2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

A Geometric Tool for Two-Phase Multiplayer Reach-Avoid Games: Ellipses

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***Abstract*—This paper focuses on the utilization of geometric tools in two-phase multiplayer reach-avoid games. In these games, a thief aims to enter a target region and subsequently reach a safe region while evading capture by guarders. We first obtain the solution to the game of kind in the game scenario where the safe region is a single point by using ellipses. Then we show that ellipses are also useful to solve the game of kind when two guarders cooperatively play against the thief. Furthermore, we study the dominance region for two-phase games with the help of ellipses. The construction method of the boundary of dominance regions is provided and illustrated with a numerical example.**

***Index Terms*—Reach-avoid games, ellipse, cooperative defense, dominance regions.**

I. Introduction

Reach-avoid games have garnered significant interest among researchers in various engineering domains, such as air traffic control [1], path planing [2] and robot surveillance [3]. In a reach-avoid game, one team of agents aims to enter a sequence of target regions in the state space while avoiding being captured by their opponents.

There have been numerous advancements in the field of single-phase reach-avoid games, in which agents only need to reach one target region. These game scenarios encompass various problems such as target defense games [4]–[7], active defense problems [8]–[10], perimeter defense problems [11]–[13], and so on. In these studies, geometric tools are usually useful to solve the game of kind and obtain equilibrium strategies. The employed geometric shapes include Voronoi diagrams [14], Apollonian circles [5], [6], [15], and Cartesian ovals [7], [16]. Leveraging geometric tools significantly simplifies the analysis of single-phase reach-avoid games involving players with simple motion.

Researchers have also conducted a series of studies on two-phase reach-avoid games, also known as capture-the-flag games in literature [17]–[21]. In these games, a player must sequentially reach a target region (referred to as the flag region in literature) and a safe region (referred to as the return region). [17] and [18] investigate a two-player capture-the-flag game played within a rectangular region containing a circular target region and a strip safe region. Numerical Hamilton-Jacobi reachable set calculations are employed to construct winning regions and winning strategies. In [19], a similar game problem is considered using the conventional method of differential games [22]. The first and second phase are analyzed separately in cases of zero and positive capture radius scenarios, but no solution is provided for the overall game problem. [20] solves the game of kind and game of degree for a specific case involving a point target region and a point safe region. Distances between several key points play a major role in the solving process. [21] studies a game problem where the target is a point and the safe region is a half plane. The optimal strategy for the two-phase game is linked to the solution of a constrained nonlinear optimization problem. All these studies focus on two player games.

This paper aims to discuss the potential application of geometric tools in two-phase reach-avoid games, addressing a gap in the existing literature. We will demonstrate the utility of ellipses in analyzing two-phase multiplayer reach-avoid games. Specifically, the game of kind in certain game scenarios can be directly solved using ellipses. Additionally, we introduce the concept of analogy to dominance regions in single-phase games into two-phase games. This concept can be used to analyze general two-phase reach-avoid games. We will illustrate that the boundary of dominance regions can be constructed using ellipses.

The paper is organized as follows. Section II provides the problem formulation. A special reach-avoid game with a point safe region is analyzed in Section III. Next, in Section IV, we discuss a game scenario that requires cooperative defense. The dominance region is introduced and characterized in Section VI. Finally, we provide concluding remarks and discuss future work in Section VI.

II. Problem Formulation

We consider a two phase reach-avoid game that takes place in $\mathbb{R}^2$. The game involves two key regions in the plane: a target region $\mathcal{G} \subset \mathbb{R}^2$, and a safe region $\mathcal{S} \subset \mathbb{R}^2$. Two team of players engage in the game. A thief aims to steal a treasure from the target region and then reach the safe region. The guarders try to prevent the thief via interception. Fig. 1 provides an illustration of the game problem.

All the players are assumed to have first-order dynamics. Let $\mathbf{x}_T$ and $\mathbf{x}_{D_i}$ be the positions of the thief and the ith guarder, respectively. The equations of motion are given by

$$\begin{aligned} \dot{\mathbf{x}}_T &= v_T \mathbf{u}_T, \\ \dot{\mathbf{x}}_{D_i} &= v_{D_i} \mathbf{u}_{D_i},\ i = 1, \dots, N_D, \end{aligned} \tag{1}$$

This work was supported by the Science and Technology Innovation 2030-Key Project of "New Generation Artificial Intelligence" under Grant 2020AAA0108200.

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where N_D is the number of guarders; v_T and v_{D_i} are the speeds of the thief and guarder i; $\mathbf{u}_T$ and $\mathbf{u}_{D_i}$ are their control inputs respectively, satisfying that $\|\mathbf{u}_T\|_2 \leq 1$ and $\|\mathbf{u}_{D_i}\|_2 \leq 1$. The initial positions of the thief and the ith guarder are denoted as $\mathbf{x}_T^0$ and $\mathbf{x}_{D_i}^0$, respectively. The state of the system consists of positions of all players and is denoted as $\mathbf{X} \in \mathbb{R}^{2(N_D+1)}$. The initial state is denoted as $\mathbf{X}^0 = [\mathbf{x}_T^{0\top}, \mathbf{x}_{D_1}^{0\top}, \ldots, \mathbf{x}_{D_{N_D}}^{0\top}]^\top$. The capture radius of guarder D_i is denoted as r_{D_i}. If $\|\mathbf{x}_T - \mathbf{x}_{D_i}\|_2 \leq r_{D_i}$, the thief is captured by D_i.

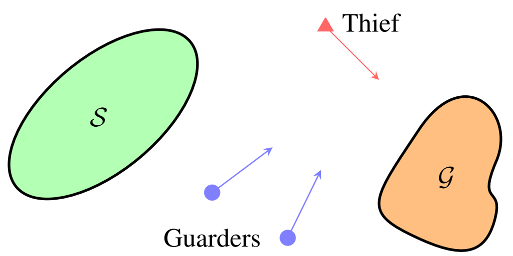

Fig. 1. Illustration of the two-phase reach-avoid game.

It is assumed that the state is fully observable to all players. Each player can make decisions based on all the available state information up to the current moment. However, they do not have access to the current control of their opponents.

This paper focus on the game of kind. The game terminates when the thief is captured by the guarders or when it reaches $\mathcal{S}$ passing through $\mathcal{G}$. The thief wins the game if it brings the treasure to the safe region without being captured, or in other words, if there exists $t_f \in \mathbb{R}$ such that: $\mathbf{x}_T(t_f) \in \mathcal{S}$; $\exists t_c \in [0, t_f]$, $\mathbf{x}_T(t_c) \in \mathcal{G}$; $\forall t \in [0, t_f]$, $\min_{1\leq i\leq N_D} \|\mathbf{x}_T(t) - \mathbf{x}_{D_i}(t)\|_2 - r_{D_i} > 0$. Otherwise, the guarders win the game.

We aim to investigate the utilization of geometric methods in the analysis of two-phase game problems. We would like to show that with the help of ellipses, solutions to the game of kind can be easily obtained in certain problems with specific settings. Additionally, we would like to generalize the concept of dominant regions to two-phase game problems, and obtain the representation of the thief's dominant region using geometric tools.

III. Game with Point Safe Region

In this section, we consider the specific case where the safe region is composed of a single point, which is denoted as $\mathbf{x}_S$.

In the case of point safe region, the guarders can directly move towards the position $\mathbf{x}_S$. Let's consider zero capture radii first. The shortest time for the guarders to reach $\mathbf{x}_S$ can be calculated by

$$t_m = \min_{1\leq i\leq N_D} \frac{\|\mathbf{x}_S - \mathbf{x}_{D_i}^0\|_2}{v_{D_i}}. \quad (2)$$

In order to win the game, the thief must steal the treasure and bring it to $\mathbf{x}_S$ before time t_m. For a point $\mathbf{x} \in \mathcal{G}$, the shortest time for the thief to arrive at $\mathbf{x}_S$ passing through $\mathbf{x}$ is

$$t(\mathbf{x}_T, \mathbf{x}, \mathbf{x}_S) = \frac{\|\mathbf{x}_S - \mathbf{x}\|_2 + \|\mathbf{x}_T - \mathbf{x}\|_2}{v_T}.$$

The thief can bring the treasure to $\mathbf{x}_S$ before t_m if and only if there is a point $\mathbf{x}$ in $\mathcal{G}$ such that $t(\mathbf{x}_T^0, \mathbf{x}, \mathbf{x}_S) < t_m$. This condition can be expressed equivalently as

$$\inf_{\mathbf{x}\in\mathcal{G}} \|\mathbf{x}_S - \mathbf{x}\|_2 + \|\mathbf{x}_T^0 - \mathbf{x}\|_2 < v_T t_m. \quad (3)$$

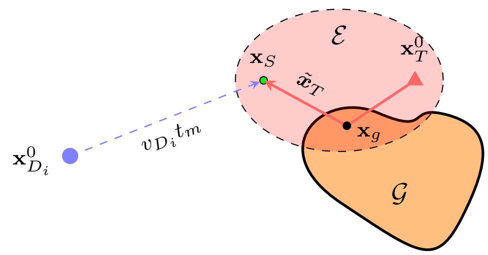

Fig. 2. In this example, the red ellipse region $\mathcal{E}(\mathbf{x}_T^0, \mathbf{x}_S, v_T t_m)$ intersects with the target region $\mathcal{G}$. The thief can successfully steal the treasure at $\mathbf{x}_g$ and then reach the safe region by travelling along the trajectory $\tilde{\boldsymbol{x}}_T$ which is shown with red lines.

The inequality (3) has a geometric explanation. Denote the interior of the ellipse with focal points $\mathbf{x}_1$ and $\mathbf{x}_2$ and a major axis length of d as $\mathcal{E}(\mathbf{x}_1, \mathbf{x}_2, d)$. This region can be expressed explicitly as

$$\mathcal{E}(\mathbf{x}_1, \mathbf{x}_2, d) = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x} - \mathbf{x}_1\|_2 + \|\mathbf{x} - \mathbf{x}_2\|_2 < d\}. \quad (4)$$

If $\|\mathbf{x}_1 - \mathbf{x}_2\|_2 \geq d$, $\mathcal{E}(\mathbf{x}_1, \mathbf{x}_2, d)$ is a empty set. It can be deduced that the condition (3) is equal to

$$\mathcal{E}(\mathbf{x}_T^0, \mathbf{x}_S, v_T t_m) \cap \mathcal{G} \neq \emptyset, \quad (5)$$

or in other words, the target region intersects with an open ellipse region. An illustration of the geometric condition (5) is shown in Fig. 2. We have the following conclusion.

Theorem 1. *Assume that $\mathcal{S} = \{\mathbf{x}_S\}$, $\mathcal{G}$ is a connected non-singleton set, and $r_{D_i} = 0$ for all i. Let t_m be defined by (2), and $\mathcal{E}(\mathbf{x}_T^0, \mathbf{x}_S, v_T t_m)$ be defined by (4). The thief can win the reach-avoid game if and only if the target region intersects with $\mathcal{E}(\mathbf{x}_T^0, \mathbf{x}_S, v_T t_m)$, namely, (5) is satisfied.*

Proof. Necessity: If $\mathcal{E}(\mathbf{x}_T^0, \mathbf{x}_S, v_T t_m) \cap \mathcal{G}$ is empty, then regardless of the strategy the thief adopts, at least one guarder can reach $\mathbf{x}_S$ before the thief bring the treasure to the safe region. Thus, the thief cannot win.

Sufficiency: Choose a point $\mathbf{x}_g \in \mathcal{E}(\mathbf{x}_T^0, \mathbf{x}_S, v_T t_m) \cap \mathcal{G}$. Let $\tilde{\boldsymbol{x}}_T : [0, t_m) \to \mathbb{R}^2$ be a trajectory of the thief such that $\tilde{\boldsymbol{x}}_T(t_1) = \mathbf{x}_g$ and $\tilde{\boldsymbol{x}}_T(t_2) = \mathbf{x}_S$, where $t_1 = \frac{\|\mathbf{x}_g - \mathbf{x}_T^0\|_2}{v_T}$ and $t_2 = t_1 + \frac{\|\mathbf{x}_g - \mathbf{x}_S\|_2}{v_T}$. Obviously, $\tilde{\boldsymbol{x}}_T$ consists of two straight line segments. An illustration of such a trajectory is given in Fig. 2. We claim that the thief can win the game by travelling along a trajectory sufficiently close to $\tilde{\boldsymbol{x}}_T$. First, consider the guarders with speeds equal to or larger than v_T. The thief can reach any point on trajectory $\tilde{\boldsymbol{x}}_T$ before these guarders. Otherwise, there is a guarder that can reach $\mathbf{x}_S$ before the thief by first approaching a point on $\tilde{\boldsymbol{x}}_T$ and then following this trajectory. Thus, these guarders cannot prevent the thief from bring the treasure to $\mathcal{S}$ along $\tilde{\boldsymbol{x}}_T$. Next, consider the guarders slower than the thief. Denote the index

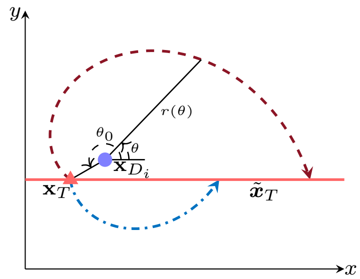

Fig. 3. Illustration of bypassing paths. The clockwise and counterclockwise paths are shown with a dashdotted line and a dashed line, respectively. In this example, the clockwise bypassing path is the shorter one.

set of these guarders as $\mathcal{N}_s$. The maximum speed of these guarders is $\bar{v}_s = \max_{i\in\mathcal{N}_s} v_{D_i}$. Such a guarder may reach a point on $\tilde{\boldsymbol{x}}_T$ before the thief. However, the thief can avoid being captured by slightly changing the trajectory to bypass a slower guarder, e.g. D_s, once its distance from $\mathbf{x}_{D_s}$ is less than an adjustable parameters δ. Denote the start time of the bypassing maneuvering as t_p. The bypassing path can be expressed in polar coordinates, with $\mathbf{x}_{D_s}(t_p)$ as the origin, as $r(\theta) = \delta e^{k_p(\theta-\theta_0)}$, where θ_0 is the direction angle of vector $\tilde{\boldsymbol{x}}_T(t_p) - \mathbf{x}_{D_s}(t_p)$; $k_p = \frac{\bar{v}_s}{\sqrt{v_T^2-\bar{v}_s^2}}$ if the thief bypasses clockwise and $k_p = -\frac{\bar{v}_s}{\sqrt{v_T^2-\bar{v}_s^2}}$ otherwise. An illustration of such a bypassing path is shown in Fig. 3. The bypassing path returns to $\tilde{\boldsymbol{x}}_T$ while θ changes by less than 2π. In most cases, the thief should select the clockwise or counterclockwise direction that results in a shorter bypassing path. However, there is a special scenario where the thief needs to avoid a guarder near $\mathbf{x}_g$. In this case, the thief should choose a direction that allows the bypassing path to intersect with $\mathcal{G}$. It can be checked that $r(\theta)$ satisfies the differential equation $\frac{1}{\bar{v}_s}\left|\frac{dr}{d\theta}\right| = \frac{1}{v_T}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$. The time needed by the thief to reach a position on the path with angle θ is

$$\begin{aligned}\frac{1}{v_T}\left|\int_{\theta_0}^{\theta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta\right| &= \frac{1}{\bar{v}_s}\left|\int_{\theta_0}^{\theta}\frac{dr}{d\theta}d\theta\right| \\ &= \frac{r(\theta)-r(\theta_0)}{\bar{v}_s} \le \frac{r(\theta)}{v_{D_s}}.\end{aligned}$$

It follows that the thief can reach any position on the bypassing path before guarder D_s. Thus, the thief can bypass a slower guarder by travelling along a bypassing path. By adjusting the parameter δ, the bypassing time can be made arbitrarily small. It may also be possible that the thief encounters another slower guarder while following a bypassing path. In such a case, the parameter δ of the new bypassing path should be small enough than that of the current one. If the parameters of all bypassing paths are chosen to be sufficiently small, the thief will only need to bypass each slower guarder at most once on each straight line of $\tilde{\boldsymbol{x}}_T$. As a result, the thief can reach $\mathbf{x}_S$ before t_m and win the game by travelling along a trajectory arbitrarily close to $\tilde{\boldsymbol{x}}_T$. $\square$

Theorem 1 is valid for zero capture radii but regardless of the guarders' speeds. If we consider only the guarders with speeds not smaller than v_T, a similar conclusion holds for non-zero capture radii. Hereafter, the shortest time $\min_{1\le i\le N_D}\frac{\|\mathbf{x}-\mathbf{x}_{D_i}\|_2-r_{D_i}}{v_{D_i}}$ will be denoted as $\tilde{t}(\mathbf{x},\mathbf{X})$ for brevity.

Theorem 2. *Assume that $\mathcal{S} = \{\mathbf{x}_S\}$, and $v_{D_i} \ge v_T$ for all i. The thief can win the reach-avoid game if and only if*

$$\mathcal{E}\left(\mathbf{x}_T^0, \mathbf{x}_S, v_T\tilde{t}\left(\mathbf{x}_S,\mathbf{X}^0\right)\right) \cap \mathcal{G} \neq \emptyset. \tag{6}$$

Proof. The proof of necessity is the same as that of Theorem 1. For sufficiency, assume that (6) is satisfied and consider the trajectory $\tilde{\boldsymbol{x}}_T$ defined in the proof of Theorem 1. The thief will not be captured if it travels along $\tilde{\boldsymbol{x}}_T$. Otherwise, there is a guarder D_i that has a strategy such that $\|\mathbf{x}_{D_i}(t_c) - \tilde{\boldsymbol{x}}_T(t_c)\|_2 \le r_{D_i}$ for $t_c < \tilde{t}\left(\mathbf{x}_S,\mathbf{X}^0\right)$. Then, D_i can arrive within the range of r_{D_i} around $\mathbf{x}_S$ before $\tilde{t}\left(\mathbf{x}_S,\mathbf{X}^0\right)$ by first arriving within the range of r_{D_i} around $\tilde{\boldsymbol{x}}_T(t_c)$ and then following a trajectory parallel to $\tilde{\boldsymbol{x}}_T$. Thus, it is satisfied that $v_{D_i}\tilde{t}\left(\mathbf{x}_S,\mathbf{X}^0\right) > \|\mathbf{x}_S - \mathbf{x}_{D_i}^0\|_2 - r_{D_i}$, which contradicts the definition of $\tilde{t}\left(\mathbf{x}_S,\mathbf{X}^0\right)$. Therefore, the thief can win the game by travelling along $\tilde{\boldsymbol{x}}_T$. $\square$

Theorems 1 and 2 show that the solution to the game of kind of the reach-avoid game with point safe region can be determined by checking a geometric condition: whether the target region intersects with an open ellipse region. Thus, ellipses play a similar role as Apollonius circles in single-phase reach-avoid games.

IV. Game with Two Guarders

In this section, we discuss a two-phase reach-avoid game with a non-singleton safe region. It will be shown that ellipses are still useful in certain scenarios.

Let $N_D = 2$. Assume that $\mathcal{S}$ is a connected closed convex set. The boundary of $\mathcal{S}$ is denoted as $\partial\mathcal{S}$. The guarders' speeds are such that $v_{D_i} \ge v_T$. Let the initial state is such that the curve

$$\left\{\mathbf{x}\in\mathbb{R}^2 \;\middle|\; \frac{\|\mathbf{x}-\mathbf{x}_{D_1}^0\|_2 - r_{D_1}}{v_{D_1}} = \frac{\|\mathbf{x}-\mathbf{x}_{D_2}^0\|_2 - r_{D_2}}{v_{D_2}}\right\},$$

which can be proved to be convex [7], intersects $\partial\mathcal{S}$ at two points $\mathbf{x}_1$ and $\mathbf{x}_2$. The line segment with endpoints $\mathbf{x}_1$ and $\mathbf{x}_2$ divides $\mathcal{S}$ into two convex regions $\mathcal{S}_1$ and $\mathcal{S}_2$, as shown in Fig. 4. It is assumed that $\mathcal{S}_1$ is the region close to $\mathbf{x}_{D_1}^0$. Thus, the guarder D_i can reach the boundary $\partial\mathcal{S}\cap\mathcal{S}_i$ in less time than the other guarder.

To analyze the reach-avoid game, define a time function $h_i : \mathcal{S}\times\mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$ for each guarder such that

$$\begin{aligned}&h_i(\mathbf{x},\mathbf{x}_T,\mathbf{x}_{D_i}) \\ &\quad= \inf_{\mathbf{z}\in\mathcal{G}}\frac{\|\mathbf{x}-\mathbf{z}\|_2+\|\mathbf{x}_T-\mathbf{z}\|_2}{v_T} - \frac{\|\mathbf{x}-\mathbf{x}_{D_i}\|_2-r_{D_i}}{v_{D_i}}.\end{aligned} \tag{7}$$

If $h_i(\mathbf{x},\mathbf{x}_T^0,\mathbf{x}_{D_i}^0) < 0, \forall\mathbf{x}\in\mathcal{S}$, then the thief can bring the treasure to $\mathbf{x}$ without being captured by guarder D_i, as demonstrated by Theorem 2. In this section, we consider the case where two guarders must cooperative with each other to win the game. Specifically, we assume that $\min_{\mathbf{x}\in\mathcal{S}} h_i(\mathbf{x},\mathbf{x}_T^0,\mathbf{x}_{D_i}^0) < 0$ for $i = 1, 2$, namely, no guarder

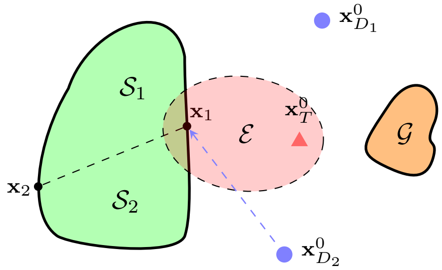

Fig. 4. The safe region is divided into $\mathcal{S}_1$ and $\mathcal{S}_2$. Under the assumption of Theorem 4, the ellipse region $\mathcal{E}\left(\mathbf{x}_T^0, \mathbf{x}_1, v_T\tilde{t}\left(\mathbf{x}_1, \mathbf{X}^0\right)\right)$ can be used to solve the game of kind. In this example, the target region does not intersect with $\mathcal{E}\left(\mathbf{x}_T^0, \mathbf{x}_1, v_T\tilde{t}\left(\mathbf{x}_1, \mathbf{X}^0\right)\right)$. Thus, the thief cannot win.

can successfully defend against the thief alone. Throughout this section, we make the following assumption.

Assumption 1. *There exists a unique point* $\mathbf{x}^*$ *such that* $h_i(\mathbf{x}^*, \mathbf{x}_T, \mathbf{x}_{D_i}) = \min_{\mathbf{x}\in\mathcal{S}_i} h_i(\mathbf{x}, \mathbf{x}_T, \mathbf{x}_{D_i})$, $\forall \mathbf{x}_T$, $\forall \mathbf{x}_{D_i}, i = 1, 2$.

With respect to the game of kind, the following conclusion holds.

Theorem 3. *If* $\min_{\mathbf{x}\in\mathcal{S}} h_i(\mathbf{x}, \mathbf{x}_T^0, \mathbf{x}_{D_i}^0) < 0$ *for* $i = 1, 2$*, and Assumption 1 holds, then the thief can win the game if and only if* $\min_{\mathbf{x}\in\mathcal{S}} \max_i h_i(\mathbf{x}, \mathbf{x}_T^0, \mathbf{x}_{D_i}^0) < 0$*. The condition is equal to that* $\min_{\mathbf{x}\in\mathcal{S}_i} h_i(\mathbf{x}, \mathbf{x}_T^0, \mathbf{x}_{D_i}^0) < 0$ *for* $i = 1$ *or* 2.

Proof. We first prove the equivalence of two conditions. In $\mathcal{S}_1$, it holds that $\frac{\|\mathbf{x}-\mathbf{x}_{D_1}^0\|_2 - r_{D_1}}{v_{D_1}} < \frac{\|\mathbf{x}-\mathbf{x}_{D_2}^0\|_2 - r_{D_2}}{v_{D_2}}$. Thus, $h_1(\mathbf{x}, \mathbf{x}_T^0, \mathbf{x}_{D_1}^0) > h_2(\mathbf{x}, \mathbf{x}_T^0, \mathbf{x}_{D_2}^0)$ if $\mathbf{x} \in \mathcal{S}_1$, which means that $\max_i h_i(\mathbf{x}, \mathbf{x}_T^0, \mathbf{x}_{D_i}^0) = h_1(\mathbf{x}, \mathbf{x}_T^0, \mathbf{x}_{D_1}^0)$ in $\mathcal{S}_1$. Similarly, $\max_i h_i(\mathbf{x}, \mathbf{x}_T^0, \mathbf{x}_{D_i}^0) = h_2(\mathbf{x}, \mathbf{x}_T^0, \mathbf{x}_{D_2}^0)$ in $\mathcal{S}_2$. It follows that

$$\min_{\mathbf{x}\in\mathcal{S}} \max_i h_i(\mathbf{x}, \mathbf{x}_T^0, \mathbf{x}_{D_i}^0) = \min_i \min_{\mathbf{x}\in\mathcal{S}_i} h_i(\mathbf{x}, \mathbf{x}_T^0, \mathbf{x}_{D_i}^0).$$

The equivalence of two conditions is then obvious. Below, we will prove sufficiency and necessity separately.

Sufficiency: If $\min_{\mathbf{x}\in\mathcal{S}} \max_i h_i(\mathbf{x}, \mathbf{x}_T^0, \mathbf{x}_{D_i}^0) < 0$, there exists $\mathbf{x}_S \in \mathcal{S}$ such that $h_i(\mathbf{x}_S, \mathbf{x}_T^0, \mathbf{x}_{D_i}^0) < 0$ for $i = 1$ and 2. It follows from Theorem 2 that the thief can successfully bring the treasure to $\mathbf{x}$ without being captured. Thus, the thief wins the game.

Necessity: If $\min_{\mathbf{x}\in\mathcal{S}_i} h_i(\mathbf{x}, \mathbf{x}_T^0, \mathbf{x}_{D_i}^0) \geq 0$, it can be proved that guarder D_i can prevent the thief from bringing the treasure into $\mathcal{S}_i$. Let $\mathbf{x}_i^*(\mathbf{x}_T, \mathbf{x}_{D_i})$ be such that $h_i\left(\mathbf{x}_i^*(\mathbf{x}_T, \mathbf{x}_{D_i}), \mathbf{x}_T, \mathbf{x}_{D_i}\right) = \min_{\mathbf{x}\in\mathcal{S}_i} h_i(\mathbf{x}, \mathbf{x}_T, \mathbf{x}_{D_i})$. As assumed in Assumption 1, $\mathbf{x}_i^*(\mathbf{x}_T, \mathbf{x}_{D_i})$ is the unique minimum point. Thus, according to Danskin's Theorem [23], it holds that

$$\frac{\partial}{\partial \mathbf{x}_T} \min_{\mathbf{x}\in\mathcal{S}_i} h_i(\mathbf{x}, \mathbf{x}_T, \mathbf{x}_{D_i}) = \frac{\partial}{\partial \mathbf{x}_T} h_i\left(\mathbf{x}_i^*, \mathbf{x}_T, \mathbf{x}_{D_i}\right)$$

and

$$\frac{\partial}{\partial \mathbf{x}_{D_i}} \min_{\mathbf{x}\in\mathcal{S}_i} h_i(\mathbf{x}, \mathbf{x}_T, \mathbf{x}_{D_i}) = \frac{\partial}{\partial \mathbf{x}_{D_i}} h_i\left(\mathbf{x}_i^*, \mathbf{x}_T, \mathbf{x}_{D_i}\right).$$

If guarder D_i moves towards $\mathbf{x}_i^*(\mathbf{x}_T, \mathbf{x}_{D_i})$, it is satisfied that

$$\begin{aligned}
&\frac{d}{dt} \min_{\mathbf{x}\in\mathcal{S}_i} h_i(\mathbf{x}, \mathbf{x}_T, \mathbf{x}_{D_i}) \\
&= v_{D_i} \mathbf{u}_{D_i}^\top \frac{\partial h_i\left(\mathbf{x}_i^*, \mathbf{x}_T, \mathbf{x}_{D_i}\right)}{\partial \mathbf{x}_{D_i}} + v_T \mathbf{u}_T^\top \frac{\partial h_i\left(\mathbf{x}_i^*, \mathbf{x}_T, \mathbf{x}_{D_i}\right)}{\partial \mathbf{x}_T} \\
&= 1 + \frac{d}{dt} \inf_{\mathbf{z}\in\mathcal{G}} \frac{\|\mathbf{x}^* - \mathbf{z}\|_2 + \|\mathbf{x}_T - \mathbf{z}\|_2}{v_T} \geq 0,
\end{aligned}$$

no matter what strategy the thief adopts. Therefore, when the thief arrives in $\mathcal{G}$, it is true that

$$\min_{\mathbf{x}\in\mathcal{S}_i} \frac{\|\mathbf{x}_T - \mathbf{x}\|_2}{v_T} - \frac{\|\mathbf{x} - \mathbf{x}_{D_i}\|_2 - r_{D_i}}{v_{D_i}} = \min_{\mathbf{x}\in\mathcal{S}_i} h_i(\mathbf{x}, \mathbf{x}_T, \mathbf{x}_{D_i}),$$

the right side of which is non-negative. Thus, according to the result of single-phase reach-avoid game [24], D_i can successfully prevent the thief from reaching $\mathcal{S}_i$. Therefore, the thief cannot bring the treasure to $\mathcal{S}_1$ or $\mathcal{S}_2$, which means that the thief loses the game. $\square$

It can be seen that guarder D_i has an advantage to protect the region $\mathcal{S}_i$. For the thief, the dividing points between $\mathcal{S}_1$ and $\mathcal{S}_2$ could potentially be a weak point in the defense of the guarders. Thus, it may be a optimal choice for the thief to bring the treasure to one of the dividing points, $\mathbf{x}_1$ or $\mathbf{x}_2$. If it is indeed the case, the ellipse regions $\mathcal{E}\left(\mathbf{x}_T^0, \mathbf{x}_1, v_T\tilde{t}\left(\mathbf{x}_1, \mathbf{X}^0\right)\right)$ and $\mathcal{E}\left(\mathbf{x}_T^0, \mathbf{x}_2, v_T\tilde{t}\left(\mathbf{x}_2, \mathbf{X}^0\right)\right)$ can be used to solve the game of kind. Below, we will show that the aforementioned hypothesis holds true in a certain scenario.

Let $f_i : \mathcal{S}_i \times \mathcal{G} \to \mathbb{R}$ be defined by $f_i(\mathbf{x}, \mathbf{z}) = \frac{\|\mathbf{x}-\mathbf{z}\|_2}{v_T} - \frac{\|\mathbf{x}-\mathbf{x}_{D_i}^0\|_2 - r_{D_i}}{v_{D_i}}$. If $f_i(\mathbf{x}, \mathbf{z}) < 0$, then the thief starts from $\mathbf{z}$ can arrive at $\mathbf{x} \in \mathcal{S}_i$ before guarder D_i. We have the following conclusion.

Theorem 4. *Assume that Assumption 1 holds. If for all* $i \in \{1, 2\}$, $\mathbf{z} \in \mathcal{G}$*, it holds that* $f_i(\mathbf{x}_1, \mathbf{z}) = \min_{\mathbf{x}\in\mathcal{S}_i} f_i(\mathbf{x}, \mathbf{z})$*, then the thief can win the game if and only if*

$$\mathcal{E}\left(\mathbf{x}_T^0, \mathbf{x}_1, v_T\tilde{t}\left(\mathbf{x}_1, \mathbf{X}^0\right)\right) \cap \mathcal{G} \neq \emptyset. \tag{8}$$

Proof. At $\mathbf{x}_1$, $\tilde{t}\left(\mathbf{x}_1, \mathbf{X}^0\right) = \frac{\|\mathbf{x}-\mathbf{x}_{D_1}^0\|_2 - r_{D_1}}{v_{D_1}} = \frac{\|\mathbf{x}-\mathbf{x}_{D_2}^0\|_2 - r_{D_2}}{v_{D_2}}$. According to the assumptions, it holds that

$$\begin{aligned}
\min_{\mathbf{x}\in\mathcal{S}_i} h_i(\mathbf{x}, \mathbf{x}_T^0, \mathbf{x}_{D_i}^0) &= \min_{\mathbf{x}\in\mathcal{S}_i} \inf_{\mathbf{z}\in\mathcal{G}} \frac{\|\mathbf{x}_T^0 - \mathbf{z}\|_2}{v_T} + f_i(\mathbf{x}, \mathbf{z}) \\
&= \inf_{\mathbf{z}\in\mathcal{G}} \frac{\|\mathbf{x}_T^0 - \mathbf{z}\|_2}{v_T} + f_i(\mathbf{x}_1, \mathbf{z}) \\
&= \inf_{\mathbf{z}\in\mathcal{G}} \frac{\|\mathbf{x}_T^0 - \mathbf{z}\|_2 + \|\mathbf{z} - \mathbf{x}_1\|_2}{v_T} - \tilde{t}\left(\mathbf{x}_1, \mathbf{X}^0\right).
\end{aligned}$$

Thus, the condition $\min_{\mathbf{x}\in\mathcal{S}_i} h_i(\mathbf{x}, \mathbf{x}_T^0, \mathbf{x}_{D_i}^0) < 0$ means that

$$\inf_{\mathbf{z}\in\mathcal{G}} \|\mathbf{x}_T^0 - \mathbf{z}\|_2 + \|\mathbf{z} - \mathbf{x}_1\|_2 < v_T\tilde{t}\left(\mathbf{x}_1, \mathbf{X}^0\right),$$

which is equal to (8). The conclusion follows from Theorem 3. $\square$

Theorem 4 shows that under certain conditions, the point $\mathbf{x}_1$ is a critical defense position. The game of kind is then determined by the geometric relationship between the target region and the ellipse region $\mathcal{E}(\mathbf{x}_T^0, \mathbf{x}_1, v_T\tilde{t}_1)$.

V. Thief's Dominance Region

For general game settings, the ellipses cannot be directly used to solve the game of kind. In this section, we generalize the ellipse regions to the concept of dominance regions in two-phase reach-avoid games, which will be useful in the analysis of general game problems. Specifically, we consider only the case where $v_{D_i} \geq v_T, \forall i \in \{1, \dots, N_D\}$.

Definition 1. *The dominance region $\mathcal{D}(\mathbf{X}; \mathcal{S})$ of the thief is defined by*

$$\mathcal{D}(\mathbf{X}; \mathcal{S}) = \big\{\mathbf{x} \in \mathbb{R}^2 \mid \exists \mathbf{x}_S \in \mathcal{S}, \|\mathbf{x} - \mathbf{x}_T\|_2 + \|\mathbf{x} - \mathbf{x}_S\|_2 < v_T\tilde{t}(\mathbf{x}_S, \mathbf{X})\big\}.$$

The dominance region $\mathcal{D}(\mathbf{X}; \mathcal{S})$ is a set where the thief has an open-loop strategy to pass through a point within the set and then reach the safe area without being captured. It is obvious that when $\mathcal{S}$ contains only one point, $\mathcal{D}(\mathbf{X}; \mathcal{S})$ is the ellipse region $\mathcal{E}(\mathbf{x}_T^0, \mathbf{x}_S, v_T t_m)$. When $\mathcal{S}$ is not a singleton, it is no longer an ellipse region. We can immediately obtain the following conclusion.

Theorem 5. *The thief can win the game if $\mathcal{D}(\mathbf{X}^0; \mathcal{S}) \cap \mathcal{G} \neq \emptyset$.*

Proof. If $\mathcal{D}(\mathbf{X}^0; \mathcal{S}) \cap \mathcal{G} \neq \emptyset$, then there exist a point $\mathbf{x}_g \in \mathcal{G}$ and a point $\mathbf{x}_S \in \mathcal{S}$ such that

$$\|\mathbf{x}_g - \mathbf{x}_T^0\|_2 + \|\mathbf{x}_g - \mathbf{x}_S\|_2 < v_T\tilde{t}\left(\mathbf{x}_S, \mathbf{X}^0\right).$$

According to the proof of Theorem 2, the thief can win the game by travelling along the trajectory $\tilde{\boldsymbol{x}}_T$ defined in the proof Theorem 1. ☐

The investigation into the necessity of the intersection condition in Theorem 5 will be pursued as a future endeavor.

To study the properties of the dominance boundary, it is useful to express the dominance region using ellipse regions. We have the following result.

Lemma 1. *The dominance region of the thief satisfies*

$$\mathcal{D}(\mathbf{X}; \mathcal{S}) \setminus \mathcal{S} = \cup_{\mathbf{x} \in \partial\mathcal{S}} \mathcal{E}\left(\mathbf{x}_T, \mathbf{x}, v_T\tilde{t}(\mathbf{x}, \mathbf{X})\right) \setminus \mathcal{S}. \quad (9)$$

Proof. Denote the right-hand side of (9) as $\mathcal{E}_U$. It is obvious that $\mathcal{E}\left(\mathbf{x}_T, \mathbf{x}, v_T\tilde{t}_{\mathbf{x}}\right) \subset \mathcal{D}(\mathbf{X}; \mathcal{S})$ if $\mathbf{x} \in \partial\mathcal{S}$. Thus, $\mathcal{E}_U \subset \mathcal{D}(\mathbf{X}; \mathcal{S}) \setminus \mathcal{S}$.

If $\mathbf{z} \in \mathcal{D}(\mathbf{X}; \mathcal{S}) \setminus \mathcal{S}$, there exists a point $\mathbf{x} \in \mathcal{S}$, such that $\|\mathbf{z} - \mathbf{x}_T\|_2 + \|\mathbf{z} - \mathbf{x}\|_2 < v_T\tilde{t}(\mathbf{x}, \mathbf{X})$. According to the proof of Theorem 2, the thief can reach $\mathbf{x}$ without being captured by travelling along a trajectory passing through $\mathbf{z}$. Let $\mathbf{y} \in \partial\mathcal{S}$ be the intersection point between $\partial\mathcal{S}$ and the line segment connecting $\mathbf{z}$ and $\mathbf{x}$. The thief can also reach $\mathbf{y}$ passing through $\mathbf{z}$ without being captured. Thus, it holds that $\mathbf{z} \in \mathcal{E}(\mathbf{x}_T, \mathbf{y}, v_T\tilde{t}(\mathbf{y}, \mathbf{X}))$. It follows that $\mathcal{D}(\mathbf{X}; \mathcal{S}) \setminus \mathcal{S} \subset \mathcal{E}_U$. ☐

We can use the above conclusion to characterize the boundary of the dominance region, which will be referred to as the dominance boundary. Before proceeding, it is helpful to clarify some terminology that will be used. Let $\boldsymbol{x}_s : \mathcal{I}_S \to \mathbb{R}^2$ be a parametric representation of $\partial\mathcal{S}$, where $\mathcal{I}_S$ is an interval of the real line $\mathbb{R}$. Assume that $\boldsymbol{x}_s$ is piecewise continuously differentiable. Define a function $\tilde{t}_s : \mathcal{I}_S \to \mathbb{R}$ such that $\tilde{t}_s(s) = \tilde{t}(\boldsymbol{x}_s(s), \mathbf{X}^0)$ for $s \in \mathcal{I}_S$. It can be easily seen that $\tilde{t}_s$ is also piecewise continuously differentiable. The right (left) derivative of $\boldsymbol{x}_s$ at $s \in \mathcal{I}_s$, if existing, is denoted as $\boldsymbol{v}_s^+(s)$ (respectively, $\boldsymbol{v}_s^-(s)$). Similarly, the right (left) derivative of $\tilde{t}_s$ is denoted as $\tilde{t}_s^+$ (respectively, $\tilde{t}_s^-$). We focus on the part of dominance boundary outside the safe region, namely, $\partial D(\mathbf{X}^0; \mathcal{S}) \setminus \mathcal{S}$, which will be denoted as $\mathcal{B}_D(\mathbf{X}^0)$.

Lemma 2. *For each $\mathbf{x} \in \mathcal{B}_D(\mathbf{X}^0)$, there exists $s \in \mathcal{I}_S$ such that $\mathbf{x} \in \partial\mathcal{E}\left(\mathbf{x}_T^0, \boldsymbol{x}_s(s), v_T\tilde{t}_s(s)\right)$. It holds that*

$$\frac{(\boldsymbol{x}_s(s) - \mathbf{x})^\top}{\|\mathbf{x} - \boldsymbol{x}_s(s)\|_2}\boldsymbol{v}_s^+(s) \geq v_T\tilde{t}_s^+(s) \quad (10)$$

when the right derivatives of $\boldsymbol{x}_s$ and $\tilde{t}_s$ exist, and

$$\frac{(\boldsymbol{x}_s(s) - \mathbf{x})^\top}{\|\mathbf{x} - \boldsymbol{x}_s(s)\|_2}\boldsymbol{v}_s^-(s) \leq v_T\tilde{t}_s^-(s) \quad (11)$$

when the left derivatives exist.

Proof. Consider a point $\mathbf{x} \in \mathcal{B}_D(\mathbf{X}^0)$. It is obvious that $\mathbf{x} \in \partial\mathcal{E}\left(\mathbf{x}_T^0, \boldsymbol{x}_s(s), v_T\tilde{t}_s(s)\right)$ for a certain $s \in \mathcal{I}_s$. Otherwise, $\mathbf{x}$ is an interior point of $\mathcal{E}\left(\mathbf{x}_T^0, \boldsymbol{x}_s(s), v_T\tilde{t}_s(s)\right) \subset \mathcal{D}(\mathbf{X}^0; \mathcal{S})$. It follows that

$$\|\mathbf{x}_T^0 - \mathbf{x}\|_2 + \|\mathbf{x} - \boldsymbol{x}_s(s)\|_2 = v_T\tilde{t}_s(s). \quad (12)$$

For any $\epsilon > 0$ such that $(s - \epsilon, s + \epsilon) \subset \mathcal{I}_s$, it holds that $\mathbf{x} \notin \mathcal{E}\left(\mathbf{x}_T^0, \boldsymbol{x}_s(s'), v_T\tilde{t}_s(s')\right)$, $\forall s' \in (s - \epsilon, s + \epsilon)$. Thus, the function $h(s') = \|\mathbf{x}_T^0 - \mathbf{x}\|_2 + \|\mathbf{x} - \boldsymbol{x}_s(s')\|_2 - v_T\tilde{t}_s(s')$ attains a local minimum at s. The inequalities (10) and (11) follow from the first-order minimum condition. ☐

When $\boldsymbol{x}_s$ and $\tilde{t}_s$ are both differentiable at s, the following equation holds:

$$\frac{(\boldsymbol{x}_s(s) - \mathbf{x})^\top}{\|\mathbf{x} - \boldsymbol{x}_s(s)\|_2}\frac{d\boldsymbol{x}_s(s)}{ds} = v_T\frac{d\tilde{t}_s(s)}{ds}. \quad (13)$$

The dominance boundary $\mathcal{B}_D(\mathbf{X}^0)$ can be constructed by simultaneously solving equations (12) and (13). Let $\alpha(s)$ be the direction angle of $\frac{d\boldsymbol{x}_s(s)}{ds}$. There are two solutions to these equations which can be explicitly expressed by

$$\mathbf{x}_\pm(s) = \boldsymbol{x}_s(s) - l_\pm(s)\mathbf{e}_{\theta_\pm(s)}, \quad (14)$$

where $\mathbf{e}_{\theta_\pm(s)} = \begin{bmatrix}\cos\theta_\pm(s)\\ \sin\theta_\pm(s)\end{bmatrix}$ with $\theta_\pm(s)$ satisfying that

$$\theta_\pm(s) = \alpha(s) \pm \arccos\left(v_T\frac{d\tilde{t}_s(s)}{ds}\Big/\left\|\frac{d\boldsymbol{x}_s(s)}{ds}\right\|_2\right),$$

and

$$l_\pm(s) = \frac{v_T^2\tilde{t}_s^2(s) - \|\mathbf{x}_T^0 - \boldsymbol{x}_s(s)\|_2^2}{2\left((\mathbf{x}_T^0 - \boldsymbol{x}_s(s))^\top\mathbf{e}_{\theta_\pm(s)} + v_T\tilde{t}_s(s)\right)}.$$

There may be at most two different solutions to equations (12) and (13). A point $\mathbf{x} \in \mathcal{B}_D(\mathbf{X}^0)$ needs also to satisfy the second-order condition $\frac{d^2h(s)}{ds^2} > 0$, where $h(s')$ is defined in the proof of Lemma 2. When s is an endpoint

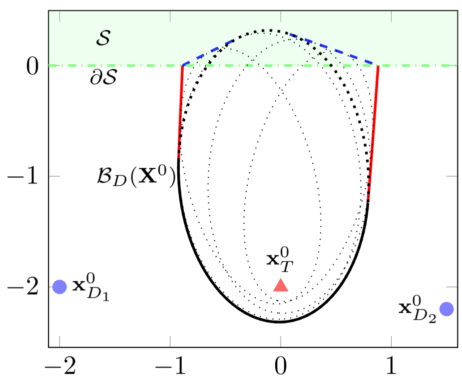

Fig. 5. Example of dominance boundary. $\mathcal{B}_D(\mathbf{X}^0)$ is shown with solid lines.

of $\mathcal{I}_s$ or a point where $\boldsymbol{x}_s$ or $\tilde{t}_s$ is not differentiable, the part of $\partial\mathcal{E}\left(\mathbf{x}_T^0, \boldsymbol{x}_s(s), v_T\tilde{t}_s(s)\right)$ where (10) and (11) are satisfied may also be included in $\mathcal{B}_D(\mathbf{X}^0)$. An example is given as follows to illustration the results in this section.

Example 1. Let $\partial\mathcal{S}$ be a straight line with a parametric equation $\boldsymbol{x}_s(s) = [s, 0]^\top$ and $\mathcal{S}$ be the half plane above $\partial\mathcal{S}$. The initial position of the thief is $\mathbf{x}_T^0 = [0, -2]^\top$. There is two guarders whose initial positions are $\mathbf{x}_{D_1}^0 = [-2, -2]^\top$ and $\mathbf{x}_{D_2}^0 = [1.5, -2.2]^T$. The capture radii are $r_{D_1} = r_{D_2} = 0.1$. All players have a speed of 1. The boundary $\mathcal{B}_D(\mathbf{X}^0)$ is shown in Fig. 5. $\mathcal{B}_D(\mathbf{X}^0)$ consists of two parts: the curve $\mathbf{x}_+(s)$ shown with red lines which is discontinuous at $s_c = 0.13$, and an ellipse arc of $\partial\mathcal{E}(\mathbf{x}_T^0, \boldsymbol{x}_s(s_c), v_T\tilde{t}_s(s_c))$ shown with a black curve. The curve $\mathbf{x}_-(s)$, which is shown with blue dashed lines, lies in $\mathcal{S}$ so that it is not included in $\mathcal{B}_D(\mathbf{X}^0)$. The dominance region outside $\mathcal{S}$ is the region surrounded by $\mathcal{B}_D(\mathbf{X}^0)$ and $\partial\mathcal{S}$. A set of ellipses $\mathcal{E}(\mathbf{x}_T^0, \boldsymbol{x}_s(s), v_T\tilde{t}_s(s))$ are also shown by dotted lines in the figure to confirm that the obtained curve $\mathcal{B}_D(\mathbf{X}^0)$ is indeed the boundary of the dominance region. It can be seen from the figure that these ellipses are tangent to the curve $\mathbf{x}_+(s)$.

VI. Conclusion

In this paper, we have explored the utilization of ellipses in two-phase multiplayer reach-avoid games. Specifically, we have employed ellipses to solve two particular game scenarios. Furthermore, we have introduced the concept of dominance region to enhance the understanding of two-phase reach-avoid games. The connection between dominance regions and ellipses has been thoroughly examined. Additionally, we have presented a methodology for constructing the boundary of dominance regions. In the future, we hope to study the further application of dominance regions in general two-phase reach-avoid games.

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