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Local stabilization of systems with time- and state-dependent perturbations using super-twisting integral sliding-mode control

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***Abstract*— We consider a super-twisting sliding-mode controller with an integral sliding variable. The system is subject to time- and state-dependent perturbations. Our stability analysis yields an estimate for the region of attraction as well as bounds for the control signal and the sliding variables. We demonstrate the results with a numerical example.**

I. Introduction

The super-twisting algorithm is a well-established extension to the conventional first-order sliding-mode controller exhibiting a continuous control signal [2]. In this setup, time-dependent Lipschitz perturbations are fully compensated in sliding-mode. Finite-time convergence to the sliding surface is established by [3], [4] using non-smooth Lyapunov functions. In [7] stability of well-established parameter settings in the presence of time-dependent perturbations is proven.

Integral sliding-mode control is a powerful extension to this concept that allows for the initialisation of the controller on the sliding manifold such that perturbations can be fully compensated at initial time if the initial perturbation is known [12], [14]. However, applying the super-twisting controller to systems subject to state-dependent perturbations is challenging, as a bound on the time-derivative of the perturbation is required prior to design. This requirement poses a fundamental problem for the super-twisting control design as the time-derivative may depend on the derivative of the system state, which again depends on the control signal to be designed. In [5] and [6] this fundamental problem is called an algebraic loop. An attempt to avoid this is implied by the approach in [8], [9], where a global bound for the total time-derivative of the perturbation is known a priori. An explicit approach to break this algebraic loop for first-order systems is presented in [5] and [6]. The gains of the super-twisting controller are chosen such that global finite-time convergence is achieved for time- and state-dependent perturbations.

In this contribution we consider systems of arbitrary order subject to time- and state-dependent perturbations with bounded derivatives in time and state, respectively. We combine the super-twisting algorithm with an integral sliding variable and analyse their local stability properties. We employ the well-established Lyapunov function of [4] and exploit the super-twisting controller proposed in [10], where an augmented parametrisation of the controller is used for the local stability analysis of a higher-order system with state-dependent perturbations. Using this parametrisation and the bounds of the perturbation's derivatives we guarantee a bound on the state-trajectory and determine a set of feasible initial conditions. Furthermore, we give bounds for the control signal as well as the sliding variables.

The paper is structured as follows. Section II gives a formal problem definition and introduces bounds on the perturbation considered. In Section III we introduce the integral sliding variable, a conventional sliding variable and the parametrisation of the super-twisting controller. The main results are presented in Section IV. In Section IV-A we formulate a set of initial states for which the solution of the closed-loop system remains within a compact set if the sliding variable is bounded. Subsequently, we consider the dynamics of the integral sliding variable and the controller state in Section IV-B. Following the approach taken in [10] we specify bounds for the integral sliding variable and the controller state. By choosing a sufficiently small scaling parameter we obtain a compact set bounding the solution of the closed-loop system. Section IV-C considers the dynamics of the integral sliding variable to specify a bound on the conventional sliding-variable. Combining these findings we obtain our main stability result in Section IV-D. We show that the bounds on the integral and the conventional sliding variable established in the previous sections hold for the initial states of the system specified in Section IV-A. The example in Section V illustrates the proposed control design of an super-twisting integral sliding-mode controller and the calculation of its region of attraction as well as the bounding set of the closed-loop system state. We summarise our findings in the conclusions in Section VI.

II. Problem Definition

Consider the nonlinear system in normal form given by

$$\dot{x}_i(t) = x_{i+1}(t), \qquad i = 1, ..., n-1, \tag{1a}$$
$$\dot{x}_n(t) = u(t) + \delta(x,t), \tag{1b}$$

where $x(t) = [x_1(t), ..., x_n(t)]^\top \in \mathbb{R}^n$ denotes the system state with initial value $x_0 \in \mathbb{R}^n$, and $u(t) \in \mathbb{R}$ is the input. The perturbation

$$\delta(x,t) = \delta_t(t) + \delta_x(x) \tag{2}$$

consists of a time-dependent disturbance $\delta_t(t) : \mathbb{R} \mapsto \mathbb{R}$ and a state-dependent uncertainty $\delta_x(x) : \mathbb{R}^n \mapsto \mathbb{R}$. This decomposition allows for the formulation of the following assumption.

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Assumption 1. *Let $\mathcal{D} \subset \mathbb{R}^n$ be a compact subset of the state space that contains the origin. The perturbation $\delta(x,t)$ and its first partial derivatives with respect to t and x, are bounded by positive constants ρ, ρ_t and ρ_{x_i} such that*

$$\max_{x\in\mathcal{D},\, t\in\mathbb{R}^+} |\delta(x,t)| \le \rho,$$

$$\max_{t\in\mathbb{R}^+} \left|\frac{\mathrm{d}\delta_t(t)}{\mathrm{d}t}\right| \le \rho_t, \quad \max_{x\in\mathcal{D}} \left|\frac{\partial \delta_x(x)}{\partial x_i}\right| \le \rho_{x_i},\ i=1,\dots,n.$$

The goal is to design a super-twisting sliding-mode controller to stabilize the origin $x=0$ in presence of the time- and state-dependent perturbation (2), and provide an estimate for the region of attraction of the closed-loop system.

III. Control Design

This section introduces our considered control design using a combination of an integral sliding surface and a super-twisting controller with parametrisation proposed in [10]. The design of the integral sliding surface follows [11] and [12]. First a conventional sliding variable

$$s_1 : \mathbb{R}^n \mapsto \mathbb{R},\ x \mapsto s_1(x) = \begin{bmatrix} m^\top & 1 \end{bmatrix} x, \tag{3}$$

with $m = [m_1, ..., m_{n-1}]^\top \in \mathbb{R}^{n-1}$ is chosen such that the polynomial

$$\lambda^{n-1} + m_{n-1}\lambda^{n-2} + m_{n-2}\lambda^{n-3} + ... + m_2\lambda + m_1$$

specifying the dynamics in sliding-mode is Hurwitz.

Based on the conventional sliding variable s_1 and the initial state x_0 of (1) an integral sliding variable is designed with

$$s(x(t),t) = s_1(x(t)) - s_1(x_0) + k_s \int_0^t s_1(x(\tau))\,\mathrm{d}\tau, \tag{4}$$

such that $s(x_0,0)=0$, and k_s is some positive constant.

Taking the time-derivate of s along the solution of system (1) yields

$$\dot{s} = \dot{s}_1 + k_s\, s_1, \tag{5}$$

where

$$\dot{s}_1 = \begin{bmatrix} 0 & m^\top \end{bmatrix} x + u + \delta(x,t). \tag{6}$$

We consider the following control law

$$u = u_0 + u_1, \tag{7}$$

which consists of a linear state feedback

$$\begin{aligned} u_0 &= -\begin{bmatrix} 0 & m^\top \end{bmatrix} x - k_s\, s_1(x) && \text{(8a)}\\ &= -\Big(\begin{bmatrix} 0 & m^\top \end{bmatrix} + k_s \begin{bmatrix} m^\top & 1 \end{bmatrix} \Big) x =: k^\top x, && \text{(8b)} \end{aligned}$$

with $k \in \mathbb{R}^n$ and a dynamic super-twisting control law

$$\begin{aligned} u_1 &= -\frac{\alpha_1}{\mu} |s|^{1/2} \operatorname{sgn}(s) + v, && \text{(9a)}\\ \dot{v} &= -\frac{\alpha_2}{2\,\mu^2} \operatorname{sgn}(s), \quad v(0) = v_0 = 0, && \text{(9b)} \end{aligned}$$

as proposed in [10], where the gains $\alpha_1, \alpha_2 > 0$ and the controller-gain scaling $\mu > 0$ are to be chosen.

Substituting (6) into (5) with the control law in (7), (8), (9) yields the dynamics of the sliding variable

$$\begin{aligned} \dot{s} &= -\frac{\alpha_1}{\mu} |s|^{1/2} \operatorname{sgn}(s) + v + \delta(x,t), \quad s(x_0) = s_0, && \text{(10a)}\\ \dot{v} &= -\frac{\alpha_2}{2\,\mu^2} \operatorname{sgn}(s), \quad v(0) = v_0. && \text{(10b)} \end{aligned}$$

Remark 2. *Note, if suitable design parameters α_1, α_2, μ enforce that $s(t) = 0$, and thus $\dot{s}(t) = 0$, for all $t \ge t_r$ after a finite time $t_r > 0$, then the dynamics (5) reduce to the stable scalar differential equation $\dot{s}_1 + k_s\, s_1 = 0$ for $t \ge t_r$. Then the solution x in sliding-mode satisfies*

$$s_1(x(t)) = e^{-k_s(t-t_r)}\, s_1(x(t_r)), \quad t \ge t_r. \tag{11}$$

The choice of the coefficient vector $m \in R^{n-1}$ and the gain parameter k_s specifies the set $\mathcal{S} = \{x \in \mathbb{R}^n \,|\, s_1(x) = 0\}$ and the convergence rate of s_1 for $t > t_r$, respectively.

The following stability analysis considers the closed-loop system consisting of (1), (7) and (10). The solutions x and (s,v) are trajectories in the sense of Filippov [13]. Following [10] we obtain that the gains $\alpha_1, \alpha_2 > 0$ can be chosen arbitrary. The scaling factor $\mu > 0$ will be selected such that the local stability of the closed-loop system is guaranteed while preventing the appearance of the algebraic loop concerning the control signal, which was outlined in [5].

IV. Region of Attraction

Following established approaches, we divide the analysis of the closed-loop system (1), (7) and (10) into two parts: the dynamics of the state x of (1) and the dynamics of the sliding variable (10).

The controller (9) is designed to force the system onto the sliding manifold in finite time t_r such that $s(x(t)) = 0$ and $v(t) = -\delta(x(t),t)$ for all $t \ge t_r$. In order to achieve sliding-mode for $t_0 = 0$, and thus a vanishing reaching phase we require $v_0 = -\delta(x_0, 0)$ as initialisation in (9). However, $\delta(x_0,0)$ is unknown, and thus the sliding motion cannot be established by the super-twisting integral sliding-mode controller (7) from the initial time $t_0 = 0$ if $v_0 \ne -\delta(x_0,0)$, as pointed out in [14, Remark 1] and [12, Section 5].

Therefore, a region in $\mathcal{D}$ cannot be invariant for arbitrary $v_0 \ne -\delta(x_0,0)$. In order to accommodate this fact we shall introduce two sets: the set Ω_{c_1} in (17) bounding the trajectories x for $v_0 = 0$, and the set Ψ_{c_0} of feasible initial states x_0 in (19).

A. State space invariance

Consider the reduced state $z := [x_1, ..., x_{n-1}]^\top \in \mathbb{R}^{n-1}$. Using the conventional sliding variable s_1 in (3) the dynamics can be written as

$$\dot{z} = A_0\, z + B_0\, s_1, \tag{12}$$

with $A_0 \in \mathbb{R}^{(n-1)\times(n-1)}$ and $B_0 \in \mathbb{R}^{n-1}$ given by

$$A_0 = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & & 0 & 1 \\ -m_1 & \dots & \dots & -m_{n-1} \end{bmatrix} \quad \text{and} \quad B_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Since A_0 is Hurwitz by design of (3), the Lyapunov equation

$$A_0^\top P_0 + P_0 A_0 = -q I \tag{13}$$

has a unique solution $P_0 = P_0^\top > 0$ for every $q > 0$. Along the solution z the time-derivative of the function

$$V_0(z) = \left(z^\top P_0 z\right)^{1/2}$$

is given as

$$\begin{aligned}\dot{V}_0 &= \frac{1}{2 V_0(z)}\left(- q z^\top z + 2 z^\top P_0 B_0 s_1 \right) \\ &\leq \frac{\|z\|_2}{2 V_0(z)}\left(- q \|z\|_2 + 2 \|P_0 B_0\|_2 |s_1| \right),\end{aligned}$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Certainly the derivative is negative if $\|z\|_2 \geq 2 q^{-1} \|P_0 B_0\|_2 |s_1|$. Since

$$V_0(z) \leq \lambda_{\max}^{1/2}(P_0) \|z\|_2$$

we obtain the following implication

$$V_0(z) \geq 2 q^{-1} \lambda_{\max}^{1/2}(P_0) \|P_0 B_0\|_2 |s_1| \Rightarrow \dot{V}_0 \leq 0, \tag{14}$$

where $\lambda_{\max}(P_0)$ denotes the largest eigenvalue of P_0. Inspired by the approach in [15] we introduce two constants that will be used to parametrise the region of attraction of the closed-loop system (1), (7) and (10) in Section IV-D. Let

$$c_1 = 2(c_s + c_0) \quad \text{for} \quad c_s, c_0 > 0, \tag{15}$$

and

$$a = 2 q^{-1} \lambda_{\max}^{1/2}(P_0) \|P_0 B_0\|_2 a_1, \tag{16}$$

with $a_1 \geq 1$ such that

$$\Omega_{c_1} := \left\{ x \in \mathbb{R}^n \,\middle|\, |s_1(x)| \leq c_1 \wedge V_0(z) \leq a c_1 \right\} \subseteq \mathcal{D}. \tag{17}$$

The constants c_s and c_1 can be considered as the bounds of the integral- and conventional sliding variable, s and s_1, respectively, along the solution x of (1), i.e.

$$|s(x(t), t)| \leq c_s \quad \text{and} \quad |s_1(x(t))| \leq c_1 \quad \forall\, t \geq 0. \tag{18}$$

Using c_0 we further define the set of initial states

$$\Psi_{c_0} := \left\{ x_0 \in \Omega_{c_1} \,\middle|\, |s_1(x_0)| \leq c_0 \right\} \subset \Omega_{c_1}. \tag{19}$$

Proposition 3. *Consider the closed-loop system* (1), (7) *and* (10). *Given some c_1, let there exist α_1, α_2, μ in* (9) *such that $\lim_{t\to\infty} s_1(x(t)) = 0$ and $|s_1(x(t))| \leq c_1$ for all $t \geq 0$. Then for all initial states $x_0 \in \Psi_{c_0}$ it follows that $x(t) \in \Omega_{c_1}$ for all $t \geq 0$ with $\lim_{t\to\infty} x(t) = 0$.*

Proof. The initial state x_0 is an element of the set $\Psi_{c_0} \subset \Omega_{c_1}$. Since $|s_1(x(t))| \leq c_1$ for all $t \geq 0$, we have $V_0(\bar{z}) > a c_1$ for points $\bar{x} = [\bar{z}^\top, \bar{x}_n]^\top \notin \Omega_{c_1}$ satisfying $|s_1(\bar{x})| \leq c_1$. Thus with (16) and (14) we have $\dot{V}_0(\bar{z}) \leq 0$. Therefore Ω_{c_1} is invariant with respect to initial conditions $x_0 \in \Omega_{c_1}$. It follows from (12) that $\lim_{t\to\infty} z(t) = 0$ for $\lim_{t\to\infty} s_1(x(t)) = 0$ from (11), since A_0 is Hurwitz. Therefore, $\lim_{t\to\infty} x_n(t) = \lim_{t\to\infty}(s_1(x(t)) - m^\top z(t)) = 0$ and thus $\lim_{t\to\infty} x(t) = 0$. □

Remark 4. *Note that the above analysis of the state space by means of the reduced dynamic in z requires A_0 to be Hurwitz. This property is guaranteed by design of the sliding variable s_1 in* (3). *Since the super-twisting algorithm cannot establish $s \equiv 0$ in general, the stability analysis of the closed-loop system is based on the fact that the asymptotic convergence $\lim_{t\to\infty} s_1(x(t)) = 0$ resulting from* (11) *implies the asymptotic convergence of the state x. Choosing $m^\top = 0$, i.e. $s_1(x) = x_n$, as proposed in [16] for first-order integral sliding-mode control, however, does not yield convergence of the state x for the super-twisting controller.*

Proposition 3 postulates that $|s_1(x(t))| \leq c_1$ holds along every solution x with initial state $x_0 \in \Psi_{c_0}$. In the next three subsections we show that solutions x with initial conditions $x_0 \in \Psi_{c_0}$ will not leave the set Ω_{c_1}.

B. Bounds for the sliding variable s and the control signal u

To analyse the stability of (10) including the combined perturbation $\delta(x,t)$, we introduce the auxiliary variable

$$\bar{v} = v + \delta(x, t), \tag{20}$$

such that (10) can be written as

$$\dot{s} = -\frac{\alpha_1}{\mu} |s|^{1/2} \operatorname{sgn}(s) + \bar{v}, \quad s(x_0) = s_0, \tag{21a}$$

$$\dot{\bar{v}} = -\frac{\alpha_2}{2\mu^2} \operatorname{sgn}(s) + \frac{\mathrm{d}\delta(x,t)}{\mathrm{d}t}, \quad \bar{v}(0) = \bar{v}_0. \tag{21b}$$

With $v_0 = 0$ and Assumption 1 we have

$$\bar{v}_0 = \delta(x_0, 0) \text{ with } |\bar{v}_0| \leq \rho \text{ for all } x_0 \in \Psi_{c_0} \subset \mathcal{D}. \tag{22}$$

Following [10] and [17] we consider the state transformation

$$\chi_1 = |s|^{1/2} \operatorname{sgn}(s) \qquad \text{and} \qquad \chi_2 = \mu \bar{v} \tag{23}$$

and define $\chi := [\chi_1, \chi_2]^\top$. The dynamics (10) are then given by

$$\dot{\chi} = \frac{1}{\mu |\chi_1|} \left(A_1 \chi + \mu^2 B_2 \frac{\mathrm{d}\delta(x,t)}{\mathrm{d}t} \right), \tag{24}$$

for all times $t \geq 0$ with $s(x(t), t) \neq 0$, where

$$A_1 = \frac{1}{2}\begin{bmatrix} -\alpha_1 & 1 \\ -\alpha_2 & 0 \end{bmatrix} \qquad \text{and} \qquad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{25}$$

The matrix A_1 is Hurwitz for all $\alpha_1, \alpha_2 > 0$ and the Lyapunov equation $A_1^\top P_1 + P_1 A_1 = -I$ has the unique solution

$$P_1 = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = P_1^\top > 0 \quad \text{with} \quad p_{12} = -1. \tag{26}$$

For the stability analysis of (21) we choose the controller-gain scaling μ such that we obtain an invariant set in the s-$\bar{v}$-plane. Using that we show the finite-time convergence of the sliding variable s.

Consider the absolutely continuous function

$$V_1(s, \bar{v}) = p_{11}|s| + 2 p_{12} \mu \bar{v} |s|^{1/2} \operatorname{sgn}(s) + p_{22} \mu^2 \bar{v}^2, \tag{27}$$

which can be written as quadratic form $V_1(s, \bar{v}) = \chi^\top P_1 \chi$ in the coordinates χ.

By construction we have $s(x_0, 0) = 0$ for all $x_0 \in \mathbb{R}^n$. Therefore the initial value $V_1(s(x_0, 0), \bar{v}_0)$ for all $x_0 \in \mathbb{R}^n$ is

$$V_1(s(x_0, 0), \bar{v}_0) = V_1(0, \bar{v}_0) = p_{22}\, \mu^2\, \bar{v}_0^2.$$

Further we introduce the two level-sets of V_1

$$\Gamma_0 \coloneqq \left\{ \begin{bmatrix} s \\ \bar{v} \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; V_1(s, \bar{v}) \leq V_1(0, \bar{v}_0) \right\}, \tag{28}$$

$$\Gamma_\rho \coloneqq \left\{ \begin{bmatrix} s \\ \bar{v} \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; V_1(s, \bar{v}) \leq V_1(0, \rho) \right\}. \tag{29}$$

The former is defined by the initial value $\bar{v}_0$, whereas the latter by the bound $|\bar{v}_0| \leq \rho$, and thus $\Gamma_0 \subseteq \Gamma_\rho$.

Lemma 5. *Let the controller-gain scaling be bounded by*

$$\mu \leq \frac{1}{\rho\, p_{22}} \Big(\big(p_{11}\, p_{22} - p_{12}^2\big)\, c_s \Big)^{1/2} =: \mu_0. \tag{30}$$

Then for all points in Γ_ρ we have

$$\max_{[s,\bar{v}]^\top \in \Gamma_\rho} |s| \leq c_s \;\; \textit{and} \;\; \max_{[s,\bar{v}]^\top \in \Gamma_\rho} |\bar{v}| \leq \frac{1}{\mu} \sqrt{\frac{p_{11}}{p_{22}}\, c_s}. \tag{31}$$

Proof. By design of V_1 in (27) the level set Γ_ρ is symmetric with respect to the origin of the s-$\bar{v}$-plane. Consider the boundary of the set Γ_ρ with $s > 0$, where

$$V_1(s, \bar{v}) = p_{11} s + 2 p_{12} \mu \bar{v} s^{\frac{1}{2}} + p_{22} \mu^2 \bar{v}^2 = p_{22} \mu^2 \rho^2. \tag{32}$$

We shall calculate the points $[s^\star,\, \bar{v}^\star]^\top$ with $\max_{[s,\bar{v}]^\top \in \Gamma_\rho} |s| = s^\star$ by considering the derivative of (32) with respect to $\bar{v}$

$$\frac{\mathrm{d}V_1}{\mathrm{d}\bar{v}} = p_{11} \frac{\mathrm{d}s}{\mathrm{d}\bar{v}} + 2\, p_{12}\, \mu\, s^{\frac{1}{2}} + 2\, p_{12}\, \mu\, \bar{v}\, \frac{\mathrm{d}s^{\frac{1}{2}}}{\mathrm{d}\bar{v}} + 2\, p_{22}\, \mu^2\, \bar{v}.$$

At the boundary of Γ_ρ we have $\frac{\mathrm{d}V_1}{\mathrm{d}\bar{v}} = 0$. Furthermore, for the points $[s^\star,\, \bar{v}^\star]^\top$ the derivative $\frac{\mathrm{d}s}{\mathrm{d}\bar{v}}$, and thus $\frac{\mathrm{d}s^{1/2}}{\mathrm{d}\bar{v}}$, vanishes. We obtain

$$s^\star = \frac{p_{22}^2}{p_{11}\, p_{22} - p_{12}^2}\, \mu^2\, \rho^2 \quad \text{and} \quad \bar{v}^\star = -\frac{1}{\mu}\, \frac{p_{11}}{p_{22}}\, \sqrt{s^\star}.$$

Hence, for $\mu \leq \mu_0$ we have $\max_{[s,\bar{v}]^\top \in \Gamma_\rho} |s| \leq c_s$.

The second inequality of (31) is obtained similarly by differentiating V_1 with respect to s. □

Remark 6. *Since $[s^\star, \bar{v}^\star]^\top$ is an extremum of the level-set Γ_ρ we have equality in* (31) *only for $\mu = \mu_0$. Further*

$$\max_{[s,\bar{v}]^\top \in \Gamma_\rho} |\bar{v}| = \rho \sqrt{\frac{p_{11}\, p_{22}}{(p_{11}\, p_{22} - p_{12}^2)}},$$

which is independent of the parameter $c_s > 0$.

Remark 7. *Our approach is inspired by [10] and [18], where a similar analysis can be found for the s-v-plane associated with* (10)*. However [10] poses a more conservative estimate of a level-set defined in the s-v-plane, which is based on the eigenvalues of the matrix P_1. In [18, Section 3] a similar geometric argument is used to calculate a maximum positive invariant level-set regarding the phase plane.*

Next we shall employ V_1 as a Lyapunov function to show that Γ_0 is invariant for the closed-loop system (1), (7) and (21). However Γ_0 is not suitable for an estimate of the region of attraction in practice as the initial value $\bar{v}_0 = \delta(x_0, 0)$ is unknown. Therefore we establish invariance of $\Gamma_\rho \supseteq \Gamma_0$ for a sufficiently small μ.

It turns out that the desired analysis of the derivative of V_1 requires a known bound for the control signal $u(t)$. With respect to the sets Ω_{c_1} and Γ_ρ we obtain a maximal value for the control signal considering (8) and (10). With

$$|v(t)| \leq |\bar{v}(t)| + \rho \;\; \text{for} \;\; x(t) \in \Omega_{c_1}$$

the absolute value of the control signal u is bounded for $x(t) \in \Omega_{c_1}$ and $[s(x(t), t),\; \bar{v}(t)]^\top \in \Gamma_\rho$ by

$$u_{\max} = \max_{x \in \Omega_{c_1}} \left| k^\top x \right| + \frac{\alpha_1}{\mu} \max_{[s,\bar{v}]^\top \in \Gamma_\rho} |s|^{\frac{1}{2}} + \max_{[s,\bar{v}]^\top \in \Gamma_\rho} |\bar{v}| + \rho\,. \tag{33}$$

Using (31) of Lemma 5 we obtain the estimate

$$u_{\max} \leq \max_{x \in \Omega_{c_1}} \left| k^\top x \right| + \frac{1}{\mu} \left(\alpha_1 + \sqrt{\frac{p_{11}}{p_{22}}} \right) \sqrt{c_s} + \rho. \tag{34}$$

Note that (34) retrieves the equality (33) for $\mu = \mu_0$. For some $\mu < \mu_0$, (34) poses a more conservative estimate of the constant $u_{\max}$.

Based on the estimate (34) of the control signal it is possible to establish the desired invariance of the set Γ_ρ.

Proposition 8. *Let $\mu < \min\{\mu_0, \mu_1\}$ with*

$$\mu_1 \coloneqq \frac{1}{2\, \gamma\, \|P_1\, B_2\|_2} \tag{35}$$

for some $\gamma \geq \gamma_0$ with

$$\gamma_0 \coloneqq \Big(\rho_t + \big(2\rho + \max_{x \in \Omega_{c_1}} \left| k^\top x \right| \big) \rho_{x_n} + \sum_{i=2}^{n} \max_{x \in \Omega_{c_1}} |x_i|\, \rho_{x_{i-1}} \Big) \mu_0 + \rho_{x_n} \Big(\alpha_1 + \sqrt{p_{11}/p_{22}} \Big) \sqrt{c_s}.$$

If $x(t) \in \Omega_{c_1}$ for all $t \geq 0$, then Γ_ρ in (29) *is an invariant set for the solutions $(s, \bar{v})$ of* (1), (7) *and* (21)*. Furthermore, the solutions $(s, \bar{v})$ reach the origin in finite time.*

Proof. Consider the time-derivative of V_1 in (27). For all times t with $s(x(t), t) \neq 0$ we have

$$\begin{aligned} \dot{V}_1 &= \frac{1}{\mu |\chi_1|} \left(-\chi^\top \chi + 2\, \mu^2\, \chi^\top P_1\, B_2\, |\chi_1|\, \frac{\mathrm{d}\delta(x,t)}{\mathrm{d}t} \right) \\ &\leq \frac{1}{\mu |\chi_1|} \left(-\|\chi\|_2^2 + 2\mu^2 \|\chi\|_2\, \|P_1\, B_2\|_2 |\chi_1| \left| \frac{\mathrm{d}\delta(x,t)}{\mathrm{d}t} \right| \right). \end{aligned} \tag{36}$$

The time-derivative of the perturbation $\delta(x, t)$ is given by

$$\frac{\mathrm{d}\delta(x,t)}{\mathrm{d}t} = \frac{\mathrm{d}\delta_t(t)}{\mathrm{d}t} + \frac{\partial \delta_x(x)}{\partial x} \Big[x_2, ..., x_n, u + \delta(x,t) \Big]^\top.$$

With Assumption 1 and $|u(t)| \leq u_{\max}\ \forall\ t \geq 0$ in (33), we obtain the following estimate for $x(t) \in \Omega_{c_1} \subseteq \mathcal{D}$:

$$\left| \frac{\mathrm{d}\delta(x(t),t)}{\mathrm{d}t} \right| \leq \rho_t + \rho_{x_n}(u_{\max} + \rho) + \sum_{i=2}^{n} \max_{x \in \Omega_{c_1}} |x_i|\, \rho_{x_{i-1}}. \tag{37}$$

As suggested in [10] there exists a constant $\gamma_0 \geq 0$ such that

$$\left|\frac{\mathrm{d}\delta(x,t)}{\mathrm{d}t}\right| \leq \frac{\gamma_0}{\mu} \quad \text{for} \quad \mu \leq \mu_0. \tag{38}$$

Choosing γ_0 as in Proposition 8 above yields a bound for the closed-loop system (1), (7) and (21), which is readily verified by substituting the estimate (34) into (37) for $\mu = \mu_0$.

For $\gamma \geq \gamma_0$ we have $\frac{\gamma_0}{\mu} \leq \frac{\gamma}{\mu}$. Thus, for (36) we obtain

$$\begin{aligned}\dot{V}_1 &\leq \frac{1}{\mu\,|\chi_1|}\left(-\|\chi\|_2^2 + 2\,\mu^2\,\|P_1\,B_2\|_2\,\|\chi\|_2^2\,\frac{\gamma}{\mu}\right)\\ &\leq \frac{1}{\mu\,|\chi_1|}\big(-1 + 2\,\mu\,\|P_1\,B_2\|_2\,\gamma\big)\|\chi\|_2^2,\end{aligned}$$

and $\mu < \min\{\mu_0, \mu_1\}$, with $\mu_1 = (2\,\gamma\,\|P_1\,B_2\|_2)^{-1}$ ensures that $\dot{V}_1$ is negative definite along the solutions of the closed-loop system (1), (7) and (21). Thus, the level-set Γ_ρ of V_1 is invariant.

To show finite-time convergence, we note that $\|\chi\|_2 \geq |\chi_1|$ and $\|\chi\|_2 \geq (V_1/\lambda_{\max}(P_1))^{1/2}$, since

$$V_1(s,\bar{v}) \leq \lambda_{\max}(P_1)\,\|\chi\|_2^2.$$

Therefore, we have

$$\dot{V}_1 \leq -\varepsilon\,V_1^{1/2}, \quad \text{with} \quad \varepsilon = \frac{1 - 2\,\mu\,\|P_1\,B_2\|_2\,\gamma}{\mu\,\lambda_{\max}^{1/2}(P_1)}. \tag{39}$$

For an initial value $[s(x_0,0),\bar{v}_0]^\top = [0,\bar{v}_0]^\top$ the solutions $(s,\bar{v})$ of the closed-loop system (1), (7) and (21) reach the origin in finite time without leaving the set $\Gamma_0 \subseteq \Gamma_\rho$. □

Remark 9. *Note that the standard estimate for the reaching time is* $t_r \leq 2\,\varepsilon^{-1}\,V_1(0,\bar{v}_0)$. *However,* $\bar{v}_0$ *involves the combined perturbation, and thus is unknown. With* (22) *we obtain*

$$t_r \leq 2\,\varepsilon^{-1}\,V_1(0,\bar{v}_0) \leq 2\,\varepsilon^{-1}\,V_1(0,\rho). \tag{40}$$

By construction, the initial value $[s(x_0,0),\ \bar{v}_0]^\top$ lies within the set $\Gamma_0 \subseteq \Gamma_\rho$, see (28). Therefore, the choice of the controller-gain scaling $\mu < \min\{\mu_0,\mu_1\}$ renders Γ_0 invariant, and thus implies that the solutions s and $\bar{v}$ along (1), (7) and (21) are bounded by (31), respectively. Hence, Proposition 8 establishes bounds on the integral sliding variable s as well as the auxiliary variable $\bar{v}$ and the controller state v that were obtained by the geometrical consideration of Lemma 5.

Remark 10. *The transformation* (23) *results in a singularity of the time-derivative* (24) *at* $s = 0$. *As mentioned in [10, Remark 3] the change of variables to* χ *is done only to facilitate the choice of the Lyapunov function and the calculation of its derivative.* V_1 *is continuous and continuously differentiable everywhere except on the set* $\{[s,\ \bar{v}]^\top \in \mathbb{R}^2 \,|\, s = 0\}$. *In [4, Appendix 1] it is shown that* V_1 *can still be used as a Lyapunov function of the closed loop.*

Remark 11. *As pointed out in [10, Remark 2] scaling the gains* α_1,α_2 *of the super-twisting controller* (9) *in conjunction with the scaled state* $\chi_2 = \mu\,\bar{v}$ *of the Lyapunov function* (27) *results in the definiteness of the time-derivative* $\dot{V}_1$ *being independent of the controller-gain scaling* μ *if* $\mu < \mu_0$. *The scalar* μ *is chosen small enough to dominate the time-derivative of the combined perturbation* $\delta(x,t)$.

C. Bounds of the conventional sliding variable s_1

In this section we show that the bounds on the integral sliding variable s established by Proposition 8 imply the boundedness of the conventional sliding variable s_1.

Recall the definition of the bounds c_0 and c_1 in (15) and (18), as well as the definition of the set of initial states Ψ_{c_0} in (19). The following result establishes boundedness of s_1.

Proposition 12. *Given the closed-loop system* (1), (7) *and* (10), *and some* $c_0, c_s > 0$. *Let* $x_0 \in \Psi_{c_0}$ *and* $|s(x(t),t)| \leq c_s$ *for all* $t \geq 0$. *Then*

$$|s_1(x(t))| \leq 2(c_0 + c_s) = c_1 \quad \textit{for all } t \geq 0.$$

Proof. Consider (4) as a BIBO-stable scalar differential equation in $\sigma(t) := \int_0^t s_1(x(\tau))\mathrm{d}\tau$ with the input $s(x(t),t) + s_1(x_0)$ and the initial conditions $\sigma(0) = 0$ and $\dot{\sigma}(0) = s_1(x_0)$. Thus

$$\dot{\sigma} + k_s\,\sigma = s(x(t),t) + s_1(x_0). \tag{41}$$

Assuming that $|s(x(t),t)| \leq c_s$ and $x_0 \in \Psi_{c_0}$ the right-hand side of (41) is bounded by $|s(x(t),t) + s_1(x_0)| \leq c_s + c_0 = \frac{c_1}{2}$.

Consider the Lyapunov function $V_{s_1}(\sigma) = \frac{1}{2}\,\sigma^2$. Its time-derivative along the trajectory σ is given by $\dot{V}_{s_1} = \sigma\,\dot{\sigma}$. Substituting $\dot{\sigma}$ from (41) yields

$$\begin{aligned}\dot{V}_{s_1} &= -k_s\,\sigma^2 + \sigma\,(s(x(t),t) + s_1(x_0))\\ &\leq -k_s\,\sigma^2 + |\sigma|\,(c_s + c_0).\end{aligned}$$

$\dot{V}_{s_1}$ is negative for $|\sigma| \geq \frac{c_s + c_0}{k_s} = \frac{c_1}{2\,k_s}$. Thus,

$$|\sigma(0)| \leq \frac{c_1}{2\,k_s} \;\Rightarrow\; |\sigma(t)| \leq \frac{c_1}{2\,k_s} \quad \text{for all } t \geq 0.$$

Therefore, the compact set

$$\Sigma := \left\{\begin{bmatrix}\sigma\\ \dot{\sigma}\end{bmatrix} \in \mathbb{R}^2 \;\middle|\; |\dot{\sigma} + k_s\,\sigma| \leq \frac{c_1}{2} \,\wedge\, |\sigma| \leq \frac{c_1}{2\,k_s}\right\} \tag{42}$$

is invariant with respect to (41). Since $|s_1(x_0)| \leq c_0 < \frac{c_1}{2}$, the initial value $[\sigma(0),\ \dot{\sigma}(0)]^\top = [0,\ s_1(x_0)]^\top$ is in Σ. Thus, the trajectory $(\sigma,\dot{\sigma})$ remains in Σ. It follows that

$$|s_1(x(t))| \leq \max_{[\sigma,\dot{\sigma}]^\top \in \Sigma} |\dot{\sigma}| = c_1 \quad \text{for all } t \geq 0.$$

□

D. Main stability statement and discussion

We shall now combine the Propositions 3, 8 and 12 for our main result.

Theorem 13. *Consider the closed-loop system* (1), (7) *and* (10) *satisfying Assumption 1. Given some arbitrary values for the controller parameters* $\alpha_1, \alpha_2 > 0$ *and* $\mu < \min\{\mu_0, \mu_1\}$ *with* μ_0 *in* (30) *and* μ_1 *in* (35) *as well as the bounds* $c_s, c_0, c_1 > 0$ *in* (15) *such that* $\Omega_{c_1} \subseteq \mathcal{D}$ *in* (17).

Then $x(t) \in \Omega_{c_1}$ *for all* $t \geq 0$ *and* $\lim_{t\to\infty} x(t) = 0$ *for all initial states* $x_0 \in \Psi_{c_0}$. *Further there exists a* $t_r > 0$ *such that* $s(x(t),t) = 0$ *for all* $t \geq t_r$.

Proof. By construction of the sets Ψ_{c_0} and Σ in (19) and (42) for $x_0 \in \Psi_{c_0} \subseteq \Omega_{c_1}$, we have $x_0 \in \Omega_{c_1}$ and $[0, s_1(x_0)]^\top \in \Sigma$. Further, with Assumption 1 we have $\delta(x_0, 0) \le \rho$ for all $x_0 \in \Psi_{c_0} \subset \mathcal{D}$, and thus for Γ_ρ in (29) we have $[s(x_0, 0), \bar{v}(0)]^\top = [0, \delta(x_0, 0)]^\top \in \Gamma_\rho$ with $\mu < \mu_0$ and $x_0 \in \Psi_{c_0}$.

Note that in the proof of Proposition 3 it is established that the set Ω_{c_1} in (17) is invariant if $|s_1(x(t))| \le c_1$ for all $t \ge 0$. Thus if the continuous solution x with initial state $x_0 \in \Psi_{c_0}$ leaves the set Ω_{c_1} at time $t = t_1$ then $|s_1(x(t_1))| = c_1$.

To prove the statement of the theorem we show that $x(t)$ remains in Ω_{c_1} for all $t \ge 0$. Applying the Propositions 3, 8 and 12 then yields asymptotic stability.

1) $x(t) \in \Omega_{c_1}$ by contradiction: Suppose that there is some $t_1 > 0$ such that $x(t) \in \Omega_{c_1}$ for all $t \le t_1$, and $|s_1(x(t_1))| = c_1$. For convenience, we consider the case $s_1(x(t_1)) = c_1$ only. The case $s_1(x(t_1)) = -c_1$ can be handled similarly.

Considering Proposition 8 for the finite time-horizon $t \le t_1$ with $x(t) \in \Omega_{c_1}$ for all $t \le t_1$ and $\mu < \min\{\mu_0, \mu_1\}$, we obtain that the solution $(s, \bar{v})$ satisfies $[s(x(t), t), \bar{v}(t)]^\top \in \Gamma_\rho$ for all $t \le t_1$. Similarly, we obtain from Proposition 12 that $[\int_0^t s_1(x(\tau)) \mathrm{d}\tau, s_1(x(t))]^\top \in \Sigma$ for all $t \le t_1$ with Σ in (42). Note that $[\int_0^{t_1} s_1(x(\tau)) \mathrm{d}\tau, s_1(x(t_1))]^\top \in \Sigma$ with $s_1(x(t_1)) = c_1$ if and only if $\int_0^{t_1} s_1(x(\tau)) \mathrm{d}\tau = -\frac{c_1}{2k_s}$.

Note that, $|s_1(x_0)| \le c_0$ for $x_0 \in \Psi_{c_0}$ as assumed. Consider $s(x(t_1), t_1)$ in (4). With $|s_1(x_0)| \le c_0$, $s_1(x(t_1)) = c_1$, $\int_0^{t_1} s_1(x(\tau)) \mathrm{d}\tau = -\frac{c_1}{2k_s}$, we obtain $s(x(t_1), t_1) \ge c_s$, using the relation of constants $c_1 = 2(c_s + c_0)$ in (15). With Lemma 5 for $[s(x(t_1), t_1), \bar{v}(t_1)]^\top \in \Gamma_\rho$, we have $|s(x(t_1), t_1)| \le c_s$. These two requirements are only satisfied if $s(x(t_1), t_1) = c_s$. However, $[c_s, \bar{v}(t_1)]^\top$ is on the boundary of Γ_ρ and thus $\mu = \mu_0$, see Remark 6. This contradicts $\mu < \min\{\mu_0, \mu_1\}$.

Hence, there exists no $t_1 \ge 0$ for which the solution x leaves the set Ω_{c_1} and thus we have $x(t) \in \Omega_{c_1}$ for all $t \ge 0$.

2) Asymptotic stability: With solution x bounded in Ω_{c_1} and $\mu < \min\{\mu_0, \mu_1\}$, Proposition 8 establishes finite-time convergence $s(x(t), t) = 0$ for all $t \ge t_r$. Using Remark 2 and the boundedness $|s_1(x(t))| \le c_1$ for all $t \ge 0$, we have $\lim_{t\to\infty} s_1(x(t)) = 0$. Finally, Proposition 3 establishes $\lim_{t\to\infty} x(t) = 0$ and $x(t) \in \Omega_{c_1}$ for all $t \ge 0$. □

Remark 14. *The set Ψ_{c_0} can be considered as the region of attraction of the proposed super-twisting integral sliding-mode control scheme since $\lim_{t\to\infty} x(t) = 0$ for all $x_0 \in \Psi_{c_0}$.*

The key aspect of the proposed local stability criterion here lies in the fact that the control signal u can be estimated by (33) for bounded solutions x, s and $\bar{v}$, and thus x, s and v. The choice of the integral sliding variable (4) ensures that we always start on the $\bar{v}$-axis in the s-$\bar{v}$-plane. Moreover, we have a bound for $\bar{v}(0)$. Lemma 5 and Proposition 8 give a bound Γ_ρ on the trajectory in the s-$\bar{v}$-plane.

The boundedness of the integral sliding variable s implies the boundedness of (3) (see Proposition 12), which in turn implies $x(t) \in \Omega_{c_1}$ for the solution (see Proposition 3).

The boundedness of x and (s, v) provides a bound on $\frac{\mathrm{d}}{\mathrm{d}t}\delta(x(t), t)$ scaled by μ. Thus the choice $\mu < \mu_1$ dominates the time-derivate of the combined perturbation $\delta(x, t)$. In the context of higher-order systems this local analysis similar to [10] prevents the occurrence of the algebraic loop, which was resolved in [5], [6] for first-order systems globally.

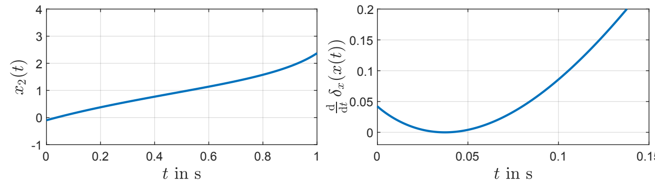

Fig. 1. Simulation of the second system state x_2 and the time derivative of the perturbation $\delta_d(x)$ for an inappropriate rule of thumb control design. The closed loop is not stable.

V. Illustrative Example

To illustrate the design procedure and the performance of the proposed control we consider the second-order system

$$\dot{x}_1 = x_2, \tag{43a}$$

$$\dot{x}_2 = u + \delta_x(x) + \delta_t(t), \tag{43b}$$

with $\delta_x(x) = 0.5\, x_2^3$ and $\delta_t(t) = 3\cos(t/30)$. Note that we have $\delta_t(0) \ne 0$, such that instantaneous sliding motion does not occur for arbitrary initialisation $x_0 = [x_{10},\ 0]^\top \in \mathbb{R}^2$, as discussed at the beginning of Section IV.

We shall design a super-twisting integral sliding-mode controller for the system (43) and establish local stability of the closed-loop system with an estimate of its region of attraction. For the conventional s_1 and the integral sliding variable s in (3) and (4), respectively, we choose $m = 0.25$ and $k_s = 0.5$. The linear state feedback u_0 in (8) is chosen as $k = [k_1,\ k_2]^\top = -[0.125,\ 0.75]^\top$.

Note that the perturbation $\delta(x, t) = \delta_x(x) + \delta_t(t)$ depends on system states and thus boundedness of its time-derivative is difficult to assess. Choosing the gains of the super-twisting integral sliding-mode controller (9) as proposed in [2], [4] and [19] may therefore lead to unstable behaviour. Choosing $L = 0.1$ certainly bounds the time-derivative of δ_t for all $t \ge 0$, and we also observe that the time-derivative for $t = 0$ of the combined perturbation

$$\frac{\mathrm{d}\delta(x_0, 0)}{\mathrm{d}t} = \tfrac{3}{2}\, x_{20}^2\, (u_0 + \tfrac{1}{2}\, x_{20}^3 + 3) \approx 0.042,$$

with initial state $x_0 = [2,\ -0.1]^\top$ is bounded by L. Applying the well established design rules $\alpha_1 = 1.5\sqrt{L} = 0.47$ and $\alpha_2 = 1.1L = 0.11$ proposed in [2] and studied in [7], yields the super-twisting controller

$$u_1 = -0.47\, |s|^{1/2} \operatorname{sgn}(s) + v,\ \dot{v} = -0.11 \operatorname{sgn}(s),\ v(0) = 0.$$

The simulation of the closed-loop system is carried out in MATLAB-Simulink using the fixed-step ODE 4 solver with a step size of 0.1ms. Figure 1 shows the diverging state trajectory x_2 and the time-derivative of the perturbation $\delta_x(x)$ along the solution of the closed loop. The controller proposed above does not stabilise the system.

A. Design of a stabilising control

In order to stabilise the origin of the system (43) consider the super-twisting controller (9) with the scaling parameter μ. The gains are selected as $\alpha_1 = 1$ and $\alpha_2 = 1$. To specify the region of attraction the dynamics of the reduced state $z = x_1$ in (12) are considered. Choosing $q = 2\,m$ in (13) yields the Lyapunov function $V_0(z) = |x_1|$ for $P_0 = 1$. We chose $c_s = 0.15$ and $c_0 = 0.1$ resulting in $c_1 = 0.5$. For $a_1 = 1$ the sets Ω_c and Ψ_{c_0} from (17) and (19) are given by

$$\Omega_{c_1} = \left\{ x \in \mathbb{R}^2 \,\middle|\, |0.25\,x_1 + x_2| \leq 0.5 \wedge |x_1| \leq 2 \right\},$$
$$\Psi_{c_0} = \left\{ x \in \mathbb{R}^2 \,\middle|\, |0.25\,x_1 + x_2| \leq 0.1 \wedge |x_1| \leq 2 \right\}.$$

Note that $\Psi_{c_0} = \left\{ x \in \Omega_{c_1} \,\middle|\, |s_1(x)| \leq c_0 \right\}$ as defined in (19).

In view of Assumption 1 the combined perturbation $\delta(x,t)$ and its partial derivatives with respect to state and time can be estimated on the compact set $\mathcal{D} = \Omega_{c_1}$ by

$$\rho = \max_{x\in\mathcal{D},\, t\in\mathbb{R}^+} |\delta(x,t)| = 3.5, \;\; \rho_t = \max_{t\in\mathbb{R}^+} \left| \frac{\mathrm{d}\delta_t(t)}{\mathrm{d}t} \right| = 0.1,$$
$$\rho_{x_1} = \max_{x\in\mathcal{D}} \left| \frac{\partial \delta_x(x)}{\partial x_1} \right| = 0, \;\; \rho_{x_2} = \max_{x\in\mathcal{D}} \left| \frac{\partial \delta_x(x)}{\partial x_2} \right| = 1.5.$$

1) Controller-gain scaling: To determine the upper bound for the controller-gain considered in Theorem 13 we calculate the bounds μ_0 and μ_1 given by (30) and (35). Regarding the matrices A_1 and P_1 defined in (25) and (26) it holds that $A_1^\top P_1 + P_1 A_1 = -I$ for $P_1 = \left[\begin{smallmatrix} 2 & -1 \\ -1 & 3 \end{smallmatrix}\right]$ and $A_1 = \frac{1}{2}\left[\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Evaluating (30) yields $\mu_0 = 0.0825$. To calculate the second bound μ_1 the time-derivative of the perturbation is estimated as discussed in Section IV-B. Defining $\kappa_1 := \max_{x\in\Omega_{c_1}} |x_1| = 2$ and $\kappa_2 := \max_{x\in\Omega_{c_1}} |x_2| = 1$ we obtain the conservative estimate of the state feedback: $\max_{x\in\Omega_{c_1}} |k^\top x| \leq |k_1|\,\kappa_1 + |k_2|\,\kappa_2$. Based on this estimate we chose $\gamma = 2.05$ to satisfy the inequality (38) by calculating

$$\gamma = \left(\rho_t + \rho_{x_1}\kappa_2 + (2\rho + |k_1|\,\kappa_1 + |k_2|\,\kappa_2)\,\rho_{x_2}\right)\,\mu_0$$
$$+\rho_{x_2}\left(\alpha_1 + \sqrt{p_{11}/p_{22}}\right)\sqrt{c_s}.$$

Evaluation of (35) thus yields $\mu_1 = 0.077$. Therefore, $\mu = 0.95\min\{\mu_0, \mu_1\} = 0.95\mu_1 = 0.073$ is chosen, resulting in the scaled controller-gains $\alpha_1/\mu = 13.67$ and $\alpha_2/(2\mu^2) = 93.43$.

2) Estimated reaching time and control effort: In regard to the stability analysis presented in Section IV, the reaching time as well as the absolute value of the control signal can be estimated. The choice $\mu = 0.95\,\mu_1$ leads to (39) being satisfied for $\varepsilon = 0.36$. An estimate of the reaching time can be obtained as proposed in Remark 9. Evaluating the second inequality of (40) yields $t_r \leq 2\,\varepsilon^{-1}\,V_1(0,\rho) = 1.09$. The absolute value of the control signal u is bounded by the constant $u_{\max}$ given in (33). Calculating the set Γ_ρ from (29) for $\mu = 0.073$ yields $\max_{[s,\bar{v}]^\top\in\Gamma_\rho} |s| = 0.12$ and $\max_{[s,\bar{v}]^\top\in\Gamma_\rho} |\bar{v}| = 3.83$. We obtain $|u| \leq 13.03$ by evaluating (33) with $\max_{x\in\Omega_{c_1}} |k^\top x| \leq |k_1|\,\kappa_1 + |k_2|\,\kappa_2$. Further, we get $\left|\frac{\mathrm{d}}{\mathrm{d}t}\delta(x(t),t)\right| \leq 28.07$ for all $t \geq 0$ from (38) with $\gamma = 2.05$.

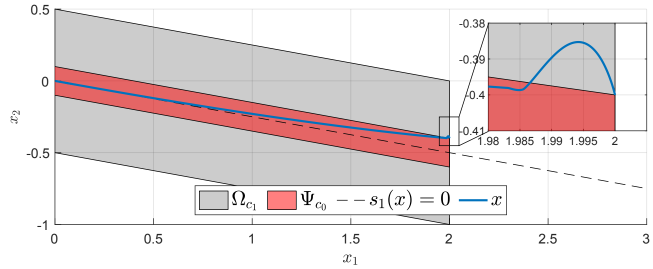

Fig. 2. System state x in the phase plane. The estimated region of attraction Ψ_{c_0} and the set Ω_{c_1} bounding the trajectory are given for $c_1 = 0.5$, $c_0 = 0.1$ and $m = 0.25$.

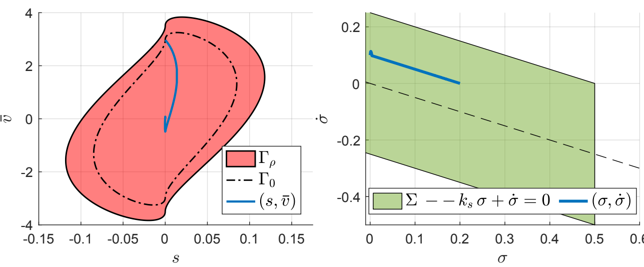

Fig. 3. Trajectories of the sliding variables $(s, \bar{v})$ and $(\sigma, \dot{\sigma})$, respectively. The invariant sets Γ_0, Γ_ρ and Σ are given for $\mu = 0.073$ and $c_1 = 0.5$. It holds that $\max_{[s,\bar{v}]^\top\in\Gamma_\rho} |s| = 0.11 < 0.15 = c_s$.

B. Simulation of the closed loop

Figures 2, 3 and 4 show the simulation results for the closed-loop system (43), (7) and (10) for the initial state $x_0 = [2, -0.4]^\top \in \Psi_{c_0}$ and controller design as described in the previous subsection.

Figure 2 shows the state-space with the sets Ψ_{c_0} and Ω_{c_1} representing the specified region of attraction and the bounds on the trajectories, respectively. Note that the trajectory indeed leaves the set Ψ_{c_0} before converging onto $s_1(x) = 0$ and the origin asymptotically.

Figure 3 depicts the solution $(s, \bar{v})$ of the dynamics of the sliding variable (21) as well as the solution $(\sigma, \dot{\sigma})$ of the dynamics (41) in their respective phase-planes. The invariant sets Γ_0, Γ_ρ and Σ, given by (28), (29) and (42), are shown. By definition, the initial value $[s(x_0, 0), \bar{v}(0)]^\top = [0,\ 2.968]^\top$ lies on the boundary of the set Γ_0. Since $\bar{v}_0 < \rho$, it follows that $\Gamma_0 \subset \Gamma_\rho$. The trajectory converges to the origin without leaving the set Γ_0. Choosing $\mu < \mu_0$ results in (31) given as a strict inequality, i.e. $\max_{[s,\bar{v}]^\top\in\Gamma_\rho} |s| = 0.11 < 0.15 = c_s$ and $\max_{[s,\bar{v}]^\top\in\Gamma_\rho} |\bar{v}| = 3.83 < 4.32 = \mu^{-1}\sqrt{p_{11}/p_{22}\,c_s}$. Further it holds that $[\sigma(t),\ \dot{\sigma}(t)]^\top \in \Sigma$ for all $t \geq 0$.

The solutions of the integral and the conventional sliding variable s and s_1 as well as the control signal u and the time-derivative of the perturbation $\delta_x(x)$ are given in Figure 4. The reaching time $t_r \approx 0.04$ marked in each plot is significantly smaller than its estimate 1.09, calculated above.

Note that the integral sliding variable s is not identically zero although we initialise on the sliding surface $s(x_0, 0) = 0$. This prevents s_1 from decaying exponentially for all $t \geq 0$, i.e. $\max_{t\in\mathbb{R}^+} |s_1(x(t))| > |s_1(x_0)|$ as pointed out in Remark 2, and causes the state x to leave the set Ψ_{c_0}, see detail in

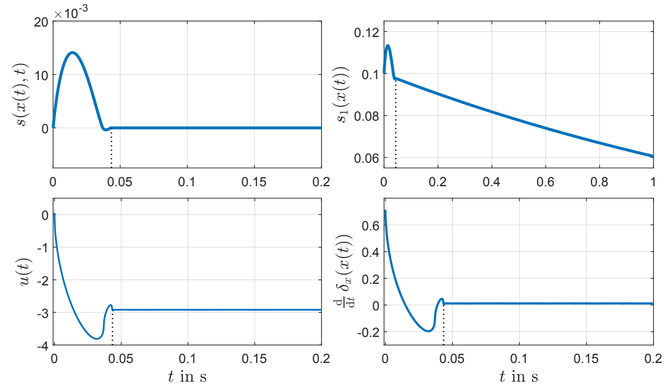

Fig. 4. Simulation of the sliding variables s, s_1 (top) and the control signal u and the time derivative of $\delta_x(x)$ (bottom) in the time domain. The absolute values of the trajectories are limited by $|s(x,t)| \leq c_s = 0.15$, $|s_1(x)| \leq c_1 = 0.5$, $|u| \leq 13.03$ and $|\frac{\mathrm{d}}{\mathrm{d}t}\delta_x(x)| \leq 28.07$.

Figure 2. For $t \geq t_r$ the conventional sliding variable s_1 converges exponentially as in (11), see top right plot in Figure 4.

We observe that both sliding variables are bounded by the design parameters c_s and c_1, respectively as proposed in (18):

$$\max_{t\in\mathbb{R}^+} |s(x(t),t)| = 0.014 < c_s = 0.15,$$
$$\max_{t\in\mathbb{R}^+} |s_1(x(t))| = 0.11 < c_1 = 0.5.$$

The time-derivative $\frac{\mathrm{d}}{\mathrm{d}t}\delta(x,t)$ is given by $\frac{\mathrm{d}}{\mathrm{d}t}\delta(x,t) = \frac{\mathrm{d}}{\mathrm{d}t}\delta_x(x) + \frac{\mathrm{d}}{\mathrm{d}t}\delta_t(t)$ with $\frac{\mathrm{d}}{\mathrm{d}t}\delta_x(x)$ shown in Figure 4 and $\frac{\mathrm{d}}{\mathrm{d}t}\delta_t(t) = -0.1\sin(t/30)$. The control signal u and the time-derivative $\frac{\mathrm{d}}{\mathrm{d}t}\delta(x,t)$ satisfy the inequalities $\max_{t\in\mathbb{R}^+}|u(t)| = 4.1 \leq 13.03$ and $\max_{t\in\mathbb{R}^+}\left|\frac{\mathrm{d}}{\mathrm{d}t}\delta(x(t),t)\right| = 0.71 \leq 28.07$ from Section V-A.

In sliding-mode for $t \geq t_r$ we have $s(t) = 0$ such that the control signal $u(t) = u_0(t) + u_1(t) = k^\top x(t) + v(t)$ compensates the perturbation $\delta(x(t),t)$ with the controller state $v(t) = -\delta(x(t),t)$. Thus $u(t)$ converges to $-\delta_t(t)$, and also $\frac{\mathrm{d}}{\mathrm{d}t}\delta_x(x(t)) \to 0$ for $t \to \infty$.

VI. Conclusion

We propose a super-twisting sliding-mode design based on an integral sliding variable for systems of arbitrary order with time- and state-dependent perturbations. The approach does not required an a priori known bound for the time-derivative of the control signal. The parametrisation of the super-twisting controller, proposed in [10], is used to overcome the problem of an algebraic loop [5] by means of a local stability analysis. As part of the design process an estimate for the region of attraction as well as a guaranteed bound of the solution for the closed-loop system are obtained. The analysis provides bounds for the integral sliding variable and the controller state as well as for the control effort needed to compensate the perturbation.

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