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Asymptotic Stabilization of Passive Nonlinear Systems with Finite Countable Control Actions: mixed switching – nearest action control approach

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***Abstract*— This paper studies the global asymptotic stabilization of passive nonlinear systems with finite, countable control actions. We show that for nonlinear passive systems that are large-time norm observable and admit a finite control input set whose convex hull contains the origin, the origin can be globally asymptotically stabilized and locally exponentially stabilized by means of relaxed control and nearest-action control approaches. In particular, we improve on a recent result of practical stabilization via nearest-action control by utilizing switching controllers that can synthesize extra control actions from an existing control input set. Three switching methodologies are proposed to enlarge the control set and enable global asymptotic and local exponential stabilization. These three methodologies vary in the cardinality of the expanded control set. These methods are validated in numerical simulations where a comparison of the convergence rate is provided.**

I. Introduction

Modern control theories and methods have often relied on the assumption that the input space is a continuum that can be arbitrarily assigned/accessed by any control laws. This assumption is based on the availability of actuator systems that can be actuated in a continuous mode. However, it is no longer applicable for applications where the control inputs are limited to a finite or infinite countable set. For example, the Ocean Grazer's multi-piston-pump systems [1] are designed to have only a finite number of constant actuation forces. This limits the device to realize arbitrary pumping forces, and the control input can only be chosen from a finite combination of multiple piston pumps. Other examples include the finite number of thrusters in rocket systems and the finite number of actions in stepper motors. Another class of well-studied control systems with countable control input sets are digital control systems, where the control inputs must be taken from a finite set obtained via quantization and zero-order-hold operations.

In the latter context (namely, the digital control systems and networked control systems), controller design and stability analysis of linear systems under limited information have been studied, for instance, in [2], [3]. In recent years, the generalization of these results to multi-agent systems has been presented in literature, such as [4], [5], and [6]. Correspondingly, this paper studies the related control problem with finite and minimal countable control inputs, that is, when the control actions are constrained to be from the finite control set $\mathcal{U} = \{0, u_1, \ldots, u_p\} \subset \mathbb{R}^m$.

The literature on the design of controllers for systems with limited control actuation is vast, particularly those that correspond to quantized or digital control systems. In [7], [8], [9], [10], the authors analyze the effect of quantized output feedback on the stability of the closed-loop systems. The number of control actions required in the above methods is $(2N+1)^m$, where $\mathbb{R}^m$ is the input-output space, and each dimension has $2N+1$ quantization levels. Passive systems have also attracted much interest as the presence of storage functions facilitates the design of control laws, as presented in [12], [13], and [14]. Using binary control, the papers [4], [5] present the practical stability property of the resulting feedback control systems. These controllers require 2^m+1 control actions where m is the dimension of the input space.

For passive systems with large-time norm observability, the authors of [16] and [17] propose a nearest neighbor control[1] protocol that guarantees practical stability using $m+2$ control actions along with an algorithm to construct these $m+2$ control actions.

On the one hand, the authors in [16], [17] could only establish practical stabilization, where the state converges to a ball close to the origin since it cannot realize an arbitrary small control signal. On the other hand, in the networked control systems literature, logarithmic quantizers have been utilized to guarantee asymptotic stabilization, see, for example, [2], [11]. Using logarithmic quantizers, one can realize arbitrary small control signals. Note that the sufficiency of small control property to design a continuous stabilizing control law is well-studied, e.g. [19].

Inspired by the use of a logarithmic quantizer in these works, we extend the works of [16] and [17] by expanding the minimal countable input set via appropriate switching strategy, which resembles a pulse-width modulation (PWM) approach and can be analyzed in the framework of relaxed control [22]. We present three switching strategies that enable

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[1]Throughout this paper, we will refer to this approach as nearest-action-control method to avoid the confusion with the term of neighbors that are commonly used in the multi-agent control systems literature.

us to construct an arbitrarily large countable control input set where the globally stabilizing control law can be applied.

Switching controllers have gathered immense interest since the seminal result of [18], where we can stabilize nonlinear systems that are not stabilizable using continuous feedback. We refer interested readers to [20] for a detailed exposition on switched control systems. To analyze the stability of the closed-loop system under switching controllers, we employ the relaxed control framework as recently presented in [21] and [22]. Via this framework, we can study the systems property of switched passive systems as relaxed systems where the switching ratio becomes the new control input.

In summary, the main contributions of this paper are:

1) We propose three switching controllers that achieve global asymptotic stability instead of practical stability while only utilizing the existing minimal control actions.
2) The proposed design approaches can decouple the control design methods to achieve the desired convergence rate and the region-of-attraction of the closed-loop system.

The latter contribution improves upon the control performance achieved in the previous works of [21] and [22].

The rest of the paper is organized as follows. Section II provides the preliminaries on passive systems, on stabilization with finite countable control input set, on nonsmooth analysis, and on relaxed control systems. In Section III, we present the three switching methodologies and analyze the stability property of the resulting closed-loop relaxed control systems. In Section IV, we provide numerical simulations to validate, compare, and contrast the proposed methodologies. Finally, we provide the conclusions and future work in Section V.

II. Preliminaries and Problem Formulation

A. Notation

$\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{W}, \mathbb{N}$ denote the set of real numbers and the set of non-negative real numbers, the set of whole numbers and natural numbers respectively. For a vector $x \in \mathbb{R}^n$, or a matrix $A \in \mathbb{R}^{n\times m}$, we denote the Euclidean norm and the corresponding induced matrix norm by $\|x\|$ and $\|A\|$ respectively. The inner product of two vectors $\mu, \nu \in \mathbb{R}^n$ is denoted and defined as $\langle \mu, \nu\rangle \triangleq \sum_{i=1}^n \mu_i \nu_i$. For a set S, $\overline{\mathrm{co}}(S)$ defines its convex closure. For a discrete set $\mathcal{U}$, its cardinality is denoted by $\mathrm{card}(\mathcal{U})$. The convex hull of vertices from a discrete set $\mathcal{U}$ is denoted by $\mathrm{conv}(\mathcal{U})$. For a matrix $A \in \mathbb{R}^{n\times n}$, $A^\top$ denotes the transpose of A. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class κ if it continuous, strictly increasing, and $\gamma(0) = 0$. a function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class κ_∞ if γ is of class κ and unbounded. The open ball of radius ϵ centered around the point $x \in \mathbb{R}^n$ is denoted by $\mathbb{B}_\epsilon(x)$. The notation δ_ϵ denotes a Dirac probability measure centered at ϵ.

B. Passive systems and standing assumptions

Consider the following class of nonlinear systems

$$\Sigma : \begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x). \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state, $u \in \mathbb{R}^m$ denotes the control input and $y \in \mathbb{R}^m$ is the observation from the system. The system dynamics consists of the smooth mappings $f : \mathbb{R}^n \to \mathbb{R}^n, g : \mathbb{R}^n \to \mathbb{R}^{n\times m}, h : \mathbb{R}^n \to \mathbb{R}^m$. The system Σ is called *passive* if for all pairs of input and output signals $\int_0^T \langle y(t), u(t)\rangle dt > -\infty$ for all $T > 0$. The passivity of Σ implies the existence of positive definite storage $H : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that $\langle \nabla H(x), f(x)\rangle \leq 0$ and $\langle \nabla H(x), g(x)\rangle = h^\top(x)$. We will assume throughout that H is proper, that is, all level sets are compact.

Definition 1 ([23]): The system (1) is large-time initial-state norm observable if there exists $\tau > 0$, and $\gamma, \chi \in \mathcal{K}_\infty$ such that the solution x of (1) satisfies

$$\|x(t)\| \leq \gamma(\|y\|_{[t,t+\tau]}) + \chi(\|u\|_{[t,t+\tau]}) \tag{2}$$

for all $t \geq 0, x(0) \in \mathbb{R}^n$, and locally essentially bounded and measurable inputs $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$.

The large-time norm observability of the system is used to ascertain the bounds on the system's state when the control input $u = 0$. Further, we assume the following property.

(A0) The system Σ in (1) is passive with a proper storage function H and $\Sigma|_{u=0}$ is large-time norm-observable for some $\tau > 0$ and $\gamma \in \mathcal{K}_\infty$.

C. Practical stabilization problem with finite countable control input set

As briefly described in the Introduction, using the output measurements y, we consider the stabilization of the system Σ in (1) when the control actions can only be chosen from a finite input set. Correspondingly, we consider a finite control action set $\mathcal{U}$, satisfying the following assumption

(A1) For a given set $\mathcal{U} := \{0, u_1, \dots, u_p\}$ where $u_i \in \mathbb{R}^m$, $i = 1, \dots, p$, there exists a minimal index set $\mathcal{I} \subset \{1, \dots, p\}$ such that the set $\mathcal{V} := \{u_i\}_{i\in\mathcal{I}} \subset \mathcal{U}$ defines the vertices of a convex polytope satisfying, $0 \in \mathrm{int}(\mathrm{conv}(\mathcal{V}))$.

Let us recall the following result from [16], [17] on the practical stabilization of Σ to a prescribed ball $\mathbb{B}_\epsilon$.

Proposition 1 ([16, Proposition 2]): Consider a nonlinear system Σ as in (1) satisfying (A0), along with a finite countable control input set $\mathcal{U} \supset \mathcal{V}$ satisfying (A1) and a scalar $C_\mathcal{V} = \max_{\tilde{\nu}\in\tilde{\mathcal{V}}}(\|\tilde{\nu}\|)$, where $\tilde{\mathcal{V}}$ is given by

$$\tilde{\mathcal{V}} := \{\tilde{\nu} \in \mathbb{R}^m \mid [\nu_1, \dots, \nu_q]^\top \tilde{\nu} \leq \frac{1}{2}[\|\nu_1\|^2, \dots, \|\nu_q\|^2]^\top\}$$

and $\nu_1, \dots, \nu_q \in \mathcal{V}$. For a given $\epsilon > 0$ assume that

$$\gamma(C_\mathcal{V}) \leq \epsilon,$$

where γ is the large-time norm observability function as in (A0). Then the control law

$$u = \underset{\nu \in \mathcal{U}}{\operatorname{argmin}} \|\nu + y\|$$

globally practically stabilizes Σ with respect to $\mathbb{B}_\epsilon$.

It has also been shown that the minimum number of nonzero control actions required to achieve the above results is $m+1$. The key contribution of this paper is achieving global asymptotic stability to the origin while choosing control actions from a finite set of control inputs $\mathcal{U}$. In particular, we address the following problem.

Asymptotic output-feedback stabilization with limited control (AOS - LC): For the given system Σ in (1), determine the finite set $\mathcal{U} = \{0, u_1, \ldots, u_p\} \subset \mathbb{R}^m$ and design the feedback control law $\phi : \mathbb{R}^n \to \mathcal{U}$ such that the closed-loop system of (1) with $u = \phi(y)$ satisfies $x(t) \to 0$ as $t \to \infty$ for all initial conditions $x(0) \in \mathbb{R}^n$.

As mentioned before, using ideas from switching controllers and relaxed control framework, we propose three methodologies to achieve the desired global asymptotic stability. The key idea is to switch between the permissible control actions to generate newer and finer control actions to achieve stabilization. The three methodologies differ in the construction of the extended countable control action set.

D. Nonsmooth analysis and differential inclusions

As we can only select control actions from a finite discrete set, the controller mapping is inherently discontinuous. Therefore, notions of nonsmooth solutions and nonsmooth stabilization are required. For a discontinuous map $F : \mathbb{R}^n \to \mathbb{R}^m$, we can define a set-valued map $\mathcal{K}(F) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by convexifying F as follows

$$\mathcal{K}(F(x)) := \bigcap_{\delta > 0} \overline{\text{co}}(F(x + \mathbb{B}_\delta)).$$

In this case, we can analyze the dynamic behaviour of the original differential equation $\dot{x} = F(x), x(0) = x_0$ via the corresponding differential inclusion

$$\dot{x} \in \Phi(x) \stackrel{\Delta}{=} \mathcal{K}(F(x)) \qquad x(0) = x_0. \tag{3}$$

A Krasovskii solution $x(t)$ on the interval $\mathbb{R}_{\geq 0}$ is an absolutely continuous function $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ such that (3) holds almost everywhere on $\mathbb{R}_{\geq 0}$.

E. Relaxed Systems and Relaxed input

Let $\text{rpm}(U)$ be the set of all Radon probability measures defined on U. In a relaxed control framework, one studies the system's behavior when the ordinary real-valued control input ($u \in U \subset \mathbb{R}^n$) is replaced by a measure-valued control input ($\nu \sim \mu \in \text{rpm}(U)$), where the switching input signals can be embedded. Consider a general nonlinear system $\dot{x} = f(x, u)$, $x \in X \subseteq \mathbb{R}^n$ and $u \in U$, U is a compact subset of $\mathbb{R}^n$. For a compact metric space $V \subset \mathbb{R}^q$, the space $\mathcal{R}_f(V, \text{rpm}(U))$ is the space of all functions $\mu : V \to \text{rpm}(U)$ such that the function

$$(x, \nu) \to \int_U f(x, \tau) d\mu_\nu(\tau)$$

is locally Lipschitz on $X \times V$. Here, we consider a class of probability measure for the input that is parametrized by $\nu \in V$. Let the function $f_R : X \times V \to \mathbb{R}^n$ be defined by

$$f_R(x, \nu) := \int_U f(x, \tau) d\mu_\nu(\tau).$$

When one can apply the probability measure μ_ν as the input to the original systems, the corresponding average behavior is described by

$$\dot{x} = f_R(x, \nu) \qquad x(0) = x_0. \tag{4}$$

The system (4) is called a relaxed system, and ν denotes the relaxed control input.

As a concrete example, consider a switching controller that switches between two control inputs u and 0 with a duty cycle denoted by α. Then the switching controller $\eta : \mathbb{U} \times [0,1] \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathcal{U}$ is defined as,

$$\eta(u, \alpha, \Delta, t) = \begin{cases} u & n\Delta \leq t < \alpha\Delta + n\Delta \\ 0 & \alpha\Delta + n\Delta \leq t \leq (n+1)\Delta. \end{cases}, n \in \mathbb{W} \tag{5}$$

By switching between the control actions sufficiently fast, the control input can be equivalently considered to be sampled from a probability measure-valued function μ_α (which depends on the duty cycle α) whose domain is defined on the input space $\text{span}\{0, u\}$ and is defined by $\mu_\alpha(E) = \int_E r_\alpha(\tau) d\tau$ for all $E \subset \text{span}\{0, u\}$, where

$$r_\alpha = \alpha\delta_u + (1 - \alpha)\delta_0$$

for all $\alpha \in [0,1]$ and δ_ϵ is the Dirac measure at $\epsilon \in \mathbb{R}$. Applying such measure-valued function μ_α to the ordinary control input, the corresponding average systems (4) is given by

$$\Sigma_{\text{rel}} : \begin{cases} \dot{x} & = f(x) + g(x)(\alpha u), \alpha \in [0,1] \\ y & = h(x). \end{cases}$$

This corresponds to applying a control input of αu to the system Σ. It must be noted that the control action αu might not necessarily belong to the set of all admissible control actions $\mathcal{U}$. Thus, by utilizing switching controllers, we can generate new control actions not available previously. The admissible control set has been expanded from $\mathcal{U} = \{0, u\}$ to $\mathcal{U}' = \{0, u, \alpha u\}$ which we call the *equivalent control set* here on in this paper. The proposed methodologies involve generating *equivalent control sets* $\mathcal{U}'$ to achieve asymptotic stability.

III. METHODOLOGY

A. Methodology A: Minimally Realizable Set with Logarithmic Extension

In subsection II.E, it has been shown how to generate αu given the control actions $\{0, u\}$. Considering all pairs $\{0, u_i\}, u_i \in \mathcal{U}$, we can generate new control actions in the set $\alpha\mathcal{U}$ using measure-valued control input μ_v. Correspondingly, if we consider the switching frequency value to be $\alpha, \alpha^2, \alpha^3, \ldots$ then we can generate control actions in the set $\alpha\mathcal{U},\ \alpha^2\mathcal{U},\ ,\ldots$ respectively. In this case, the *effective control set* is given by $\mathcal{U}_\mathrm{A} = \bigcup_{k\in\mathbb{N}} \alpha^k\mathcal{U}$. Define the nearest action map as follows

$$\phi(y,\ \mathcal{D}) = \underset{d\in\mathcal{D}}{\operatorname{argmin}}\|d-y\|$$

which picks all the points from the set $\mathcal{D}$ closest to the point y. As the system (1) is passive, a linear output feedback $u=-y$ stabilizes the system. Given that the control action can only be chosen from the set $\mathcal{U}_\mathrm{A}$, we propose a nearest action controller defined as follows

$$u = \phi_\mathrm{A}(-y) = \phi(-y,\ \mathcal{U}_\mathrm{A}) = \underset{u\in\mathcal{U}_A}{\operatorname{argmin}}\|u+y\|. \tag{6}$$

Although the control actions in $\mathcal{U}_\mathrm{A}\setminus\mathcal{U}$ are not directly realizable, it can be achieved by using the switching controller (5) with an appropriate choice of $\alpha \in [0,1]$ and $u \in \mathcal{U}$. It is first shown that the output of the nearest action control $\mathcal{W}_\mathrm{A} = \phi_\mathrm{A}(-y)$ is a finite set, even though $\mathcal{U}_\mathrm{A}$ is a countably infinite set.

Lemma 1: Let $\mathcal{U}$ be a minimal countable set satisfying (A1). The cardinality of the set $\mathcal{W}_\mathrm{A} = \phi_\mathrm{A}(-y)$ given by (6) is upper bounded by $2(m+1)$.

Proof: Firstly we recall that the cardinality of $\mathcal{U}\setminus\{0\}$ is given by $m+1$ as shown in [16] and [17]. Consider now three points $p_1 = \alpha^{k_1}u_d,\ p_2 = \alpha^{k_2}u_d,\ p_3 = \alpha^{k_3}u_d$ for some $u_d \in \mathcal{U}$ and all $k_1 \neq k_2 \neq k_3 \in \mathbb{N}$. Trivially $p_1,\ p_2,\ p_3 \in \mathcal{U}_\mathrm{A}$. The points p_1, p_2, p_3 cannot have a point $z \in \mathbb{R}^m$ equidistant from all three of them. Thus, we can conclude that $\operatorname{card}(\mathcal{W}_\mathrm{A}) \leq 2(m+1)$. ■

Next, we show some desirable properties of the set $\mathcal{W}_\mathrm{A} = \phi_\mathrm{A}(-y)$. These properties will be used to show the asymptotic stability of the closed-loop system Σ under (6).

Lemma 2: $\mathcal{W}_\mathrm{A} = \{0\} \iff y = 0$.

Proof: If $y=0$, then we can trivially conclude that $\mathcal{W}_\mathrm{A} = \{0\}$. Let us prove the "⇒" part by contradiction. In this case, consider $\mathcal{W}_\mathrm{A} = \{0, u\}, u \neq 0, u \in \mathcal{U}_\mathrm{A}$ and $y \neq 0$. This implies that 0 and u are the closest points to $-y$, i.e. $-y$ is in the middle-point between 0 and u. However, the point $\alpha u \in \mathcal{U}_\mathrm{A}$ is closer to $-y$ than 0 or u, which is a contradiction. ■

In addition to above properties of the effective control set $\mathcal{U}_\mathrm{A}$, the following is an important property of the proposed controller (6).

Lemma 3: Consider the nearest-action mapping ϕ_A in (6) and let $\mathcal{W}_\mathrm{A} = \phi_\mathrm{A}(-y) = \{w_i\}_{i\in\mathcal{I}} \subset \mathcal{U}_\mathrm{A}$ for some index set $\mathcal{I} \subset \{1,\ldots,2(m+1)\}$. Then the inequality

$$-\|w_i\|\|y\| \leq \langle w_i, y\rangle \leq -\frac{1}{2}\|w_i\|^2 \tag{7}$$

holds for all $i \in \mathcal{I}$.

Proof: By the definition of ϕ_A, the inequality $\|w_i + y\|^2 \leq \|w_j + y\|^2$ holds for all $i \in \mathcal{I}$ and for all $j \in \{0,1,\ldots,(m+1)\}$. By noting that $\|w_i+y\|^2 = \langle w_i+y, w_i+y\rangle = \|w_i\|^2 + 2\langle w_i, y\rangle + \|y\|^2$ and fixing $w_j = 0$, we have that $\langle w_i, y\rangle \leq -\frac{1}{2}\|w_i\|^2$. Moreover $\langle w_i, y\rangle \geq -\|w_i\|\|y\|$. Hence, the inequality (12) holds for every $y \in \mathbb{R}^m$. ■

The following proposition shows that the closed-loop system under the control law (6) is globally asymptotically stable.

Proposition 2: Given an admissible control set $\mathcal{U}$ satisfying (A1), the control law $u = \phi_\mathrm{A}(-y)$ given by (6) asymptotically stabilizes the system Σ in (1) satisfying (A0) to the origin.

Proof: Let $\mathcal{W}_\mathrm{A} = \phi_\mathrm{A}(y)$, from *Lemma* 1 and *Lemma* 2, $\mathcal{W}_\mathrm{A}$ can be one of the two cases:

(i) $0 \notin \mathcal{W}_\mathrm{A}$ for $y \neq 0$. As $\phi_\mathrm{A}(-y)$ is a non-smooth operator, we consider instead the following differential inclusion

$$\begin{aligned}&\dot{x} \in \mathcal{K}(f(x) + g(x)\phi(y)) = f(x) + g(x)\mathcal{K}(\phi_\mathrm{A}(-y))\\&y = h(x).\end{aligned} \tag{8}$$

Computing the time derivative of the storage function of the original system (1), along the solutions of (8),

$$\begin{aligned}\dot{H}(x) &= \langle\nabla H(x), \dot{x}\rangle \in \langle\nabla H(x), f(x) + g(x)\mathcal{K}(\phi(y))\rangle\\ &= \langle\nabla H(x), f(x)\rangle + \langle y, \operatorname{conv}(\mathcal{W}_\mathrm{A}).\rangle\end{aligned}$$

Consider any $x \in \mathcal{W}_\mathrm{A} = \{w_i\}_{j\in\mathcal{J}},\ \mathcal{J} \subset \{1,\ldots,2(m+1)\}$ can be written as,

$$x = \sum_{j\in\mathcal{J}}\lambda_j w_j,\ \ \sum_{j\in\mathcal{J}}\lambda_j = 1,\ \lambda_j \in [0,1], \forall\, j \in \mathcal{J}.$$

Then using *Lemma* 3, we get

$$\sum_{j\in\mathcal{J}}\lambda_j\|w_j\|\|y\| \leq \langle y, x\rangle \leq \sum_{j\in\mathcal{J}}\lambda_j\|w_j\|^2.$$

Therefore $\langle y, \operatorname{conv}(\mathcal{W}_\mathrm{A})\rangle \in [\ \|w_{y,\max}\|\|y\|, -\frac{1}{2}\|w_{y,\min}\|^2\ ]$, and $\|w_{y,\max}\| = \max_{w\in\mathcal{W}_\mathrm{A}}\|w\|, \|w_{y,\min}\| = \min_{w\in\mathcal{W}_\mathrm{A}}\|w\|$. Giving,

$$\dot{H}(y) < -\frac{1}{2}\|w_{y,\min}\|^2 < 0.$$

(ii) $\mathcal{W}_\mathrm{A} = \{0\}$, when $y=0$. We obtain

$$\dot{H}(t) = 0.$$

As $H(x(t))$ is non-increasing and since H is proper, all solutions of $x(t)$ are bounded. By the LaSalle invariance principle, all of such compact trajectories converge to the largest invariant set $M \in \mathbb{R}^n$ where $h(M) \subset Z$, where $Z := \{y \in \mathbb{R}^m \mid 0 = \phi_\mathrm{A}(-y)\}$. $0 \in \phi_\mathrm{A}(-y) \iff y = 0$, yielding $M = Z = \{0\}$. Using the large-time norm-observability of $\Sigma|_{u=0}$, it follows that,

$$\lim_{t\to\infty}\|x(t)\| = 0.$$

■

Remark 1: The proposed method for generating new control actions requires the duty cycle to become arbitrarily small to ensure asymptotic stability. The minimum duty cycle required is calculated if it is sufficient to ensure practical stability to a given $\mathbb{B}_\epsilon$. If the allowed control actions are given as $\mathcal{U} = \{0, u_1 \dots, u_p\}$, for a given accuracy level $\epsilon > 0$, it is sufficient to generate the *effective control set* given by

$$\mathcal{U}_{\mathrm{A},\epsilon} = \bigcup_{i=1}^{k_{\max}} \alpha^i \mathcal{U}$$

where $k_{\max}$ is calculated as

$$k_{\max} = \left\lceil -\log_\alpha \left(\frac{\gamma^{-1}(\epsilon)}{\max\limits_{\tilde{v}\in\tilde{\mathcal{V}}} \|\tilde{v}\|} \right) \right\rceil$$

where $\tilde{\mathcal{V}}$ is defined as follows

$$\tilde{\mathcal{V}} = \{\tilde{v} \in \mathbb{R}^m \mid [u_1, \dots, u_p]^\top \tilde{v} \leq \frac{1}{2}[\|u_1\|^2, \dots, \|u_p\|^2]^\top\}$$

The optimization procedure given in (6) returns a control action in $\mathcal{U}_{\mathrm{A}}$. The switching controller is utilized with an appropriate duty cycle α^k and appropriate control action u_p to implement this control action. The parameter k and switching control input u_p have to be determined such that $u = \phi_{\mathrm{A}}(-y) = \alpha^k u_p$. The block diagram in Fig. 1 represents the closed-loop system.

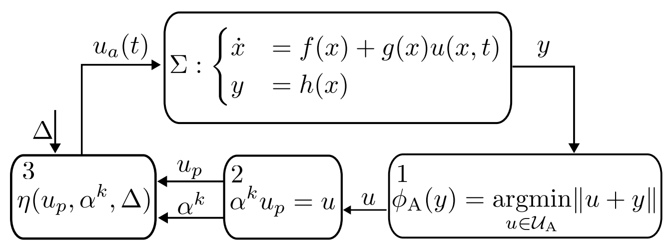

Fig. 1. Block diagram depiction of the closed-loop system under methodology A. Block-1 computes the desired control input from the effective control set $\mathcal{U}_{\mathrm{A}}$. Block-2 calculates the parameters u_p, α^k required for the switching controller. The switching control in Block-3 performs a PWM between the control inputs u_p and 0 at a α^k frequency. The output of the switching controller $u_a(t) \in \mathcal{U}$ is then given as feedback control to the system Σ.

B. Methodology B: Minimally Realizable Rays with Limited Length

Instead of the duty cycle taking discrete values $\alpha, \alpha^2, \dots$, we consider in this sub-section a continuum of values in $\alpha \in [0, 1]$. Thus the *effective control set* is given by

$$\mathcal{U}_{\mathrm{B}} = \bigcup_{i=1}^{p} \{\alpha u_i : \alpha \in [0,1], u_i \in \mathcal{U}\}. \tag{9}$$

The controller selects the control action closest to $-y$ from the set $\mathcal{U}_B$ and is defined as follows.

$$u = \phi_{\mathrm{B}}(-y) := \phi(-y, \mathcal{U}_{\mathrm{B}}) = \underset{u\in\mathcal{U}_{\mathrm{B}}}{\operatorname{argmin}}\|u + y\|. \tag{10}$$

Choosing the control input $u \in \mathcal{U}_{\mathrm{B}}$ is equivalent to choosing $\alpha \in [0, 1]$ and $u_p \in \mathcal{U}$ where $u = \alpha u_p$. The optimization problem (10) is converted from $u \in \mathcal{U}_{\mathrm{B}}$ to $\alpha \in [0, 1], u_p \in \mathcal{U}$. To ensure $\alpha \in [0, 1]$ define the following saturation function given in (11),

$$\mathrm{sat}_{[0,1]}(x) = \begin{cases} 1 & x \geq 1 \\ x & 0 < x < 1 \\ 0 & x \leq 0 \end{cases}. \tag{11}$$

The solution to the optimization problem (10) is as follows:

$$u_p = \begin{cases} \underset{u_i\in\,\mathcal{U}\setminus\{0\}}{\operatorname{argmin}} \left\| y + u_i\,\mathrm{sat}_{[0,1]}\left(\frac{\langle -y, u_i\rangle}{\|u_i\|}\right) \right\| & y \neq 0 \\ 0 & y = 0, \end{cases}$$

$$\alpha = \begin{cases} \mathrm{sat}_{[0,1]}\left(\frac{\langle -y, u_p\rangle}{\|u_p\|}\right) & y \neq 0 \\ 0 & y = 0. \end{cases}$$

Similar to *Lemma* 1, 2, and 3, we have the following lemma on the set $\mathcal{W}_{\mathrm{B}} = \phi_{\mathrm{B}}(-y)$.

Lemma 4: Consider the nearest action mapping ϕ_{B} in (10) and let $\mathcal{W}_{\mathrm{B}} = \phi_{\mathrm{B}}(-y) = \{w_i\}_{i\in\mathcal{I}} \subset \mathcal{U}_{\mathrm{B}}$ for some index set $\mathcal{I} \subset \{1, \dots, 2(m+1)\}$. Then the following statements hold

i $\mathrm{card}(\mathcal{W}_{\mathrm{B}}) \leq 2(m+1)$

ii $\mathcal{W}_{\mathrm{B}} = \{0\} \iff y = 0$

iii $-\|w_i\|\|y\| \leq \langle w_i, y\rangle \leq -\frac{1}{2}\|w_i\|^2 \qquad \forall i \in \mathcal{I}. \qquad (12)$

Proof: The proof of (i) follows similar arguments as given in *Lemma* 1. We can use prove by contradiction to show (ii) as before. Consider that $\mathcal{W}_{\mathrm{B}} = \{0, u\}$ and $y \neq 0$, then the point $\beta u \in \mathcal{U}_{\mathrm{B}}$ for any $0 < \beta < 1$ is closer to $-y$ than either 0 or u, which is a contradiction. Therefore we have $\mathcal{W} = \{0\} \iff y = 0$. Using arguments similar to the proof of *Lemma* 3, we can conclude (iii). ■

Proposition 3: Given an admissible control set $\mathcal{U}$ satisfying (A1), the control law $u = \phi_{\mathrm{B}}(-y)$ given by (10) asymptotically stabilizes the system Σ in (1) satisfying (A0) to the origin.

Proof: The set $\mathcal{W}_{\mathrm{B}} = \phi_{\mathrm{B}}(-y)$ satisfies the three properties in *Lemma* 4. In this case, we need to analyze two different cases:
(i) When $0 \notin \mathcal{W}_{\mathrm{B}}$, proceeding similar to the proof of *Proposition* 2 and computing the time derivative of the storage function along the closed-loop system trajectories, we obtain $\dot{H}(t) < 0$. Otherwise,
(ii) when $y = 0$ we have that $\mathcal{W}_{\mathrm{B}} = \{0\}$ by Lemma 4. It implies that $\dot{H}(t) = 0$. Using the same arguments of the storage function H being proper, H does not increase along the system trajectories, the application of La-Salle invariance principle and the large-time norm-observability of the system imply that the origin is globally asymptotically stable. ■

Similar to *Remark 1*, the following remark provides a lower bound on α if practical stability to $\mathbb{B}_\epsilon$ is sufficient.

Remark 2: If the allowed control actions are given by $\mathcal{U} = \{0, u_1 \dots, u_{m+1}\}$, for a given accuracy level $\epsilon > 0$, generate the *effective control set* given by

$$\mathcal{U}_{\mathrm{B},\epsilon} = \{\alpha u \mid u \in \mathcal{U},\ \alpha \in [\alpha^*_{\min}, 1]\}$$

where α^*_{min} is calculated as

$$\alpha^*_{\text{min}} = \frac{\gamma^{-1}(\epsilon)}{\max\limits_{\tilde{v}\in\tilde{\mathcal{V}}}\|\tilde{v}\|}$$

where $\tilde{\mathcal{V}}$ is defined as follows

$$\tilde{\mathcal{V}} = \{\tilde{v} \in \mathbb{R}^m \mid [u_1, \dots, u_{m+1}]^\top \tilde{v} \le \frac{1}{2}[\|u_1\|^2, \dots, \|u_{m+1}\|^2]^\top\}.$$

Then the controller defined by

$$u = \phi(-y, \mathcal{U}_{\text{B},\epsilon})$$

practically stabilizes the system Σ given in (1) to $\mathbb{B}_\epsilon$.

The closed-loop block diagram of the system Σ under the control methodology B is given in Fig. 2

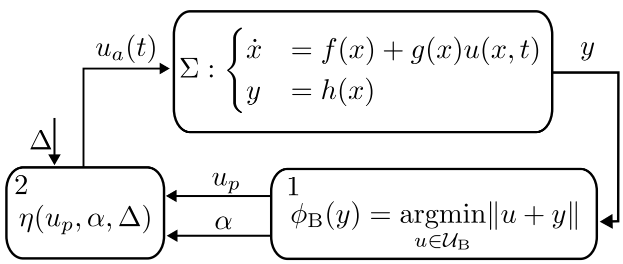

Fig. 2. Block diagram depiction of the closed-loop system under methodology A. Block-1 computes the desired control input from the effective control set $\mathcal{U}_{\text{A}}$. Block-2 calculates the parameters u_p, α^k required for the switching controller. The switching control in Block-3 performs a PWM between the control inputs u_p and 0 at a α^k frequency. The output of the switching controller $u_a(t) \in \mathcal{U}$ is then given as feedback control to the system Σ.

C. Methodology C: Stabilization Using Solid Simplices

The previous two methodologies generated finer control actions by switching between the control inputs u and 0. In this subsection, we present another methodology where the controller switches between all the control actions in the set $\mathcal{U}\backslash\{0\}$. This can be thought of as a PWM between $m+1$ control actions $\{u_1, \dots, u_{m+1}\}$. Define the vector-valued duty cycle $\alpha = [\alpha_1, \dots, \alpha_{m+1}]^\top \in [0,1]^{m+1}$ and $\sum_{i=1}^{m+1} \alpha_i = 1$. The $m+1$-switching controller is then defined as follows

$$\eta(\alpha, \Delta, t) = \begin{cases} u_1 & 0 \le \{t/\Delta\} < \Delta\alpha_1 \\ u_2 & \Delta\alpha_1 \le \{t/\Delta\} \le \Delta(\alpha_1 + \alpha_2) \\ \vdots & \vdots \\ u_{m+1} & \Delta \sum\limits_{i=1}^{m} \alpha_i < \{t/\Delta\} \le \Delta. \end{cases} \tag{13}$$

Analyzing this through the framework of relaxed control input as $\Delta \to 0$, the control input can be formulated as sampled from a probability measure-valued function μ_α (which depends on the vector-valued duty cycle α). The domain of μ_α is the input space $\text{span}\{u_1, \dots, u_{m+1}\}$ and it is defined by $\mu_\alpha(E) = \int_E r_\alpha(\tau)d\tau$ for all $E \subset \text{span}\{u_1, \dots, u_{m+1}\}$, where

$$r_\alpha = \sum_{i=1}^{m+1} \alpha_i \delta_{u_i}$$

for all $\alpha_i \in [0,1]$. Converting the input to (1) from u to α, computing we get

$$\Sigma_{\text{rel}} : \begin{cases} \dot{x} &= f(x) + g(x) \sum\limits_{i=1}^{m+1} \alpha_i u_i, \quad \sum\limits_{i=1}^{m+1} \alpha_i = 1 \\ y &= h(x). \end{cases}$$

It can be observed that the relaxed control input is any convex combination of the allowed control inputs. The equivalent allowed control input set is as follows

$$\mathcal{U}_{\text{C}} = \left\{ \sum_{i=1}^{m+1} \alpha_i u_i \;\middle|\; u_i \in \mathcal{U} \setminus \{0\},\ \sum_{i=1}^{m+1} \alpha_i = 1 \right\}$$
$$= \text{conv}(u_1, \dots, u_{m+1})$$

The element in the set U_{C} closest to $-y$ is chosen as the control input, and the controller is formulated as an optimization problem,

$$u = \phi_{\text{C}}(-y) := \phi(-y, \mathcal{U}_{\text{C}}) = \underset{u\,\in\,\mathcal{U}_{\text{C}}}{\text{argmin}}\|u + y\|. \tag{14}$$

The solution to this optimization problem is the projection of the point $-y$ on the compact convex set $\mathcal{U}_{\text{C}}$. If $-y \in \mathcal{U}_{\text{C}}$, then its projection is $-y$ itself. In order to generate the control input $-y$, the vector-valued duty cycle α is calculated as follows,

$$\sum_{i=1}^{m+1} \alpha_i u_i = -y \ \text{ subject to } \ \sum_{i=1}^{m+1} \alpha_i = 1$$

which yields

$$u_{m+1} + \sum_{i=1}^{m} \alpha_i (u_i - u_{m+1}) = -y$$
$$\begin{bmatrix}(u_1 - u_{m+1}) & \dots & (u_m - u_{m+1})\end{bmatrix} \alpha = -(y + u_{m+1})$$
$$\mathbb{U}\alpha = -(y + u_{m+1}).$$

As $0 \in \text{int}(\text{conv}(u_1, \dots, u_{m+1}))$ from (A1), the m vectors $(u_1 - u_m), \dots, (u_m - u_{m+1}) \in \mathbb{R}^m$ are linearly independent and span the m-dimensional space $\mathbb{R}^m$, thus the matrix $\mathbb{U} \in \mathbb{R}^{m\times m}$ is invertible. Therefore, the vector duty cycle α is given by,

$$\alpha = -\mathbb{U}^{-1}(y + u_{m+1}).$$

Define the projection operator of a point $d \in \mathbb{R}^n$ on a convex set $\mathcal{C} \subset \mathbb{R}^n$ as follows

$$\mathbb{P}(d,\, \mathcal{C}) = \underset{x\in\mathcal{C}}{\text{argmin}}\|x - d\|$$

If $-y$ lies outside the set $\mathcal{U}_{\text{C}}$, then $\mathbb{P}(-y,\, \mathcal{U}_{\text{C}})$ solves the optimization problem (14). The control input is given as

$$u(y) = \begin{cases} -y & y \in \text{conv}(u_1, \dots, u_{m+1}) \\ \mathbb{P}(-y,\, \mathcal{U}_{\text{C}}) & \text{otherwise} \end{cases}$$

and the vector duty cycle is given by $\alpha(y(t)) = \mathbb{U}^{-1}u(y)$.

Lemma 5: Consider the nearest-action mapping $\phi_{\rm C}$ in (14) and let $\mathcal{W}_{\rm C} = \phi_{\rm C}(-y) = \{w_i\}_{i\in\mathcal{I}} \subset \mathcal{U}_{\rm B}$ for some index set $\mathcal{I} \subset \{1,\ldots,2(m+1)\}$. Then the following statements hold

i $\operatorname{card}(\mathcal{W}_{\rm C}) = 1$

ii $\mathcal{W}_{\rm C} = \{0\} \iff y = 0$

iii if $\mathcal{W}_{\rm C} = w_1 \in \mathcal{U}_{\rm C}$, then

$$-\|w_1\|\|y\| \le \langle w_1, y\rangle \le -\frac{1}{2}\|w_1\|^2 \tag{15}$$

is satisfied.

Proof: The projection operation of a point on a compact convex set is unique; therefore, $\mathcal{W}_{\rm C}$ is a singleton. $\mathcal{W} = -y$ if $y \in \operatorname{conv}(u_1,\ldots,u_{m+1})$ and from the assumption (A0), statement (ii) can be concluded. The point w_1 is closer to $-y$ than to zero, giving $\langle w_1 + y\rangle < \langle y, y\rangle$, which, on further simplification, statement (iii) can be obtained. ■

Proposition 4: Given an admissible control set $\mathcal{U}$ satisfying (A1), the control law $u = \phi_{\rm C}(-y)$ given by (14) asymptotically stabilizes the system Σ in (1) satisfying (A0) to the origin.

Proof: From *Lemma* 5, there are two cases:
(i) For $y \notin \operatorname{conv}(\mathcal{U})$, $\mathcal{W}_{\rm C} = \phi_{\rm C}(-y) = \mathbb{P}(-y, \mathcal{U}_{\rm C}) = w$, is a singleton set according to *Lemma* 5. The closed-loop system is then given by,

$$\begin{aligned}\dot{x} &= f(x) + g(x)w\\ \dot{y} &= h(x).\end{aligned} \tag{16}$$

Using the storage function of the original system (1), its time derivative along the system trajectories to (16),

$$\dot{H} = -\langle w, y\rangle \le -\|w\|^2 < 0.$$

(ii) For $y \in \operatorname{conv}(\mathcal{U})$, we have $\mathcal{W}_{\rm C} = \phi_{\rm C}(-y) = -y$. In this case, the closed-loop system is smooth and given by

$$\begin{aligned}\dot{x} &= f(x) - g(x)y\\ y &= h(x).\end{aligned} \tag{17}$$

Using the storage function of the original system (1), its derivative along the solutions of (17), we have

$$\dot{H}(x) = -\|y\|^2 \le 0.$$

As $H(x(t))$ is non-increasing and since H is proper, all solutions of $x(t)$ are bounded. By the LaSalle invariance principle, all of such trajectories converge to the largest invariant set $M \in \mathbb{R}^n$ where $h(M) \subset Z$, and $Z := \{y \in \mathbb{R}^m \mid 0 = \phi(y)\}$. $0 \in \phi(y) \iff y = 0$, yielding $M = Z = \{0\}$. Using the large-time norm-observability of $\Sigma|_{u=0}$, it follows that,

$$\lim_{t\to\infty} \|x(t)\| = 0.$$

■

Similar to *Remark 1*, here, a lower bound on the α_i is provided if practical stability is sufficient.

Remark 3: If the allowed control actions are given as $\mathcal{U} = \{0, u_1 \ldots, u_{m+1}\}$, for a given accuracy level $\epsilon > 0$, generate the *effective control set* given by

$$\mathcal{U}_{{\rm C},\epsilon} = \mathcal{U}_{\rm C} \setminus \operatorname{int}(\alpha^*_{\rm min}\mathcal{U}_{\rm C}),$$

where $\alpha^*_{\rm min}$ is calculated as

$$\alpha^*_{\rm min} = \frac{\gamma^{-1}(\epsilon)}{\max\limits_{\tilde{v}\in\tilde{\mathcal{V}}}\|\tilde{v}\|},$$

and $\tilde{\mathcal{V}}$ be defined by

$$\begin{aligned}\tilde{\mathcal{V}} = \{\tilde{v} \in \mathbb{R}^m \mid [u_1,\ldots,u_{m+1}]^\top \tilde{v}\\ \le \frac{1}{2}[\|u_1\|^2,\ldots,\|u_{m+1}\|^2]^\top\}.\end{aligned}$$

Then the controller defined by

$$u = \phi(-y, \mathcal{U}_{{\rm C},\epsilon})$$

practically stabilizes the system Σ given in (1) to $\mathbb{B}_\epsilon$.

IV. Example and Simulation Results

In this section, we will apply our main results to an example and illustrate the behaviour of the closed-loop system through a numerical simulation. The following example is borrowed from [16]. Consider the following nonlinear system

$$\Sigma_{\rm ex}: \begin{cases} \dot{x} &= \begin{bmatrix} -x_2 + x_3^2 \\ x_1 + x_3^2 \\ -x_1x_3 - x_2x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u \\ y &= \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}, \end{cases} \tag{18}$$

where $x := [x_1, x_2, x_3]^\top \in \mathbb{R}^3$ and $y := [y_1, y_2]^\top, u := [u_1, u_2]^\top \in \mathbb{R}^2$. It has been shown in [16] that (18) is passive with a storage function $H(x) = \frac{1}{2}x^\top x$, and it is long-time norm observable with the function $\gamma(s) = 4(s + s^2)$ in (2). For the asymptotic stabilization of the system (18), we choose the admissible control set to be $\mathcal{U}_{\rm ex}$ given in

$$\begin{aligned}\mathcal{U}_{\rm ex} :=& \left\{0, \begin{bmatrix}\sin(0)\\ \cos(0)\end{bmatrix}, \begin{bmatrix}\sin(\frac{2\pi}{3})\\ \cos(\frac{2\pi}{3})\end{bmatrix}, \begin{bmatrix}\sin(\frac{4\pi}{3})\\ \cos(\frac{4\pi}{3})\end{bmatrix}\right\}\\ &= \{u_0,\ u_{{\rm ex},1},\ u_{{\rm ex},2},\ u_{{\rm ex},3},\}.\end{aligned}$$

It can been seen at $\tilde{\mathcal{V}} = \{u_{{\rm ex},1},\ u_{{\rm ex},2},\ u_{{\rm ex},3},\} \subset \mathcal{U}_{\rm ex}$ satisfies the assumption (A1). For methodology A, $\alpha = 0.2$ is chosen. The system starts from an initial condition of $x(0) = [-3, 3, 2]^\top$, and an Euler forward discretization method is used for numerical simulations. The numerical simulation results are summarized in Fig. 3. In this figure, the trajectories of the closed-loop system using Methodologies A, B and C converge all to zero as expected. As a comparison, the trajectory of standard NAC as given in [17] is shown in solid purple line. For all methods, it can be seen that the convergence rate at the start is linear due to the use of ordinary control action taken from $\mathcal{U}$. Once the feedback control enters the convex hull of $\mathcal{U}$, the use of measure-valued control input μ_α is introduced, leading to exponential convergence for all three methods A, B and C.

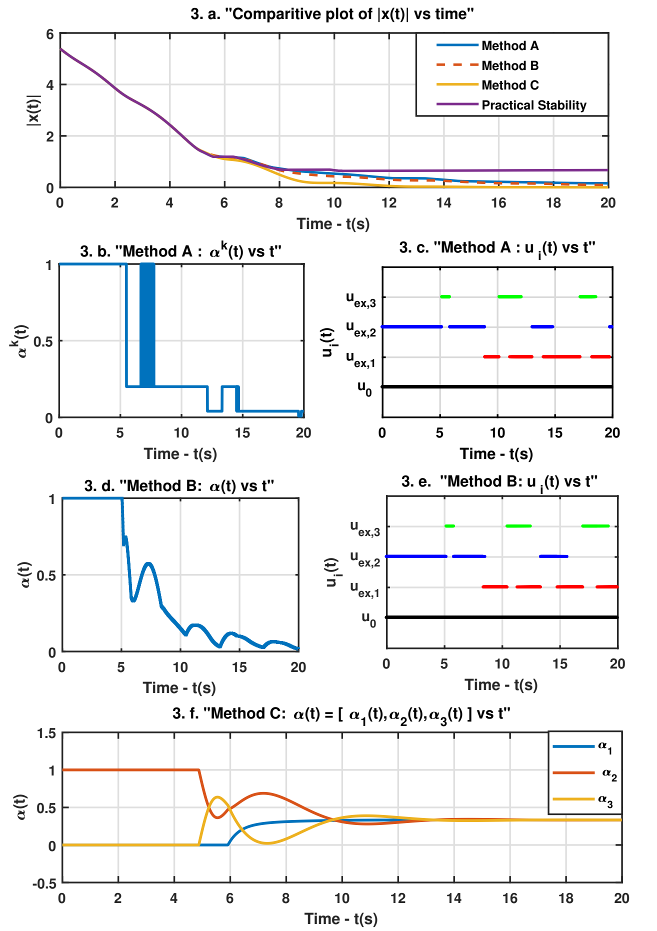

Fig. 3. This figure summarises the numerical simulations performed to validate the proposed control laws. (i) The evolution of the norm of the state ($\|x(t)\|$) is depicted in 3. a. As expected, the proposed control laws ensure asymptotic stability, whereas the previous method in [16] ensures practical stability. (ii) 3. b. shows the evolution of α^k vs t, and 3. c. shows the control input selection. At each instant, the switching controller switches between the control input $u_i(t)$ and 0 at a switching frequency of $\alpha^k(t)$ given in Section III.*A*. (iii) 3. d., shows the evolution of α vs t, and 3 .e., shows the control input selection. At each instant, the switching controller switches between the control input $u_i(t)$ and 0 at a switching frequency of $\alpha(t)$ as given in Section III.*B*. (iii) 3. f. depicts the evolution of the vector-valued PWM signal, and the controller switches between the non-zero control actions at this frequency, as mentioned in Section III.*C*.

V. Conclusions and Further Research

This paper considers the asymptotic stabilization of a continuous-time passive nonlinear system under observability assumptions, where the control inputs are chosen from a finite set. This is achieved using switching controllers and analysed using the theory of relaxed control systems. We are currently investigating the effects of a minimum dwell time requirement on the controller, thus ensuring the controller applies any control for a minimum time. Another direction is on the set-point regulation for a class of constant incremental passive nonlinear systems as in [24] via a PI controller.

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