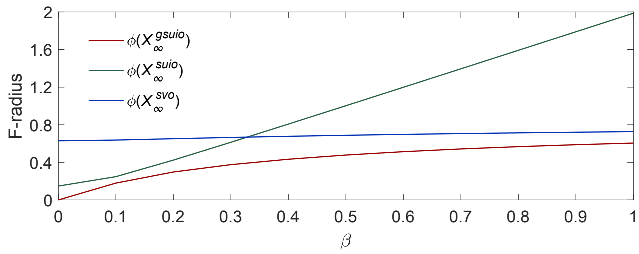
Fig. 2: The state estimation sets of the SUIO, the SVO and the GSUIO as $k \to \infty$, where the centers of all sets are moved to the origin.

Let $x_k = [x_k(1), x_k(2)]^T$ and denote $X_k(i)$ as the boundary projection of X_k on the i-th component. Fig. 1 shows the state estimation results of the SUIO, the SVO and the GSUIO under the corresponding optimal designs from $k = 0$ to $k = 50$. Since the optimal parameters are all designed analytically, the average online computation time of the three methods are all around $0.18 \sim 0.19$ms at each step. Moreover, it is observed in Fig. 1 that the state estimation bounds of the GSUIO are the tightest among the three methods.

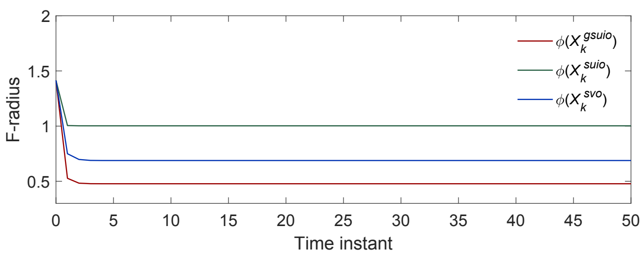
To exclude contingencies of the above simulations, we further compare the three methods under different levels of



(a) F-radius as $k \to \infty$ with different α and $\beta = 0.5$.



(b) F-radius as $k \to \infty$ with different β and $\alpha = 0.5$.



(c) F-radius from $k = 0$ to $k = 50$ with $\alpha = 0.5$ and $\beta = 0.5$.

Fig. 3: Comparison of conservatism for the SUIO, the SVO and the GSUIO under the F-radius metric

disturbances and noise in Figs. 2 and 3. Particularly, let $W = \langle 0, \alpha I_{4\times4} \rangle$, $V = \langle 0, \beta I_{2\times2} \rangle$ and keep other parameters unchanged. Then adjusting the relative magnitudes of α and β can represent the different levels of disturbances and noise.

Fig. 2 shows the state estimation sets as $k \to \infty$ for the three methods with different values of α and β. In order to compare the sizes of sets intuitively, the centers of all the sets in Fig. 2 have been moved to the origin. From Fig. 2, it can be seen that regardless of the relative levels of disturbances and noise, the state estimation sets of the GSUIO method are the smallest among the three methods.

Furthermore, Fig. 3 compares the F-radius of the state estimation sets for the three methods with different values of α, β and k, respectively. Just as indicated in Theorem 2, the F-radius of the state estimation set for the GSUIO is always the smallest among the three methods, regardless of the time instant or the relative levels of noise and disturbances.

VI. Conclusions

This paper presents a new F-radius optimal design method of SUIO for robust state estimation, which attains a milder condition for usage compared with the existing SUIO. Moreover, it is proved that under the corresponding F-radius optimal designs, the proposed method achieves a more precise estimation result than existing SVO and SUIO. To transfer the online computational burden, we further propose an offline optimal design method for SUIO to obtain the constant optimal parameters as time tends to infinity. In the future work, we intend to explore the application of the proposed method on other tasks, e.g., robust fault diagnosis.

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References

[1] C. Combastel, "An Extended Zonotopic and Gaussian Kalman Filter (EZGKF) merging set-membership and stochastic paradigms: Toward non-linear filtering and fault detection," *Annual Reviews in Control*, vol. 42, pp. 232–243, 2016.

[2] T. Alamo, J. M. Bravo, and E. F. Camacho, "Guaranteed state estimation by zonotopes," *Automatica*, vol. 41, no. 6, pp. 1035–1043, 2005.

[3] F. Blanchini and S. Miani, *Set-theoretic methods in control*. Berlin, GER: Springer, 2008, vol. 78.

[4] Y. Cong, X. Wang, and X. Zhou, "Rethinking the mathematical framework and optimality of set-membership filtering," *IEEE Transactions on Automatic Control*, vol. 67, no. 5, pp. 2544–2551, 2021.

[5] M. Althoff and J. J. Rath, "Comparison of guaranteed state estimators for linear time-invariant systems," *Automatica*, vol. 130, p. 109662, 2021.

[6] D. Efimov, W. Perruquetti, T. Raïssi, and A. Zolghadri, "On interval observer design for time-invariant discrete-time systems," in *2013 European Control Conference (ECC)*, Zurich, Switzerland, 2013, pp. 2651–2656.

[7] Z. Wang, C.-C. Lim, and Y. Shen, "Interval observer design for uncertain discrete-time linear systems," *Systems & Control Letters*, vol. 116, pp. 41–46, 2018.

[8] H. L. Smith, *Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems*. Providence, Rhode Island, USA: American Mathematical Soc., 2008, no. 41.

[9] V. T. H. Le, C. Stoica, T. Alamo, E. F. Camacho, and D. Dumur, *Zonotopes: From guaranteed state-estimation to control*. Hoboken, USA: John Wiley & Sons, 2013.

[10] W. Tang, Z. Wang, and Y. Shen, "Interval estimation for discrete-time linear systems: A two-step method," *Systems & Control Letters*, vol. 123, pp. 69–74, 2019.

[11] C. Combastel, "Zonotopes and kalman observers: Gain optimality under distinct uncertainty paradigms and robust convergence," *Automatica*, vol. 55, pp. 265–273, 2015.

[12] F. Xu, J. Tan, X. Wang, V. Puig, B. Liang, and B. Yuan, "A novel design of unknown input observers using set-theoretic methods for robust fault detection," in *2016 American Control Conference (ACC)*, Boston, MA, USA, 2016, pp. 5957–5961.

[13] F. Xu, J. Tan, X. Wang, and B. Liang, "Conservatism comparison of set-based robust fault detection methods: Set-theoretic uio and interval observer cases," *Automatica*, vol. 105, pp. 307–313, 2019.

[14] J. K. Scott, D. M. Raimondo, G. R. Marseglia, and R. D. Braatz, "Constrained zonotopes: A new tool for set-based estimation and fault detection," *Automatica*, vol. 69, pp. 126–136, 2016.

[15] M. Pourasghar, C. Combastel, V. Puig, and C. Ocampo-Martinez, "Fd-zkf: A zonotopic kalman filter optimizing fault detection rather than state estimation," *Journal of Process Control*, vol. 73, pp. 89–102, 2019.

[16] S. Li, T. Stouraitis, M. Gienger, S. Vijayakumar, and J. A. Shah, "Set-based state estimation with probabilistic consistency guarantee under epistemic uncertainty," *IEEE Robotics and Automation Letters*, vol. 7, no. 3, pp. 5958–5965, 2022.

[17] C. Combastel, "Functional sets with typed symbols: Mixed zonotopes and polynotopes for hybrid nonlinear reachability and filtering," *Automatica*, vol. 143, p. 110457, 2022.

[18] J. Chen, R. J. Patton, and H.-Y. Zhang, "Design of unknown input observers and robust fault detection filters," *International Journal of control*, vol. 63, no. 1, pp. 85–105, 1996.

[19] C. De Souza, M. Gevers, and G. Goodwin, "Riccati equations in optimal filtering of nonstabilizable systems having singular state transition matrices," *IEEE Transactions on Automatic control*, vol. 31, no. 9, pp. 831–838, 1986.