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Synthesis of Dissipative Systems Using Input-State Data

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***Abstract*— This paper deals with the data-driven synthesis of dissipative linear systems in discrete time. We collect finitely many noisy data samples with which we synthesise a controller that makes all systems that explain the data dissipative with respect to a given quadratic supply rate. By adopting the informativity approach, we introduce the notion of informativity for closed-loop dissipativity. Under certain assumptions on the noise and the system, with the help of tools for quadratic matrix inequalities, we provide necessary and sufficient conditions for informativity for closed-loop dissipativity. We also provide a recipe to design suitable controllers by means of data-based linear matrix inequalities. This main result comprises two parts, to account for both the cases that the output matrices are known or unknown. Lastly, we illustrate our findings with an example, for which we want to design a data-driven controller achieving (strict) passivity.**

I. Introduction

Motivated by the increasing complexity of modern engineering systems and the abundance of available data, there has been a recent surge of interest in data-driven analysis and control. In many cases, the data do not give rise to a unique mathematical model of the system, which makes it appealing to avoid system identification and work with direct data-driven control methods instead [1]–[3]. However, the success of any data-driven controller can only be guaranteed if the collected data are "sufficiently rich". Because of this, the concept of *data informativity* has been introduced (see [1]), but conditions for informativity vary from problem to problem.

Over the years, the notion of *dissipativity*, introduced by Jan C. Willems in [4] and [5], has proven itself to be one of the most important concepts in systems and control, that is inseparable from modelling (physical) dynamical systems and controller synthesis. A system is called dissipative if the rate of change of the stored energy in the system does not exceed the supplied energy. Originally, the stored energy is expressed as a function of the state of the system, but, later on, Willems and Trentelman expanded upon the input-state-output framework by developing the behavioural approach with quadratic differential forms laying its foundation [6]. As a prime example of its application, in the two-part paper [7], [8] the $\mathcal{H}_\infty$ control problem was posed and solved in a behavioral context.

Dissipativity for a linear system with a quadratic supply rate can be verified by the dissipation inequality, which can be rewritten as a linear matrix inequality (LMI) involving the system model. In the case that the system matrices are unknown, a number of papers have focused on the problem of verifying dissipativity properties from data. We mention the contributions [9], [10]–[12] and [13] that have tackled this problem in various scenarios involving input-state-output and input-output data that are either exact or noisy.

While [9]–[13] focus on the analysis of dissipativity properties, in this paper, we are interested in designing controllers that achieve dissipative closed-loop behaviour. We will work with a batch of noisy input-state measurements obtained from the true, data-generating system. The noise is assumed to satisfy a quadratic matrix inequality (QMI). This leads to a set of dynamical systems that are consistent with the data. Our goal is to design a single controller that makes all consistent closed-loop systems dissipative with respect to a given quadratic supply rate. If this is possible, we call the data *informative for closed-loop dissipativity*. We will work with general quadratic supply rates that satisfy a certain inertia assumption. The strength of this approach lies in the fact that appropriate choices of the supply rate lead to different relevant control problems such as data-driven feedback passivation and $\mathcal{H}_\infty$ control, the latter of which has already received attention in [1], [14] and [15].

In particular, the contributions of this paper are as follows:

1) We define the concept of *informativity for closed-loop dissipativity* for data obtained from an input-state-output system (Definitions 2 and 3).
2) In Theorem 1, we obtain necessary and sufficient LMI conditions under which noisy data are informative for closed-loop dissipativity. The theorem allows the use of prior knowledge on how the noise affects the dynamics. Furthermore, we provide an explicit formula for a controller. We consider two cases – the output and feedthrough matrices are either unknown or known. In the case that they are known, we make use of an additional projection result, namely, Proposition 1.

The outline of this paper is as follows. In Section II we recapitulate some results regarding QMIs and dissipativity. In Section III we formulate the problem. Afterwards, with the help of the preliminary results from Section IV, we formulate and prove our main result in Section V. In Section VI we consider an example of data-driven feedback passivation. Finally, the conclusions are provided in Section VII.

A. Notation

Let A be a real $n \times n$ matrix. The *Moore-Penrose pseudo-inverse* of A is denoted by $A^\dagger$. The *set of real symmetric* $n\times n$ *matrices* is denoted by $\mathbb{S}^n$. The *inertia* of a symmetric matrix

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A is denoted by $\mathrm{In}(A) = (\rho_-, \rho_0, \rho_+)$, where ρ_-, ρ_0 and ρ_+ are the number of negative, zero and positive eigenvalues of A, respectively. Let $A \in \mathbb{S}^n$. If $x^T A x > 0$, for all nonzero $x \in \mathbb{R}^n$, then A is called *positive definite*, denoted by $A > 0$. If $x^T A x \geqslant 0$, for all $x \in \mathbb{R}^n$, then A is called *positive semidefinite*, denoted by $A \geqslant 0$. *Negative definiteness* and *negative semidefiniteness* are defined similarly and denoted by $A < 0$ and $A \leqslant 0$, respectively. By $A > B$ we mean that $A - B > 0$. In addition, $A \geqslant B$, $A < B$ and $A \leqslant B$ are defined similarly. We denote the $n \times m$ zero matrix by $0_{n\times m}$ and the $n \times n$ identity matrix by I_n. The subscripts are omitted whenever the size is clear from the context.

II. Preliminaries

In this section, we review some results on quadratic matrix inequalities that will be used throughout the paper, and we recap the concept of dissipativity. For more details and proofs, we refer to [13] and [16].

A. Sets Induced by QMIs

For reasons that will become clear in the next section, we are interested in the set

$$\mathcal{Z}_r(\Pi) := \left\{ Z \in \mathbb{R}^{r\times q} \colon \begin{bmatrix} I_q \\ Z \end{bmatrix}^T \Pi \begin{bmatrix} I_q \\ Z \end{bmatrix} \geqslant 0 \right\},$$

where $\Pi \in \mathbb{S}^{q+r}$ is partitioned as $\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}$ with $\Pi_{11} \in \mathbb{S}^q$ and $\Pi_{22} \in \mathbb{S}^r$. Throughout the paper, $\Pi \in \mathbb{S}^{q+r}$ will imply this particular partitioning. The set $\mathcal{Z}_r(\Pi)$ is nonempty and convex if the following three conditions hold:

$$\Pi_{22} \leqslant 0,\ \Pi|\Pi_{22} \geqslant 0,\ \ker \Pi_{22} \subseteq \ker \Pi_{12}, \tag{1}$$

where $\Pi|\Pi_{22} := \Pi_{11} - \Pi_{12}\Pi_{22}^\dagger \Pi_{21}$ denotes the (generalised) Schur complement of Π w.r.t. Π_{22}. Define the set

$$\mathbf{\Pi}_{q,r} := \left\{ \Pi \in \mathbb{S}^{q+r} \colon (1) \text{ hold} \right\}.$$

For $\Pi \in \mathbf{\Pi}_{q,r}$, the set $\mathcal{Z}_r(\Pi)$ is bounded if and only if $\Pi_{22} < 0$.

For $W \in \mathbb{R}^{q\times p}$, $\mathcal{S} \subseteq \mathbb{R}^{r\times q}$ and $\Pi \in \mathbb{S}^{q+r}$, we define $\mathcal{S}W := \{SW \colon S \in \mathcal{S}\}$ and

$$\Pi_W := \begin{bmatrix} W^T & 0 \\ 0 & I_r \end{bmatrix} \Pi \begin{bmatrix} W & 0 \\ 0 & I_r \end{bmatrix} \in \mathbb{S}^{p+r}.$$

If $\Pi \in \mathbf{\Pi}_{q,r}$, then $\Pi_W \in \mathbf{\Pi}_{p,r}$.

Proposition 1: Let $\Pi \in \mathbf{\Pi}_{q,r}$ and $W \in \mathbb{R}^{q\times p}$. If either W has full column rank or Π_{22} is nonsingular, then $\mathcal{Z}_r(\Pi)W = \mathcal{Z}_r(\Pi_W)$.

B. Matrix Version of the S-lemma

We will recall necessary and sufficient conditions under which all solutions to one QMI also satisfy another QMI.

Proposition 2: Let $M, N \in \mathbb{S}^{q+r}$. Assume that $N \in \mathbf{\Pi}_{q,r}$ and N has at least one positive eigenvalue. Then $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r(M)$ if and only if there exists a real $\alpha \geqslant 0$ such that $M - \alpha N \geqslant 0$.

C. Dissipativity of Discrete-Time Linear Systems

Consider a linear discrete-time input-state-output system

$$\begin{aligned} \mathbf{x}(t+1) &= A\mathbf{x}(t) + B\mathbf{u}(t), \\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t), \end{aligned} \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$ and $(\mathbf{u}, \mathbf{x}, \mathbf{y}) \colon \mathbb{N} \to \mathbb{R}^{m+n+p}$.

Definition 1: Let $S \in \mathbb{S}^{m+p}$. We call system (2), or the quadruple (A, B, C, D), *dissipative w.r.t. the supply rate*

$$s(u, y) = \begin{bmatrix} u \\ y \end{bmatrix}^T S \begin{bmatrix} u \\ y \end{bmatrix} \tag{3}$$

if there exists a matrix $P \in \mathbb{S}^n$ such that $P \geqslant 0$ and the dissipation inequality

$$\mathbf{x}(t)^T P \mathbf{x}(t) + s(\mathbf{u}(t), \mathbf{y}(t)) \geqslant \mathbf{x}(t+1)^T P \mathbf{x}(t+1) \tag{4}$$

holds for all $t \geqslant 0$ and all trajectories $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ of (2).

We can rewrite (4) as

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \geqslant 0, \tag{5}$$

so asking for system (2) to be dissipative w.r.t. the supply rate (3) is equivalent to requiring the feasibility of the LMIs $P \geqslant 0$ and (5).

III. Problem Formulation

Now consider the system

$$\mathbf{x}(t+1) = A_s\mathbf{x}(t) + B_s\mathbf{u}(t) + E\mathbf{w}(t), \tag{6a}$$
$$\mathbf{y}(t) = C_s\mathbf{x}(t) + D_s\mathbf{u}(t) + F\mathbf{w}(t), \tag{6b}$$

where (A_s, B_s, C_s, D_s) denote the "true" system matrices. Henceforth, we will assume that (C_s, A_s) is observable. The state and input matrices (A_s, B_s) and the noise term $\mathbf{w} \in \mathbb{R}^d$ are unknown, whereas $(E, F) \in \mathbb{R}^{n\times d} \times \mathbb{R}^{p\times d}$ are known. The matrices E and F capture our prior knowledge on how the noise affects the dynamics of the system. If we lack such knowledge, we can take them to be the identity matrix. We will derive results for two scenarios, namely the cases that the (C_s, D_s) matrices are either unknown or known. The goal of this paper is to find a matrix $K \in \mathbb{R}^{m\times n}$ such that the static state feedback controller

$$\mathbf{u}(t) = K\mathbf{x}(t)$$

makes the closed-loop system $(A_s + B_sK, E, C_s + D_sK, F)$ dissipative w.r.t. the supply rate

$$s(w, y) = \begin{bmatrix} w \\ y \end{bmatrix}^T S \begin{bmatrix} w \\ y \end{bmatrix}. \tag{7}$$

From now on, we will assume that the matrix S has inertia $\mathrm{In}(S) = (p, 0, d)$. Examples that satisfy this condition include the passive supply rate (for $d = p$) and the ℓ_2-gain supply rate (for $\gamma > 0$):

$$\begin{bmatrix} 0 & I_d \\ I_d & 0 \end{bmatrix} \text{ and } \begin{bmatrix} \gamma^2 I_d & 0 \\ 0 & -I_p \end{bmatrix}.$$

Remark 1: Although both (6a) and (6b) use the same noise term $\mathbf{w}$, system (6) is considered a generalisation of the case where there are different process and measurement noise terms. Indeed,

$$\begin{aligned}\mathbf{x}(t+1) &= A_s\mathbf{x}(t) + B_s\mathbf{u}(t) + E\mathbf{v}(t),\\ \mathbf{y}(t) &= C_s\mathbf{x}(t) + D_s\mathbf{u}(t) + F\mathbf{z}(t),\end{aligned}$$

can be rewritten as

$$\begin{aligned}\mathbf{x}(t+1) &= A_s\mathbf{x}(t) + B_s\mathbf{u}(t) + \begin{bmatrix}E & 0\end{bmatrix}\begin{bmatrix}\mathbf{v}(t)\\ \mathbf{z}(t)\end{bmatrix},\\ \mathbf{y}(t) &= C_s\mathbf{x}(t) + D_s\mathbf{u}(t) + \begin{bmatrix}0 & F\end{bmatrix}\begin{bmatrix}\mathbf{v}(t)\\ \mathbf{z}(t)\end{bmatrix},\end{aligned}$$

which is again of the form (6) by taking $\mathbf{w} = \begin{bmatrix}\mathbf{v}\\ \mathbf{z}\end{bmatrix}$.

We collect finitely many input-state measurements of (6) in the matrices

$$\begin{aligned}U_- &:= \begin{bmatrix}u(0) & u(1) & \cdots & u(T-1)\end{bmatrix},\\ X &:= \begin{bmatrix}x(0) & x(1) & \cdots & x(T)\end{bmatrix}.\end{aligned}$$

In addition, we define the following matrices:

$$\begin{aligned}X_- &:= \begin{bmatrix}x(0) & x(1) & \cdots & x(T-1)\end{bmatrix},\\ X_+ &:= \begin{bmatrix}x(1) & x(2) & \cdots & x(T)\end{bmatrix}.\end{aligned}$$

The noise matrix $W_- := \begin{bmatrix}w(0) & w(1) & \cdots & w(T-1)\end{bmatrix}$ is assumed to satisfy

$$\begin{bmatrix}I\\ W_-^T\end{bmatrix}^T \Phi \begin{bmatrix}I\\ W_-^T\end{bmatrix} \geqslant 0, \tag{8}$$

i.e. $W_-^T \in \mathcal{Z}_T(\Phi)$, with $\Phi \in \mathbf{\Pi}_{d,T}$ and $\Phi_{22} < 0$. With an appropriate choice of Φ, the noise model can capture various assumptions on the noise such as energy bounds, sample covariance bounds, etc. (see [16]). Then, from (6a) we have that $X_+ = A_sX_- + B_sU_- + EW_-$. We first consider the setting where C_s and D_s are known. We denote the set of systems consistent with the data, i.e. all (A,B) satisfying

$$X_+ = AX_- + BU_- + EW_- \tag{9}$$

for some W_- satisfying (8), by

$$\Sigma_k := \left\{(A,B)\colon \text{(9) holds for some } W_-^T \in \mathcal{Z}_T(\Phi)\right\}.$$

Clearly, $(A_s,B_s) \in \Sigma_k$. Because we cannot distinguish the true system from other systems in Σ_k, our controller should make all systems in Σ_k dissipative w.r.t. (7), inspiring the informativity approach.

Definition 2: The data (U_-,X) are called *informative for closed-loop dissipativity* w.r.t. the supply rate (7) if there exist matrices K and $P \geqslant 0$ such that for any $(A,B) \in \Sigma_k$,

$$\begin{bmatrix}I & 0\\ \mathcal{A} & E\end{bmatrix}^T \begin{bmatrix}P & 0\\ 0 & -P\end{bmatrix} \begin{bmatrix}I & 0\\ \mathcal{A} & E\end{bmatrix} + \begin{bmatrix}0 & I\\ \mathcal{C}_s & F\end{bmatrix}^T S \begin{bmatrix}0 & I\\ \mathcal{C}_s & F\end{bmatrix} \geqslant 0, \tag{10}$$

where $\mathcal{A} := A + BK$ and $\mathcal{C}_s := C_s + D_sK$.

Next, we turn our attention to the case that C_s and D_s are unknown. We collect finitely many output measurements of (6) in the matrix

$$Y_- := \begin{bmatrix}y(0) & y(1) & \cdots & y(T-1)\end{bmatrix}.$$

Consider the system of linear equations

$$\begin{bmatrix}X_+\\ Y_-\end{bmatrix} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}X_-\\ U_-\end{bmatrix} + \begin{bmatrix}E\\ F\end{bmatrix} W_- \tag{11}$$

in the unknown (A,B,C,D). The set of systems explaining the data is now given by

$$\Sigma_u := \left\{(A,B,C,D)\colon \text{(11) holds for some } W_-^T \in \mathcal{Z}_T(\Phi)\right\}.$$

Obviously, $(A_s,B_s,C_s,D_s) \in \Sigma_u$.

Definition 3: The data (U_-,X,Y_-) are called *informative for closed-loop dissipativity* w.r.t. the supply rate (7) if there exist matrices K and $P \geqslant 0$ such that for any $(A,B,C,D) \in \Sigma_u$,

$$\begin{bmatrix}I & 0\\ \mathcal{A} & E\end{bmatrix}^T \begin{bmatrix}P & 0\\ 0 & -P\end{bmatrix} \begin{bmatrix}I & 0\\ \mathcal{A} & E\end{bmatrix} + \begin{bmatrix}0 & I\\ \mathcal{C} & F\end{bmatrix}^T S \begin{bmatrix}0 & I\\ \mathcal{C} & F\end{bmatrix} \geqslant 0, \tag{12}$$

where $\mathcal{A} := A + BK$ and $\mathcal{C} := C + DK$.

IV. Preliminary Results

In this section, we derive some results which will inspire and help prove our main result.

Lemma 1: The system $(A,B) \in \Sigma_k$ if and only if

$$\begin{bmatrix}I\\ A^T\\ B^T\end{bmatrix}^T N_k \begin{bmatrix}I\\ A^T\\ B^T\end{bmatrix} \geqslant 0, \tag{13}$$

where

$$N_k := \left[\begin{array}{c:c}I & X_+\\ \hdashline 0 & \begin{matrix}-X_-\\ -U_-\end{matrix}\end{array}\right] \begin{bmatrix}E & 0\\ 0 & I\end{bmatrix} \Phi \begin{bmatrix}E^T & 0\\ 0 & I\end{bmatrix} \left[\begin{array}{c:c}I & X_+\\ \hdashline 0 & \begin{matrix}-X_-\\ -U_-\end{matrix}\end{array}\right]^T.$$

Proof: By Proposition 1, we have that $\mathcal{Z}_T(\Phi)E^T = \mathcal{Z}_T(\Phi_{E^T})$. Then, $(A,B) \in \Sigma_k$ if and only if

$$W_-^TE^T = \begin{bmatrix}X_+\\ -X_-\\ -U_-\end{bmatrix}^T \begin{bmatrix}I\\ A^T\\ B^T\end{bmatrix} \in \mathcal{Z}_T(\Phi_{E^T})$$

or, equivalently, (13) holds. ■

Lemma 2: The system $(A,B,C,D) \in \Sigma_u$ if and only if

$$\left[\begin{array}{c}I\\ \hdashline \begin{matrix}A^T & C^T\\ B^T & D^T\end{matrix}\end{array}\right]^T N_u \left[\begin{array}{c}I\\ \hdashline \begin{matrix}A^T & C^T\\ B^T & D^T\end{matrix}\end{array}\right] \geqslant 0, \tag{14}$$

where

$$N_u := \left[\begin{array}{c:c}I & \begin{matrix}X_+\\ Y_-\end{matrix}\\ \hdashline 0 & \begin{matrix}-X_-\\ -U_-\end{matrix}\end{array}\right] \left[\begin{array}{c:c}\begin{matrix}E\\ F\end{matrix} & 0\\ \hdashline 0 & I\end{array}\right] \Phi \left[\begin{array}{c:c}\begin{matrix}E\\ F\end{matrix} & 0\\ \hdashline 0 & I\end{array}\right]^T \left[\begin{array}{c:c}I & \begin{matrix}X_+\\ Y_-\end{matrix}\\ \hdashline 0 & \begin{matrix}-X_-\\ -U_-\end{matrix}\end{array}\right].$$

The proof of Lemma 2 follows the same steps as the proof of Lemma 1, but we replace the matrix E^T with $\begin{bmatrix}E^T & F^T\end{bmatrix}$.

The quadratic matrix inequalities (13) and (14) are in terms of the transposes of the matrices A, B, C and D. This motivates the use of the following dualisation result (the proof can be found in [13, Proposition 4]).

Proposition 3: Consider a matrix $P \in \mathbb{S}^n$ with $P > 0$ and a matrix $\begin{bmatrix}A & B\\ C & D\end{bmatrix} \in \mathbb{R}^{(n+p)\times(n+m)}$. Suppose that $S \in \mathbb{S}^{m+p}$ has inertia $\operatorname{In}(S) = (p, 0, m)$. Define $Q := P^{-1}$ and

$$\hat{S} := \begin{bmatrix}0 & -I_p\\ I_m & 0\end{bmatrix} S^{-1} \begin{bmatrix}0 & -I_m\\ I_p & 0\end{bmatrix}.$$

Then,

$$\begin{bmatrix}I & 0\\ A & B\end{bmatrix}^T \begin{bmatrix}P & 0\\ 0 & -P\end{bmatrix} \begin{bmatrix}I & 0\\ A & B\end{bmatrix} + \begin{bmatrix}0 & I\\ C & D\end{bmatrix}^T S \begin{bmatrix}0 & I\\ C & D\end{bmatrix} \geqslant 0$$

if and only if

$$\begin{bmatrix}I & 0\\ A^T & C^T\end{bmatrix}^T \begin{bmatrix}Q & 0\\ 0 & -Q\end{bmatrix} \begin{bmatrix}I & 0\\ A^T & C^T\end{bmatrix} + \begin{bmatrix}0 & I\\ B^T & D^T\end{bmatrix}^T \hat{S} \begin{bmatrix}0 & I\\ B^T & D^T\end{bmatrix} \geqslant 0.$$

Note that the above dualisation result works under the assumption that $P > 0$. The following proposition reveals a condition under which $P > 0$ (the proof can be found in [17, Lemma 4.4]).

Proposition 4: Assume that S has at least p negative eigenvalues. If $P \geqslant 0$ satisfies the dissipation inequality (5), then

$$\ker P \subseteq \ker \begin{bmatrix}C^T & (CA)^T & \cdots & (CA^{n-1})^T\end{bmatrix}^T.$$

The above result shows that $\ker P = \{0\}$ and thus $P > 0$ if the pair (C, A) is observable.

V. Main Result

We are ready to state and prove the main theorem, which we divide into two parts. Part (a) is concerned with the case that (C_s, D_s) are unknown. In part (b) we assume that (C_s, D_s) are known, which requires an additional assumption on the data.

Theorem 1: Define the following matrices:

$$\hat{M}_u := \left[\begin{array}{c:c:c}\hat{R} & 0 & 0\\ \hdashline 0 & 0 & \begin{matrix}Q\\ L\end{matrix}\\ \hdashline 0 & Q \;\; L^T & Q\end{array}\right],\ \hat{N}_u := \left[\begin{array}{c:c}I & \begin{matrix}X_+\\ Y_-\end{matrix}\\ \hdashline 0 & \begin{matrix}-X_-\\ -U_-\end{matrix}\\ \hdashline 0 & 0_{n\times T}\end{array}\right] \Phi_G \left[\begin{array}{c:c}I & \begin{matrix}X_+\\ Y_-\end{matrix}\\ \hdashline 0 & \begin{matrix}-X_-\\ -U_-\end{matrix}\\ \hdashline 0 & 0_{n\times T}\end{array}\right]^T$$

as well as $\hat{M}_k$ and $\hat{N}_k$ as in (15), where

$$G := \begin{bmatrix}E^T & F^T\end{bmatrix},\ \hat{S} := \begin{bmatrix}0 & -I_p\\ I_d & 0\end{bmatrix} S^{-1} \begin{bmatrix}0 & -I_d\\ I_p & 0\end{bmatrix},$$

$$\hat{R} := \begin{bmatrix}Q & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & I\\ E^T & F^T\end{bmatrix}^T \hat{S} \begin{bmatrix}0 & I\\ E^T & F^T\end{bmatrix},$$

$$\hat{H} := \begin{bmatrix}0 & 0\\ 0 & -C_sQC_s^T - C_sL^TD_s^T - D_sLC_s^T\end{bmatrix}.$$

(a) Assume that $\hat{N}_u$ has at least one positive eigenvalue. The data (U_-, X, Y_-), generated by (6) with noise model (8), are informative for closed-loop dissipativity w.r.t. the supply rate (7) if and only if there exist a positive definite matrix $Q \in \mathbb{S}^n$, a matrix $L \in \mathbb{R}^{m\times n}$ and a scalar $\alpha \geqslant 0$ such that $\hat{M}_u - \alpha\hat{N}_u \geqslant 0$.

(b) Assume that $\operatorname{rank} \begin{bmatrix}X_-^T & U_-^T\end{bmatrix}^T = n + m$ and $\hat{N}_k$ has at least one positive eigenvalue. The data (U_-, X), generated by (6a) with noise model (8), are informative for closed-loop dissipativity w.r.t. the supply rate (7) if and only if there exist a positive definite matrix $Q \in \mathbb{S}^n$, a matrix $L \in \mathbb{R}^{m\times n}$ and a scalar $\alpha \geqslant 0$ such that $\hat{M}_k - \alpha\hat{N}_k \geqslant 0$.

In either case, if the data are informative, a controller that achieves closed-loop dissipativity for all systems in Σ_u and Σ_k, respectively, is $K = LQ^{-1}$.

Proof: We begin with the first statement. Suppose that the data (U_-, X, Y_-) are informative. Then, for all systems in Σ_u, there exist matrices K and $P \geqslant 0$ such that (12) holds for any $(A, B, C, D) \in \Sigma_u$. Because (C_s, A_s) is observable, as a consequence of Proposition 4, we know that $P > 0$. Let $(A, B, C, D) \in \Sigma_u$ and define $\mathcal{A}$ and $\mathcal{C}$ as in Definition 3. Then, by Proposition 3,

$$\begin{bmatrix}I & 0\\ \mathcal{A}^T & \mathcal{C}^T\end{bmatrix}^T \begin{bmatrix}Q & 0\\ 0 & -Q\end{bmatrix} \begin{bmatrix}I & 0\\ \mathcal{A}^T & \mathcal{C}^T\end{bmatrix} + \begin{bmatrix}0 & I\\ E^T & F^T\end{bmatrix}^T \hat{S} \begin{bmatrix}0 & I\\ E^T & F^T\end{bmatrix} \geqslant 0 \tag{16}$$

holds, where $Q := P^{-1} > 0$. The dual dissipation inequality (16) can be rewritten as

$$\left[\begin{array}{c}I\\ \hdashline A^T \;\; C^T\\ B^T \;\; D^T\end{array}\right]^T M_u \left[\begin{array}{c}I\\ \hdashline A^T \;\; C^T\\ B^T \;\; D^T\end{array}\right] \geqslant 0,$$

where

$$M_u := \left[\begin{array}{c:cc}\hat{R} & 0 & \\ \hdashline 0 & -Q & -QK^T\\ & -KQ & -KQK^T\end{array}\right].$$

$$\hat{M}_k := \left[\begin{array}{c:cc:c}\hat{R} + \hat{H} & \begin{matrix}0\\ -C_sQ\end{matrix} & \begin{matrix}0\\ -C_sL^T\end{matrix} & \begin{matrix}0\\ D_sL\end{matrix}\\ \hdashline \begin{matrix}0 & -QC_s^T\\ 0 & -LC_s^T\end{matrix} & 0 & & \begin{matrix}Q\\ L\end{matrix}\\ \hdashline 0 \;\; L^TD_s^T & Q & L^T & Q\end{array}\right],\ \hat{N}_k := \left[\begin{array}{c:c}I & X_+\\ 0 & 0_{p\times T}\\ \hdashline 0 & \begin{matrix}-X_-\\ -U_-\end{matrix}\\ \hdashline 0 & 0_{n\times T}\end{array}\right] \Phi_{E^T} \left[\begin{array}{c:c}I & X_+\\ 0 & 0_{p\times T}\\ \hdashline 0 & \begin{matrix}-X_-\\ -U_-\end{matrix}\\ \hdashline 0 & 0_{n\times T}\end{array}\right]^T \tag{15}$$

Because the data (U_-, X, Y_-) are informative for closed-loop dissipativity, we have $\mathcal{Z}_{n+m}(N_u) \subseteq \mathcal{Z}_{n+m}(M_u)$. Since $\hat{N}_u$ has a positive eigenvalue, also N_u has a positive eigenvalue and moreover it can be shown that $N_u \in \mathbf{\Pi}_{n+p,n+m}$. Therefore, by Proposition 2, $M_u - \alpha N_u \geqslant 0$ for some $\alpha \geqslant 0$. Define $L := KQ$. By using a Schur complement argument, we conclude that $\hat{M}_u - \alpha \hat{N}_u \geqslant 0$.

Conversely, suppose that there exist matrices $Q > 0$ and L and a scalar $\alpha \geqslant 0$ such that $\hat{M}_u - \alpha \hat{N}_u \geqslant 0$. Let $K := LQ^{-1}$. With a Schur complement argument w.r.t. the bottom right block, we have that $M_u - \alpha N_u \geqslant 0$. Let $(A, B, C, D) \in \Sigma_u$. After multiplying this inequality from the right by

$$\begin{bmatrix} I \\ \hline A^T \; C^T \\ B^T \; D^T \end{bmatrix} \tag{17}$$

and from the left by its transpose, we get

$$\begin{bmatrix} I \\ \hline A^T \; C^T \\ B^T \; D^T \end{bmatrix}^T M_u \begin{bmatrix} I \\ \hline A^T \; C^T \\ B^T \; D^T \end{bmatrix} \geqslant \alpha \begin{bmatrix} I \\ \hline A^T \; C^T \\ B^T \; D^T \end{bmatrix}^T N_u \begin{bmatrix} I \\ \hline A^T \; C^T \\ B^T \; D^T \end{bmatrix} \geqslant 0.$$

By Proposition 3, (12) holds with $P := Q^{-1}$. Thus, the data (U_-, X, Y_-) are informative for closed-loop dissipativity.

For the second statement, let us assume that the data (U_-, X) are informative. Applying the same reasoning, this time we can rewrite the dual dissipation inequality (10) as

$$\begin{bmatrix} I \\ \hline A^T \; 0 \\ B^T \; 0 \end{bmatrix}^T M_k \begin{bmatrix} I \\ \hline A^T \; 0 \\ B^T \; 0 \end{bmatrix}^T \geqslant 0, \tag{18}$$

where M_k is defined as in (19) and H is defined as

$$\begin{bmatrix} 0 & 0 \\ 0 & -C_sQC_s^T - C_sQK^TD_s^T - D_sKQC_s^T - D_sKQK^TD_s^T \end{bmatrix}.$$

The data (U_-, X) are informative for closed-loop dissipativity, so (18) holds for all (A, B) that satisfy (13), but due to the difference in the structures of (13) and (18), we are unable to immediately apply Proposition 2. To alleviate this problem, we will invoke Proposition 1, by noting that

$$\begin{bmatrix} A^T \\ B^T \end{bmatrix} \begin{bmatrix} I & 0_{n\times p} \end{bmatrix} = \begin{bmatrix} A^T & 0 \\ B^T & 0 \end{bmatrix}.$$

Because $\Phi_{22} < 0$ and $\begin{bmatrix} X_-^T & U_-^T \end{bmatrix}^T$ is full row rank, the $(2,2)$ block of N_k, i.e. $\begin{bmatrix} X_- \\ U_- \end{bmatrix} \Phi_{22} \begin{bmatrix} X_- \\ U_- \end{bmatrix}^T$, is negative definite and as such we have that $\mathcal{Z}_{n+m}(N_k) \begin{bmatrix} I & 0_{n\times p} \end{bmatrix} = \mathcal{Z}_{n+m}(\bar{N}_k)$, where

$$\bar{N}_k := \left[\begin{array}{cc:c} I & 0_{n\times p} & 0 \\ \hdashline & 0 & I \end{array}\right]^T N_k \left[\begin{array}{cc:c} I & 0_{n\times p} & 0 \\ \hdashline & 0 & I \end{array}\right]$$

$$= \left[\begin{array}{c:c} I & X_+ \\ 0 & 0_{p\times T} \\ \hdashline 0 & -X_- \\ & -U_- \end{array}\right] \Phi_{E^T} \left[\begin{array}{c:c} I & X_+ \\ 0 & 0_{p\times T} \\ \hdashline 0 & -X_- \\ & -U_- \end{array}\right]^T.$$

Consequently, $\mathcal{Z}_{n+m}(\bar{N}_k) \subseteq \mathcal{Z}_{n+m}(M_k)$. Given that $\hat{N}_k$ has a positive eigenvalue, it follows that $\bar{N}_k$ has a positive eigenvalue as well. It can also be shown that $\bar{N}_k \in \mathbf{\Pi}_{n+p,n+m}$. By Proposition 2, there exists a scalar $\alpha \geqslant 0$ such that $M_k - \alpha \bar{N}_k \geqslant 0$. In the same fashion as the first half of the proof of the first statement, $\hat{M}_k - \alpha \hat{N}_k \geqslant 0$ holds.

The other direction is the same as the second half of the proof of the first statement, but we replace (17) with

$$\begin{bmatrix} I \\ \hline A^T \; 0 \\ B^T \; 0 \end{bmatrix}.$$

This proves the theorem. ∎

Theorem 1 allows us to verify informativity for closed-loop dissipativity by using tools for linear matrix inequalities such as MOSEK [18] in order to check their feasibility. Furthermore, as we have not specified the matrix S, our result can be used with any supply rate that satisfies the inertia condition $\operatorname{In}(S) = (p, 0, d)$.

VI. Example

Consider the following system:

$$\mathbf{x}(t+1) = A_s\mathbf{x}(t) + B_s\mathbf{u}(t) + \begin{bmatrix} 0.534 \\ 0.233 \end{bmatrix} \mathbf{w}(t),$$
$$\mathbf{y}(t) = \begin{bmatrix} 0.573 & -0.462 \end{bmatrix} \mathbf{x}(t) + 0.857\mathbf{u}(t) + 0.474\mathbf{w}(t),$$

where the "true" state matrices are

$$A_s = \begin{bmatrix} -0.292 & 1.551 \\ -0.469 & 0.711 \end{bmatrix}, \quad B_s = \begin{bmatrix} -0.066 \\ -0.397 \end{bmatrix}.$$

We want to design a controller that renders the system *state-strictly passive* by state feedback, which can be done with an appropriate choice of the matrix S and Theorem 1.

Definition 4: We call system (2) *state-strictly passive*, if there exist an $\epsilon > 0$ and a matrix $P \in \mathbb{S}^n$ with $P > 0$ such that

$$\mathbf{x}(t)^T P\mathbf{x}(t) + 2\mathbf{u}(t)^T\mathbf{y}(t) - \epsilon||\mathbf{x}(t)||^2 \geqslant \mathbf{x}(t+1)^T P\mathbf{x}(t+1)$$

holds for all $t \geqslant 0$ and all trajectories $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ of (2).

$$M_k := \left[\begin{array}{cc:cc} \multicolumn{2}{c:}{H + \hat{R}} & \begin{matrix} 0 \\ -C_sQ - D_sKQ \end{matrix} & \begin{matrix} 0 \\ -C_sQK^T - D_sKQK^T \end{matrix} \\ \hdashline 0 & -QC_s^T - QK^TD_s^T & -Q & -QK^T \\ 0 & -KQC_s^T - KQK^TD_s^T & -KQ & -KQK^T \end{array}\right] \tag{19}$$

This notion of dissipativity is relevant in the context of Lur'e systems, where absolute stability of the system is guaranteed if the linear part of the system is state-strictly passive (see e.g. [19], [20]). Define a new output $\mathbf{z} := \begin{bmatrix}\mathbf{x}^T & \mathbf{y}^T\end{bmatrix}^T$ and note that

$$\begin{bmatrix}u\\y\end{bmatrix}^T\begin{bmatrix}0 & I_m\\I_m & 0\end{bmatrix}\begin{bmatrix}u\\y\end{bmatrix}-\epsilon||x(t)||^2=\begin{bmatrix}u\\z\end{bmatrix}^T\begin{bmatrix}0 & 0 & I_m\\0 & -\epsilon I_n & 0\\I_m & 0 & 0\end{bmatrix}\begin{bmatrix}u\\z\end{bmatrix}.$$

Thus, we can study state-strict passivity while still working within the framework of this paper by choosing

$$S=\begin{bmatrix}0 & 0 & I_m\\0 & -\epsilon I_n & 0\\I_m & 0 & 0\end{bmatrix}, \tag{20}$$

where $\mathrm{In}(S)=(n+m,0,m)$, $m=p$ and ϵ is a decision variable.

Suppose that the output matrices are known. The knowledge of the (A_s,B_s) matrices is used only for the purpose of generating $T=30$ noisy input-state measurements. We assume that the noise samples are bounded in norm at all times, i.e. $|w(t)|\leqslant 1\ \forall t$. This implies that W_- satisfies (8), where $\Phi=\begin{bmatrix}TI & 0\\0 & -I\end{bmatrix}$. The noise samples were randomly drawn from the uniform distribution in the interval $(0,1)$. The initial state was randomly drawn from the standard normal distribution. The inputs were randomly drawn from the standard normal distribution multiplied by 20 and the rank condition on the input-state data was satisfied. After verifying observability and whether $\hat{N}_k$ has at least one positive eigenvalue, we use YALMIP [21] with MOSEK in MATLAB to solve the LMI presented in Theorem 1. The controller $K=\begin{bmatrix}-0.865 & 1.33\end{bmatrix}$ makes the system state-strictly passive, i.e. dissipative w.r.t. the supply rate $s(w,z)=\begin{bmatrix}w\\z\end{bmatrix}^T S\begin{bmatrix}w\\z\end{bmatrix}$, where S is the same as in (20) with $\epsilon=0.335$. The matrix P that satisfies the dissipation inequality is $P=\begin{bmatrix}1.684 & -1.391\\-1.391 & 6.246\end{bmatrix}$.

VII. Conclusions and Discussion

Throughout this paper, we have considered a linear input-state-output system, where the unknown noise is contained in a given subspace and the matrix of noise samples satisfies a QMI. Next, depending on the presence or absence of prior knowledge on the output system matrices, we have derived necessary and sufficient LMI conditions for noisy data to be informative for closed-loop dissipativity. Additionally, from a finite number of informative data samples, we have found a controller that renders our system dissipative. Finally, we have applied our results with the aim of achieving closed-loop state-strict passivity in a numerical example.

We have assumed to have measurements of the system's state, so a natural extension of our results will involve input-output data only. In future work, similar to [22], we can adopt both the behavioural and informativity approaches in order to study dissipativity properties and design feedback controllers for linear input-output systems in auto-regressive form. Another interesting research line involves developing specific algorithms for solving the LMIs in Theorem 1.

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