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# Decentralized PI-control and Anti-windup in Resource Sharing Networks

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***Abstract*— We consider control of multiple stable first-order systems which have a control coupling described by an M-matrix. These agents are subject to incremental sector-bounded nonlinearities. We show that such plants can be globally asymptotically stabilized to a unique equilibrium using fully decentralized proportional integral anti-windup-equipped controllers subject to local tuning rules. In addition, we show that when the nonlinearities correspond to the saturation function, the closed-loop asymptotically minimizes a weighted 1-norm of the agents state mismatch. The control strategy is finally compared to other state-of-the-art controllers on a numerical district heating example.**

***Index Terms*— Energy Systems, Constrained Control, Decentralized Control**

## I. Introduction

In this paper we consider the control of agents sharing a central distribution system with limited capacity as in [1]. We investigate systems where the positive action of one agent negatively impacts others. This type of competitive structure can arise in many domains, for instance internet congestion control [2], [3] and district heating systems [4]. In the district heating scenario, the structure arises because of the hydraulic constraints of the grid. If one agent (building) locally decides to increase their heat demand by opening their control valves, this will lead to higher flow rates and greater frictional pressure losses. These losses make it so that other agents now receive lower flow rates [4]. We consider a simple description of such systems as

$$\dot{x} = -Ax + Bf(u) + w. \tag{1}$$

Here each agent has a state $x_i$, and these states are gathered in the vector $x$. The agents are subject to an external disturbance $w$ and interconnected via the matrix $B$. The nonlinear function $f(\cdot)$ can for instance represent the common phenomenon of input saturation, which motivates this work. $A$ is assumed diagonal. We will more formally describe the plant in Section II.

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In multi-agent systems such as (1), decentralized controllers are desirable. Semi-decentralized control strategies for multi-agent systems subject to input saturation have been considered in the following works. In [5], each networked agent is equipped with a local controller that receives the control input of its neighbors. In [6], semi-decentralized anti-windup was considered for stable SISO plants that are decentralized in the linear domain, but become coupled during saturation. This is demonstrated on unmanned aerial vehicles. These works focus on stabilization when the disturbance $w$ in plant (1) is energy bounded. In this work we focus instead on the asymptotic properties of plant (1), which become important when $w$ is approximately constant. Previous works considering asymptotic optimality for plants of the form (1) are [7] and [1]. In [1], it was shown that, when $B$ is an M-matrix, decentralized PI-controllers with a rank-one coordinating anti-windup scheme can minimize the cost $\max_i |x_i|$. In [7], it was shown that the static controller $u = -B^\top x$ asymptotically minimizes the cost $x^\top Ax + v^\top v$ where $v = \text{sat}(u)$. This result also extends to the case when $B$ is not an M-matrix. Both of these control strategies maintain certain scalability properties: With $u = -B^\top x$ [7], any sparsity structure in the $B$-matrix is maintained and the rank-one coordination scheme of [1] admits scalable implementations. However, the most scalable control solution is one that is fully decentralized. In this work, we analyze (1) under a fully decentralized PI (proportional-integral) control strategy. In general, it is non-trivial that decentralized PI-controllers are stabilizing, let alone fulfill any optimality criterion. However, we will not only show that our strategy minimizes asymptotic costs of the form $\sum_{i=1}^n \gamma_i |x_i|$, but also that the resulting equilibrium is globally asymptotically stable under decentralized controller tuning rules.

The paper is organized as follows. Section II presents the considered plant and control strategy. Section III presents the main results of the paper, namely equilibrium existence and uniqueness, global asymptotic stability, and equilibrium optimality for our considered closed-loop. A motivating numerical example consisting in the flow control of a simplified district-heating network is subsequently given in section IV. The proofs of the main results are presented in sections V, VI, and VII respectively. Conclusions and future work are covered in section VIII.

*Notation:* $v_i$ denotes element $i$ of vector $v \in \mathbb{R}^n$, $A_i$ denotes row $i$ of matrix $A \in \mathbb{R}^{n\times m}$, and $A_{i,j}$ denotes its $(i, j)$-th element. A matrix $A$ is strictly diagonally row-dominant if $|A_{i,i}| > \sum_{j\neq i} |A_{i,j}|$ for all $i$. $A$ is strictly diagonally column-dominant if $A^\top$, denoting the transpose of $A$, is strictly diagonally row-dominant. Let the 2-norm of a vector $x \in \mathbb{R}^n$ be given by $\|x\|_2 = (\sum_{i=1}^n x_i^2)^{1/2}$. Let the 1-and-infinity-norms of a vector $x \in \mathbb{R}^n$ be given by $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_\infty = \max_i |x_i|$ respectively. Let the norm $\|A\|_2$ of a matrix $A$ be the induced 2-norm. Let $\mathbf{1} \in \mathbb{R}^n$ be a vector of all ones, where $n$ is taken in context. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is increasing (non-decreasing) if $y > x$ implies that $f(y) > f(x)$ ($f(y) \geq f(x)$).

## II. Problem Data and Proposed Controller

We consider control of plants of the form (1) where vector $x \in \mathbb{R}^n$ gathers the states $x_i$ of each agent, $A \in \mathbb{R}^{n\times n}$, and $w \in \mathbb{R}^n$ is a constant disturbance acting on the plant. $B \in \mathbb{R}^{n\times n}$ couples the control-inputs of the agents. The input nonlinearity $f : \mathbb{R}^n \to \mathbb{R}^n$ is characterized by the following assumption.

*Assumption 1:* $f(x) = [f_1(x_1), f_2(x_2), \dots, f_n(x_n)]^\top$ has components $f_i$ satisfying $f_i(0) = 0$ and incrementally sector-bounded in the sector $[0, 1]$, namely satisfying $0 \leq (f_i(y) - f_i(x)) / (y - x) \leq 1$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$, $x \neq y$.
Note that Assumption 1 implies that $f$ is non-decreasing and Lipschitz with Lipschitz constant 1. Since $f(0) = 0$, $f$ also enjoys a sector $[0, 1]$ condition.

Stability properties for feedback with incrementally sector-bounded nonlinearities has long been considered in the literature. As far back as [8] it was used for input-output stability analysis. Both [9] and [10] consider the type of diagonally partitioned incrementally sector-bounded functions that we consider here, whereas [11]–[13] consider a richer class of incremental sector-bound constraints of the form $(f(x) - f(y) - S_1(x - y))^\top (f(x) - f(y) - S_2(x - y)) \leq 0$ for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ where $S_1$ and $S_2$ are real symmetric matrices with $0 \preceq S_1 \prec S_2$.

We will consider function pairs $f(\cdot)$, $h(\cdot)$ where $f(x) + h(x) = x$. These pairs fulfill the following property, the proof of which is in the appendix.

*Lemma 1:* Let $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfy Assumption 1. Then $h(u) = u - f(u)$ also satisfies Assumption 1.
The considered class of function pairs is well motivated by the common case $f(x) = \mathrm{sat}(x)$ where $\mathrm{sat}(x) = \max(\min(x, \mathbf{1}), -\mathbf{1})$ and $h(x) = \mathrm{dz}(x) = x - \mathrm{sat}(x)$.

We propose controlling the plant (1) with fully decentralized PI controllers having decentralized anti-windup for each agent $i = 1, \dots, n$.

$$\dot{z}_i = x_i + s_i h_i(u_i) \tag{2}$$

$$u_i = -p_i x_i - r_i z_i \tag{3}$$

where $z_i$ is the integral state, $u_i$ is the controller output, $p_i > 0$ and $r_i > 0$ are proportional and integral controller gains respectively, $s_i > 0$ is an anti-windup gain, and $h(u) = u - f(u)$ is an anti-windup signal. Note that while the notation $h$ is not needed (indeed we could equivalently replace $h(u)$ with $u - f(u)$), we will use the pair $f$, $h$ both to simplify the exposition and to highlight that $f$ is the nonlinearity acting on the plant while $h$ is the nonlinearity acting on the controller. We assume that the closed-loop system satisfies the following assumption.

*Assumption 2:* $A$ is a diagonal positive definite matrix, $B$ is an M-matrix, and $w$ is a constant disturbance. The controller parameters $p_i$, $r_i$, and $s_i$, for $i = 1, \dots, n$, are all positive.
An M-matrix $B$ as in [14] has positive diagonal elements and non-positive off-diagonal elements. The following statements for such a matrix are equivalent: $(i)$ $B$ is an M-matrix. $(ii)$ There is a diagonal positive definite matrix $Q$ such that $QB + B^\top Q \succ 0$. $(iii)$ There is a diagonal positive definite matrix $U$ such that $UB$ and $UBU^{-1}$ are strictly column-diagonally dominant. $(iv)$ $DB$ is an M-matrix for any positive definite diagonal matrix $D$. For a more extensive list of equivalent statements, refer to [14].

## III. Main Results

In this section we will cover the main results of this paper. In particular, we will consider the proposed control law (2)–(3) for the plant (1). We will show that this closed-loop system admits an equilibrium for any constant disturbance $w$. We will additionally show that this equilibrium is globally asymptotically stable and enjoys a notion of optimality. We will leave the proofs for Sections V to VII.

Let us first consider the existence of an equilibrium, which corresponds to well-posedness of the equations (1)–(3) with $\dot{x} = \dot{z} = 0$.

*Theorem 1:* (Equilibrium Existence and Uniqueness) Let $f$ satisfy Assumption 1 and let Assumption 2 hold. Then for each constant $w \in \mathbb{R}^n$, closed-loop (1)–(3) has a unique equilibrium ($x^0$, $z^0$), inducing input $u^0$ from (3), which satisfies (1)–(3) with $\dot{x} = \dot{z} = 0$.

In addition to the existence of the unique equilibrium ($x^0$, $z^0$), we can also show that it is globally asymptotically stable under the following assumption on the control parameters.

*Assumption 3:* Assume that $a_i p_i > r_i$ and $p_i s_i < 1$ for all $i$, where $a_i$ are the diagonal elements of $A$ in (1) and $p_i$, $r_i$, and $s_i$ are the controller gains in (2)–(3).

*Theorem 2:* (Global Asymptotic Stability) Let $f$ satisfy Assumption 1 and let $f(u) + h(u) = u$. Let Assumptions 2 and 3 hold. Then there is a globally asymptotically stable equilibrium for the closed-loop (1)–(3).

*Remark 1:* The tuning rules of Assumption 3 are fully decentralized. Each agent $i$ can tune their own controller gains to satisfy $r_i < a_i p_i$ and $s_i < 1/p_i$.
Let us now focus on the case where the function pair $f(\cdot)$ and $h(\cdot)$ are given by the pair $\mathrm{sat}(\cdot)$ and $\mathrm{dz}(\cdot)$ respectively, motivated by classical anti-windup for saturating controllers. Let $\gamma_i$ be positive scalar weights, and consider the problem of minimizing the weighted sum of all state errors $\sum_{i=1}^n \gamma_i |x_i|$. We can define this problem through the optimization problem

$$\underset{x,\,v}{\text{minimize}} \quad \sum_{i=1}^n \gamma_i |x_i| = \|\Gamma x\|_1 \tag{4a}$$

$$\text{subject to} \quad -Ax + Bv + w = 0, \tag{4b}$$

$$-\mathbf{1} \leq v \leq \mathbf{1}. \tag{4c}$$

where $\Gamma = \mathrm{diag}\{\gamma_1, \dots, \gamma_n\}$. The inequalities (4c) are considered componentwise. This problem can be motivated by a district heating example. Let $w$ be the outdoor temperature, $x_i$ be the deviation from the comfort temperature for each agent $i$, and let $Bv$ denote the heat provided to the agents, limited by (4c). Then if $\Gamma = I$, this corresponds to minimizing the total discomfort experienced by all agents. One could consider $\gamma_i$ to be a cost describing the severity of agent $i$ deviating from the comfort temperature, where $\gamma_i$ would be high for e.g. a hospital. Note that this cost does not capture the notion of *fairness* as considered in [1]. For instance, with $\Gamma = I$, $x = [n, 0, \dots, 0]^\top$, and $y = [1, 1, \dots, 1]^\top$ we achieve the same costs $\|\Gamma x\|_1 = \|\Gamma y\|_1$. With the problem (4) defined, the following holds.

*Theorem 3:* Let Assumption 2 hold and let $\Gamma A^{-1}B$ be a strictly diagonally column-dominant M-matrix. Let $f(u) = \text{sat}(u)$ and $h(u) = \text{dz}(u) = u - \text{sat}(u)$. Let $(x^0, z^0)$, be an equilibrium for the closed-loop system in (1)–(3), associated with input $u^0$. Then $x^* = x^0$ and $v^* = f(u^0)$ solves (4).

*Remark 2:* Since $\Gamma A^{-1}B$ must be strictly diagonally column-dominant, the weights $\gamma_i$ cannot be chosen arbitrarily. However, as $B$ is an M-matrix, there will be at least one set of weights $\gamma_i$ such that this condition is satisfied.

## IV. Numerical Example

This motivating example compares three different control strategies on a simplified, linear model of 10 buildings connected in a district heating grid. The compared strategies are the same as the ones considered in [1]. Each building $i$ has identical thermodynamics on the form

$$\dot{x}_i = -\frac{a_i}{C_i}(x_c + x_i - T_{\text{ext}}(t)) + \frac{1}{C_i}\dot{Q}_i(u), \tag{5}$$

where $x_i$ denotes agent $i$'s indoor temperature deviation from the comfort temperature $x_c$, $C_i$ is the heat capacity of each building and $T_{\text{ext}}$ is the outdoor temperature. Here the assumption of a constant disturbance $w$ is replaced with the slowly time-varying disturbance $T_{\text{ext}}$. $\dot{Q}_i$ is the heat supplied to building $i$. This heat supply is given by

$$\dot{Q} = B\text{sat}(u), \tag{6}$$

where $B$ represents the network interconnection. The simulation was conducted with $a_i = 0.167\,[\text{kW/C}^\circ]$, $C_i = 2.0\,[\text{kWh/C}^\circ]$, $p_i = 2.5\,[1/\text{C}^\circ]$, $r_i = 0.2\,[1/\text{C}^\circ\text{h}]$, and $s_i = 2.0\,[\text{C}^\circ]$ for all $i$. The parameters $a_i$, $C_i$ are chosen close to the values found in [15] which discusses parameter estimation for a single-family building. Matrix $B$ is selected as $B_{i,i} = 12\ \forall i$, $B_{i,j} = 0.15\text{min}(i,j)\ \forall i \neq j$ in units [kW]. Matrix $B$ is constructed such that fully opened control valves ($\text{sat}(u) = \mathbf{1}$) gives $\dot{Q}$ representing a reasonable peak heat demand for small houses. In this scenario, $\dot{Q}_i$ is high for buildings with $i$ small (close to the production facility). We simulate the system using the `DifferentialEquations` toolbox in Julia [16], for an outdoor temperature scenario given by data from the city of Gävle, Sweden in October 2022 during which the temperature periodically drops to almost -20°C. The data is gathered from the Swedish Meteorological and Hydrological Institute (SMHI). We compare three different controllers and three different cost functions. The first controller is the fully *decentralized* PI-controller considered in this paper. Secondly the *coordinating* controller consists of the same PI-controllers as the decentralized case, but with the coordinating rank-1 anti-windup signal $\dot{z}_i = x_i + \beta \mathbf{1}^\top \text{dz}(u)$ considered in [1]. Finally, the *static* controller is given by $u = -B^\top C^{-1} x$ as considered in [7], where $C$ is the diagonal matrix of all heat capacities $C_i$.

Figure 1 shows the resulting deviations $x$ during the simulations. At around hour 100, the outdoor temperature is critically low. At this time, the buildings do not receive sufficient heat, regardless of the control strategy. Figure 1a shows that with the decentralized strategy, the worst deviations become larger than with the coordinating strategy (Figure 1b). However, not all buildings experience temperature deviations, whereas with the coordinating strategy, all the buildings share the discomfort. Lastly, the static controller has large deviations experienced by many buildings. Even when the outdoor temperature is manageable, the static controller has a constant offset from the comfort temperature, highlighting the usefulness of the integral action. We evaluate the performance through the cost functions

$$J_1 = \frac{1}{T}\int_0^T \|x(t)\|_1 dt, \tag{7}$$

$$J_\infty = \frac{1}{T}\int_0^T \|x(t)\|_\infty dt, \tag{8}$$

$$J_2 = \frac{1}{T}\int_0^T x(t)^\top L x(t) + \text{sat}(u(t))^\top \text{sat}(u(t))\, dt. \tag{9}$$

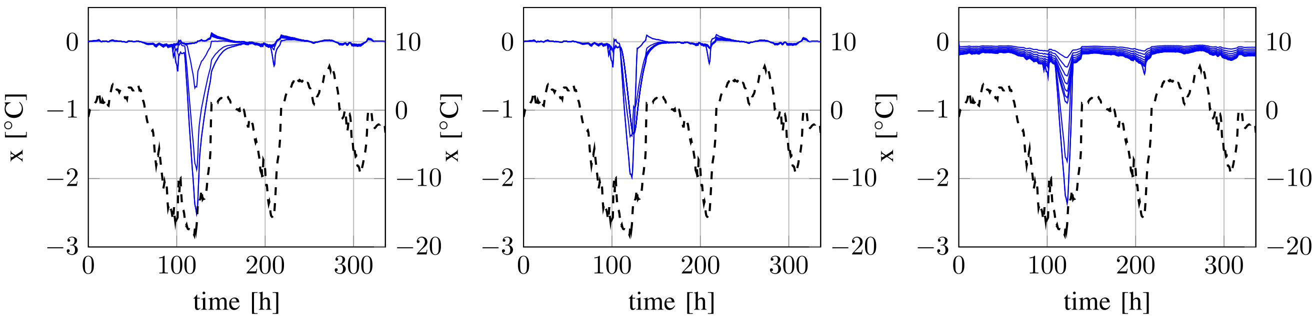
where $T$ is the simulation time and $L$ is a diagonal matrix where each element is given by $l_i = \frac{q_i}{C_i}$. The cost $J_1$ mimics the optimality notion considered in this paper, $J_\infty$ mimics the optimality notion considered in [1], and $J_2$ mimics the optimality considered in [7]. Table I shows the resulting evaluations. The coordinating controller gives minimal worst-case deviations $J_\infty$, but $J_1$ is minimized in the decentralized strategy. This result, i.e. that the total discomfort is minimized by decentralized control but the worst-case discomfort is minimized by coordination, is found also in [4] where a nonlinear model of the grid hydraulics and a 2-state model of building dynamics is employed. On the weighted cost $J_2$, all controllers provide similar performance. The static controller slightly outperforms the other two in this scenario, but it is outperformed in every other measure.

## V. Proof of Equilibrium Existence and Uniqueness

We will now prove Theorem 1 through the use of Banach's fixed-point theorem [17]. This proof requires the following two lemmas, the proofs of which are found in the appendix.

TABLE I: Costs (7)–(9) evaluated over the simulation.

| | Decentralized | Coordinating | Static |
|---|---|---|---|
| $J_\infty$ | 0.17 | 0.13 | 0.28 |
| $J_1$ | 0.67 | 0.9 | 1.96 |
| $J_2$ | 3.52 | 3.52 | 3.49 |




(a) Decentralized controller.

(b) Coordinating controller.

(c) Static controller.

Fig. 1: Temperature deviations $x$ (blue, left axis) for each strategy and the outdoor temperature $w$ (black, dotted, right axis). Around hour 100, $w$ becomes critically low and the indoor temperatures drop as the controllers saturate.

*Lemma 2:* Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^n$ where $h(x) = x - f(x)$ satisfy Assumption 1. Then $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ and $\tilde{h} : \mathbb{R}^n \to \mathbb{R}^n$ given by $\tilde{f}(x) = f(x + x^0) - f(x^0)$ and $\tilde{h}(x) = h(x + x^0) - h(x^0)$ for some $x^0 \in \mathbb{R}^n$ also satisfy Assumption 1 and $\tilde{h}(x) + \tilde{f}(x) = x$.

*Lemma 3:* Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^n$ where $h(x) = x - f(x)$ satisfy Assumption 1. Then $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ and $\tilde{h} : \mathbb{R}^n \to \mathbb{R}^n$ given by $\tilde{f}(x) = Df(D^{-1}x)$ and $\tilde{h}(x) = Dh(D^{-1}x)$ where $D$ is a diagonal positive definite matrix also satisfy Assumption 1 and $\tilde{h}(x) + \tilde{f}(x) = x$.

*Proof:* (of Theorem 1) Denote by $S$ a diagonal matrix gathering the positive anti-windup gains $s_i$, $i = 1, \dots n$. We can rearrange (1)–(3) by imposing $\dot{x} = \dot{z} = 0$, which yields

$$0 = h(u^0) + S^{-1}A^{-1}Bf(u^0) + S^{-1}A^{-1}w. \qquad (10)$$

If there is a unique $u^0$ solving (10) then $x^0 = A^{-1}\left(Bf(u^0) + w\right)$ and $z^0 = R^{-1}(-Px^0 - u^0)$ are uniquely determined by (1) and (3) respectively, where $R = \text{diag}\{r_1, \dots, r_n\}$ is invertible by Assumption 3. Hence we need only show that there is a unique $u^0$ solving (10) for the proof to be complete. Let $D$ be a diagonal positive definite matrix such that $DS^{-1}A^{-1}BD^{-1}$ is strictly diagonally column-dominant. Note that such a $D$ always exists because $A$ and $S$ are diagonal positive definite and $B$ is an M-matrix. Left-multiply (10) by $D$ and insert multiplication by $I = D^{-1}D$ before $f(u^0)$ to obtain

$$0 = Dh(u^0) + DS^{-1}A^{-1}BD^{-1}Df(u^0) + DS^{-1}A^{-1}w. \qquad (11)$$

Introduce the change of variables $\hat{B} = DS^{-1}A^{-1}BD^{-1}$, $\zeta = Du^0$, and $\hat{w} = DS^{-1}A^{-1}w$. Then (11) yields

$$0 = Dh(D^{-1}\zeta) + \hat{B}Df(D^{-1}\zeta) + \hat{w}. \qquad (12)$$

Here we can use Lemma 3 to replace $f(\cdot)$, $h(\cdot)$ with $\hat{f}(\cdot)$, $\hat{h}(\cdot)$, which satisfy Assumption 1 and $\hat{f}(\zeta) + \hat{h}(\zeta) = \zeta$. Introduce a scalar $k$ satisfying $k > \max(1, 2\max_i \hat{B}_{i,i})$. Divide (12) by $-k$, add $\zeta$ to the left-hand side, and $\zeta = \hat{f}(\zeta) + \hat{h}(\zeta)$ to the right-hand side of (12) to obtain

$$\zeta = -\frac{1}{k}\left((1-k)\hat{h}(\zeta) + (\hat{B} - kI)\hat{f}(\zeta) + \hat{w}\right). \qquad (13)$$

We define the right-hand side of this expression as $T_w(\zeta)$, defined for a specific $w$. By showing that $T_w$ is a contractive mapping for any $\hat{w}$, we can use Banach's fixed point theorem [17] to show that there is a unique solution $\zeta = T_w(\zeta)$ (and thus a unique $u^0 = D^{-1}\zeta$) for any $\hat{w}$ (and thus any $w = ASD^{-1}\hat{w}$). Consider any $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}^n$. Then

$$T_w(\alpha) - T_w(\beta) = \tfrac{-1+k}{k}\left(\hat{h}(\alpha) - \hat{h}(\beta)\right) + \tfrac{-\hat{B}+kI}{k}\left(\hat{f}(\alpha) - \hat{f}(\beta)\right). \qquad (14)$$

Here we use Lemma 2 to introduce $\tilde{h}(\alpha - \beta) = \hat{h}(\alpha) - \hat{h}(\beta)$ and $\tilde{f}(\alpha - \beta) = \hat{f}(\alpha) - \hat{f}(\beta)$. Denote $\Delta = \alpha - \beta$ and $\Delta^+ = T_w(\alpha) - T_w(\beta)$. Then

$$|\Delta_i^+| \le \tfrac{k-1}{k}|\tilde{h}_i(\Delta_i)| + \tfrac{k-\hat{B}_{i,i}}{k}|\tilde{f}_i(\Delta_i)| + \textstyle\sum_{j\neq i}\tfrac{|\hat{B}_{i,j}|}{k}|\tilde{f}_j(\Delta_j)|. \qquad (15)$$

Therefore

$$\|\Delta^+\|_1 = \sum_{i=1}^{n}|\Delta_i^+| \le \sum_{i=1}^{n}\left(\frac{k-1}{k}|\tilde{h}_i(\Delta_i)| + \frac{k-\hat{B}_{i,i}}{k}|\tilde{f}_i(\Delta_i)| + \sum_{j\neq i}\frac{|\hat{B}_{j,i}|}{k}|\tilde{f}_i(\Delta_i)|\right). \qquad (16)$$

Due to the diagonal column-dominance of $\hat{B}$ and the definition of $k$, it holds that $k > \hat{B}_{i,i} > \sum_{j\neq i}|\hat{B}_{j,i}|$. Thus, selecting $\lambda = \frac{k-1}{k} < 1$, $\mu_i = \frac{k-\left(\hat{B}_{i,i} - \sum_{j\neq i}|\hat{B}_{j,i}|\right)}{k} < 1$, $\gamma_i = \max(\lambda, \mu_i) < 1$, and $\bar{\gamma} = \max_i \gamma_i < 1$, we obtain

$$\begin{aligned}\|\Delta^+\|_1 &\le \sum_{i=1}^{n}\lambda|\tilde{h}_i(\Delta_i)| + \mu_i|\tilde{f}_i(\Delta_i)| \\ &\le \sum_{i=1}^{n}\gamma_i\left(|\tilde{h}_i(\Delta_i)| + |\tilde{f}_i(\Delta_i)|\right) \\ &\le \sum_{i=1}^{n}\bar{\gamma}|\Delta_i| = \gamma\|\Delta\|_1.\end{aligned} \qquad (17)$$

Note that $|\tilde{h}_i(\Delta_i)| + |\tilde{f}_i(\Delta_i)| = |\Delta_i|$ since $\tilde{f}_i(\Delta_i)$ and $\tilde{h}_i(\Delta_i)$ always have the same sign by Assumption 1, and sum to $\Delta_i$. This proves that $T_w$ is a contraction mapping with respect to the metric $\|\cdot\|_1$. Thus, by Banach's fixed point theorem, for each $w$ and the ensuing $\hat{w} = DS^{-1}A^{-1}w$ there is a unique $\zeta$ such that (13) holds, and thus a $u^0 = D^{-1}\zeta$ such that (10) holds, which completes the proof. ■

## VI. Proof of Global Asymptotic Stability

Given the existence of an equilibrium $(x^0, z^0)$ and the associated input $u^0$, consider the change of variables $\tilde{z} = -R(z - z^0)$, $\tilde{u} = u - u^0$, $\tilde{f}(\tilde{u}) = f(u^0 + \tilde{u}) - f(u^0)$, and $\tilde{h}(\tilde{u}) = h(u^0 + \tilde{u}) - h(u^0)$. Due to Lemma 2, $\tilde{f}(\cdot)$, $\tilde{h}(\cdot)$ satisfy Assumption 1, and $\tilde{f}(\tilde{u}) + \tilde{h}(\tilde{u}) = \tilde{u}$. This allows rewriting the (1)–(3) as

$$\begin{bmatrix} \dot{\tilde{z}} \\ \dot{\tilde{u}} \end{bmatrix} = \begin{bmatrix} -RP^{-1} & RP^{-1} \\ A - RP^{-1} & -A + RP^{-1} \end{bmatrix} \begin{bmatrix} \tilde{z} \\ \tilde{u} \end{bmatrix} - \begin{bmatrix} 0 \\ PB \end{bmatrix} \tilde{f}(\tilde{u}) - \begin{bmatrix} RS \\ RS \end{bmatrix} \tilde{h}(\tilde{u}) \tag{18}$$

where $P$, $R$, and $S$ are diagonal matrices gathering the controller parameters $p_i$, $r_i$, and $s_i$. Stabilizing this system to $\tilde{z} = \tilde{u} = 0$ is equivalent to stabilizing the original system system to the equilibrium $x = x^0$, $z = z^0$, and $u = u^0$. We will therefore now prove Theorem 2 with a Lyapunov-based argument considering system (18).

*Proof:* (of Theorem 2) Consider the Lyapunov function candidate

$$\begin{aligned} V(\tilde{z}, \tilde{u}) = &\sum_{i=1}^{n} \int_0^{\tilde{z}_i} q_i (a_i \frac{p_i}{r_i} - 1) \left( \tilde{f}_i(\zeta) + \epsilon \zeta \right) d\zeta \\ &+ \sum_{i=1}^{n} \int_0^{\tilde{u}_i} q_i \left( \tilde{f}_i(\zeta) + \epsilon \zeta \right) d\zeta \end{aligned} \tag{19}$$

where scalars $q_i > 0$ and $\epsilon > 0$ are parameters to be fixed later. For any such choice of parameters, $V$ is positive definite and radially unbounded because $\tilde{f}_i(\zeta) + \epsilon\zeta$ is increasing in $\zeta$ and zero at zero. Also $a_i \frac{p_i}{r_i} - 1 > 0$ due to Assumption 3. The time derivative of $V$ along the trajectories of system (18) is given by

$$\dot{V}(\tilde{z}, \tilde{u}) = -\left( \tilde{f}(\tilde{z}) + \epsilon\tilde{z} - \tilde{f}(\tilde{u}) - \epsilon\tilde{u} \right)^\top \tilde{D}(\tilde{z} - \tilde{u}) \tag{20a}$$

$$-\left( \tilde{f}(\tilde{z}) + \epsilon\tilde{z} \right)^\top \tilde{D}PS\tilde{h}(\tilde{u}) \tag{20b}$$

$$-\left( \tilde{f}(\tilde{u}) + \epsilon\tilde{u} \right)^\top QRS\tilde{h}(\tilde{u}) \tag{20c}$$

$$-\left( \tilde{f}(\tilde{u}) + \epsilon\tilde{u} \right)^\top QPB\tilde{f}(\tilde{u}) \tag{20d}$$

where $\tilde{D}$ is a diagonal positive definite matrix gathering the elements $q_i\,(a_i - r_i/p_i)$ and $Q$ is a diagonal positive definite matrix gathering the elements $q_i$. To simplify this expression, we split it into

$$\dot{V}(\tilde{z}, \tilde{u}) = \dot{V}_1(\tilde{z}, \tilde{u}) + \dot{V}_2(\tilde{z}, \tilde{u}) \tag{21}$$

where $\dot{V}_1(\tilde{z}, \tilde{u})$ corresponds to the terms (20a)–(20b) and $\dot{V}_2(\tilde{z}, \tilde{u})$ corresponds to the terms (20c)–(20d). Since $\tilde{D}$ and $\tilde{D}PS$ are diagonal, $\dot{V}_1$ can be analyzed for each $i$ individually. $\tilde{f}_i(\zeta_i) + \epsilon\zeta_i$ is increasing in $\zeta_i$, therefore $\text{sign}\left(\tilde{f}_i(\tilde{z}_i) + \epsilon\tilde{z}_i - \tilde{f}_i(\tilde{u}_i) - \epsilon\tilde{u}_i\right) = \text{sign}\left(\tilde{z}_i - \tilde{u}_i\right)$ and thus (20a) is negative semi-definite. If $\tilde{z}_i$ and $\tilde{u}_i$ have the same sign, (20b) contributes negatively to $\dot{V}_1$. If they have opposite signs the contribution is positive, but in this case (20a) only comprises negative terms because $\left(\tilde{f}_i(\tilde{z}_i) + \epsilon\tilde{z}_i - \tilde{f}_i(\tilde{u}_i) - \epsilon\tilde{u}_i\right) \tilde{D}_{i,i}\left(\tilde{z}_i - \tilde{u}_i\right) = \left(|\tilde{f}_i(\tilde{z}_i) + \epsilon\tilde{z}_i| + |\tilde{f}_i(\tilde{u}_i) - \epsilon\tilde{u}_i|\right) \tilde{D}_{i,i}\left(|\tilde{z}_i| + |\tilde{u}_i|\right)$. Indeed, since $p_i s_i < 1$ from Assumption 3 and $|\tilde{h}_i(\tilde{u}_i)| \le |\tilde{u}_i|$ from Assumption 1, then (20a) as developed above dominates (20b) which is upper bounded by $|\tilde{f}_i(\tilde{z}_i) + \epsilon\tilde{z}_i|\tilde{D}_{i,i}|\tilde{h}_i(\tilde{u}_i)|$. Thus $\dot{V}_1$ is negative semi-definite. We now turn our attention to $\dot{V}_2$. Note that $\tilde{u}$, $\tilde{f}(\tilde{u})$, and $\tilde{h}(\tilde{u})$ elementwise have the same sign and $QRS$ is diagonal, positive definite. Thus

$$\begin{aligned} &\left( \tilde{f}(\tilde{u}) + \epsilon\tilde{u} \right)^\top QRS\tilde{h}(\tilde{u}) = \left( \tilde{f}(\tilde{u}) + \epsilon\tilde{f}(\tilde{u}) + \epsilon\tilde{h}(\tilde{u}) \right)^\top QRS\tilde{h}(\tilde{u}) \\ &= (1+\epsilon)\tilde{f}(\tilde{u})^\top QRS\tilde{h} + \epsilon\tilde{h}(\tilde{u})^\top QRS\tilde{h}(\tilde{u}) \ge \epsilon\beta\|\tilde{h}(\tilde{u})\|_2^2 \end{aligned} \tag{22}$$

where $\beta$ is the minimum diagonal element of $QRS$. Note also that

$$\begin{aligned} \left( \tilde{f}(\tilde{u}) + \epsilon\tilde{u} \right)^\top QPB\tilde{f}(\tilde{u}) = &(1+\epsilon)\tilde{f}(\tilde{u})^\top QPB\tilde{f}(\tilde{u}) \\ &+ \epsilon\tilde{h}(\tilde{u})^\top QPB\tilde{f}(\tilde{u}). \end{aligned} \tag{23}$$

Fix now the weights $q_i$ in such a way that $QPB + B^\top PQ$ is positive definite. This is possible because $B$ is an M-matrix according to Assumption 2. Therefore $\exists\alpha > 0$ such that $QPB + B^\top PQ \succ 2\alpha I$. Thus the first term of (23) satisfies

$$(1+\epsilon)\tilde{f}(\tilde{u})^\top QPB\tilde{f}(\tilde{u}) \ge (1+\epsilon)\alpha\|\tilde{f}(\tilde{u})\|_2^2. \tag{24}$$

We also note that the second term in (23) satisfies

$$\epsilon\tilde{h}(\tilde{u})^\top QPB\tilde{f}(\tilde{u}) \ge -\epsilon\gamma\|\tilde{f}(\tilde{u})\|_2\|\tilde{h}(\tilde{u})\|_2 \tag{25}$$

where $\gamma = \|QPB\|_2$. Thus, combining the bounds in (22), (24) and (25) within (20c)–(20d), we obtain

$$\begin{aligned} \dot{V}_2(\tilde{z}, \tilde{u}) &\le -(1+\epsilon)\alpha\|\tilde{f}(\tilde{u})\|_2^2 - \epsilon\beta\|\tilde{h}(\tilde{u})\|_2^2 + \epsilon\gamma\|\tilde{f}(\tilde{u})\|_2\|\tilde{h}(\tilde{u})\|_2 \\ &= \begin{pmatrix} \|\tilde{f}(\tilde{u})\|_2 \\ \|\tilde{h}(\tilde{u})\|_2 \end{pmatrix}^\top \begin{pmatrix} -(1+\epsilon)\alpha & \frac{1}{2}\epsilon\gamma \\ \frac{1}{2}\epsilon\gamma & -\epsilon\beta \end{pmatrix} \begin{pmatrix} \|\tilde{f}(\tilde{u})\|_2 \\ \|\tilde{h}(\tilde{u})\|_2 \end{pmatrix}. \end{aligned} \tag{26}$$

We may now select the Lyapunov function parameter $\epsilon$ sufficiently small such that $\left(\alpha + \epsilon\alpha - \frac{\epsilon\gamma^2}{4\beta}\right) > 0$. This makes the quadratic form (26) negative definite. Thus $\dot{V}_2(\tilde{z}, \tilde{u}) = 0$ if and only if $\tilde{f}(\tilde{u}) = \tilde{h}(\tilde{u}) = 0$, i.e. if and only if $\tilde{u} = 0$. In this case, $\dot{V}_1(\tilde{z}, \tilde{u})$ is clearly negative definite in $\tilde{z}$. Thus $\dot{V}(\tilde{z}, \tilde{u})$ is negative definite, which implies that the origin is globally asymptotically stable for system (18). Equivalently, the equilibrium $(x^0, z^0)$, with input $u^0$, is therefore globally asymptotically stable for the original system (1)–(3). ■

## VII. Proof of Equilibrium Optimality

Here we prove Theorem 3.

*Proof:* Firstly, it is clear that $v^* = \text{sat}\left(u^0\right)$ and $x_i^* = x_i^0 = -s_i \text{dz}\left(u_i^0\right)$ for all $i$ satisfies (4b) due to $x^0$, $z^0$ being an equilibrium, and satisfies (4c) because $\text{sat}\,(\cdot)$ is bounded in the range $[-1, 1]$. Consider, for establishing a contradiction, that there exists $\mu \neq 0$ such that $v^\dagger = v^* + \mu$ and $x^\dagger = A^{-1}Bv^\dagger + A^{-1}w = x^* + A^{-1}B\mu$ is the optimal solution to (4) with a smaller cost (4a) than the one obtained by $x^*$, $v^*$. Then $\mu$ solves the optimization problem

$$\underset{\mu}{\text{minimize}} \quad \sum_{i=1}^{n} |\gamma_i x_i^* + \tilde{B}_i\mu| \tag{27a}$$

$$\text{subject to} \quad -\mathbf{1} \le v^* + \mu \le \mathbf{1}. \tag{27b}$$

where $\tilde{B}_i$ is row $i$ of the matrix $\tilde{B} = \Gamma A^{-1}B$. The equilibrium of (2) implies $x_i^* = -s_i \text{dz}\left(u_i^0\right)$. Therefore we can leverage (27b) to see that $x_i^* > 0 \implies u_i^0 < -1 \implies v_i = -1 \implies \mu_i \geq 0$ and conversely $x_i^* < 0 \implies u_i^0 > 1 \implies v_i = 1 \implies \mu_i \leq 0$. Combining this with $\Gamma$ and $A$ both being diagonal, positive definite and the fact that $B$ is an M-matrix which implies that $\tilde{B}_{i,i} > 0$, we obtain $|\gamma_i x_i + \tilde{B}_{i,i}\mu_i| = |\gamma_i x_i| + |\tilde{B}_{i,i}\mu_i|$ for all $i$. Thus (27a) can be expanded as follows

$$\begin{aligned}\sum_{i=1}^{n} |\gamma_i x_i + \tilde{B}_i \mu| &\geq \sum_{i\neq j} \left( |\gamma_i x_i + \tilde{B}_{i,i}\mu_i| - |\sum_{i\neq j} \tilde{B}_{i,j}\mu_j| \right) \\ &\geq \sum_{i=1}^{n} \left( |\gamma_i x_i| + |\tilde{B}_{i,i}||\mu_i| \right) - \sum_{i=1}^{n}\sum_{j\neq i} |\tilde{B}_{i,j}||\mu_j| \\ &= \sum_{i=1}^{n} |\gamma_i x_i| + \sum_{k=1}^{n} \left( |\tilde{B}_{k,k}| - \sum_{j\neq k} |\tilde{B}_{j,k}| \right) |\mu_k|.\end{aligned} \quad (28)$$

Since $\tilde{B}$ is diagonally column-dominant, then $|\tilde{B}_{k,k}| - \sum_{j\neq k} |\tilde{B}_{j,k}|$ is positive for all $k$. Thus this expression is minimized by $\mu = 0$, which completes the proof. ■

## VIII. Conclusions

In this paper we considered fully decentralized PI-control for a class of interconnected systems subject to incrementally sector-bounded nonlinearities. We showed that for systems where the input matrix is an M-matrix, fully decentralized PI-controllers globally asymptotically stabilize a specific equilibrium. Furthermore, this equilibrium is optimal in that it minimizes costs of the form $\sum_{i=1}^{n} \gamma_i |x_i|$. The proposed control strategy was employed in a numerical example of a simplified district heating system model. The example showed that, with our decentralized strategy, the total discomfort in the system is minimized, at the cost of higher worst-case discomforts when compared with a alternative coordinated control strategies. We have thus demonstrated that a fully decentralized and easily tuned control law constitutes a relevant design for a large class of systems.

Open questions include analysis of the transient response, and finding controller tuning rules accordingly. This could encompass the case when $w$ is not constant but slowly time-varying, such as in the simulation study in Section IV. Furthermore, to better capture the district heating application, a richer class of systems should be considered: Multi-state models for each building, as well as more complex, nonlinear models of the interconnection $B$ can be considered.

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## Appendix

We prove here suitable properties of the function class characterized by Assumption 1, as stated in Lemmas 1, 2 and 3. To simplify the exposition, we drop the index $i$.

*Proof:* (of Lemma 1)

Clearly, $h(0) = 0 - f(0) = 0$. Additionally,

$$\frac{h(y)-h(x)}{y-x} = \frac{y-f(y)-x+f(x)}{y-x} = 1 - \frac{f(y)-f(x)}{y-x} \in [0,1] \quad (29)$$

which shows that $0 \leq (h(y) - h(x))/(y - x) \leq 1$ if $x \neq y$, concluding the proof. ■

*Proof:* (of Lemma 2)

Clearly, $\tilde{f}(0) = f(x^0) - f(x^0) = 0$. In addition,

$$\frac{\tilde{f}(y) - \tilde{f}(x)}{y - x} = \frac{f(y + x^0) - f(x + x^0)}{(y + x^0) - (x + x^0)} \in [0,1] \quad (30)$$

which shows that $0 \leq (\tilde{f}(y) - \tilde{f}(x))/(y - x) \leq 1$ if $x \neq y$. Finally $\tilde{f}(x) + \tilde{h}(x) = f(x + x^0) - f(x^0) + h(x + x^0) - h(x^0) = x + x^0 - x^0 = x$, concluding the proof. ■

*Proof:* (of Lemma 3)

$\tilde{f}(0) = D^{-1}f(0) = 0$. Additionally,

$$\frac{\tilde{f}(y)-\tilde{f}(x)}{y-x} = \frac{df(y/d)-df(x/d)}{y-x} = \frac{f(y/d)-f(x/d)}{y/d-x/d} \in [0,1]. \quad (31)$$

Thus $0 \leq (\tilde{f}(y) - \tilde{f}(x))/(y - x) \leq 1$ if $x \neq y$. Finally $\tilde{f}(x) + \tilde{h}(x) = Df(D^{-1}x) + Dh(D^{-1}x) = D\left(f(D^{-1}x) + h(D^{-1}x)\right) = DD^{-1}x = x$, concluding the proof. ■