2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

Robust quadratic optimal control of linear systems with ellipsoid-set learning

Xuehui Ma[1], Yutong Chen[2], Shiliang Zhang[3], Yushuai Li[4], Fucai Qian[1], Zhiyong Sun[5]

***Abstract*— Despite the celebrated success of linear quadratic Gaussian control (LQG) for stochastic systems, LQG approaches are inefficient in handling systems with non-Gaussian noises. This paper is concerned with linear quadratic control of discrete-time systems with bounded noises and unobservable system states. We describe such noises and system states by ellipsoidal sets, enabling the establishment of boundaries for those uncertainties in the control. Further, we learn and update the ellipsoidal sets for the system states by an ellipsoidal set-membership filter. With the learned ellipsoidal sets, we derive a robust state-feedback optimal control law by solving a rendered semidefinite programming problem. Simulation results demonstrate the enhanced control performance by the proposed method.**

I. Introduction

Linear quadratic control (LQC) for systems corrupted by stochastic noises has been studied extensively from both theoretical and practical perspectives [1], [2]. The majority of LQC controller design assumes explicit distribution of noises in the system and prior knowledge about system uncertainties [3], [4]. A well-known example is linear quadratic Gaussian control (LQG) [1], where the process noises and measurement noises are hypothesized as Gaussian noises and thereafter, Kalman filter can serve as a perfect system state estimation for the control. However, practical systems are more than often subject to a shortage of information to develop probabilistic models, e.g., for the probability distribution of process or measurement noise [5].

Different from the stochastic control that depends on explicit noise distributions, robust control provides an alternative solution with bounded noises [2], [6], [7]. Specifically, robust control describes uncertainties, e.g., unknown noises, by a feasible set which represents the boundary of possible values for all the unknown parameters. Using the concept of feasible set, robust control looks into a conservative control law to guarantee the control performance in the worst-case scenarios. A representative example is the min-max control for LQC problem using robust optimization framework [2]. Nevertheless, robust optimization based methods assume that the feasible sets are fixed over the entire control horizon. As a result, the fixed sets cannot reflect the real system dynamics during the control and can lead to inefficient uncertainty representation. Further, the fixed sets are usually initiated to be large enough to cover all worst cases that might occur, which brings over-robustness (over-conservativeness) to the control and sacrifice the control accuracy.

A possible way to reduce the conservativeness is to narrow the feasible set boundary. This can be achieved using the information from observations to learn the feasible set and reduce the set size. Set-membership filter provides an applicable approach to solve the parameter estimation problems under bounded uncertainties [8], [9]. Recently, several works have integrated set-membership based estimation into robust optimization to reduce conservativeness. E.g., Lorenzen et al. [6] and Arcari et al. [10] proposed robust model predictive control for uncertain systems, where they use set-membership filter to update the feasible sets for unknown parameters of the state space models. Parsi et al. [7] developed a dual control strategy for output tracking of systems with unknown yet bounded parameters and disturbances, and utilized set-membership estimation to renew uncertainty sets during the control. However, a significant disadvantage of the set-membership approach is the usage of polytope sets in uncertainty estimation, which can cause algorithm complexity to increase dramatically with the system dimension.

To address the complexity issue, a potential solution is to use ellipsoidal sets instead of polytopes in set-membership estimation, which leads to mitigated algorithm and computation complexity [11]. Relevant studies can be found in [11], [12] where Qian et al. developed robust optimal tracking control laws for systems whose parameters and noises are described by ellipsoidal sets, and applied the ellipsoidal bounding algorithm in system parameter identification. Paris et al. [13] and Iannelli et al. [14] solved the LQC problem with ellipsoid bounded uncertainties. However, the existing ellipsoid-based approaches assume that the system states are completely observable. We remark that in most practical scenarios, the system state is not directly observable and the observations used in system state estimation are usually corrupted by noises.

Motivated by the above discussion, this work investigates the LQC problem of linear systems with process noises and observation noises described by ellipsoidal sets. We contribute to this topic in that (i) we adopt ellipsoidal sets

This work was supported by the National Natural Science Foundation of China (Grant no. 62073259) and Key R&D projects in Shaanxi Province (Grant no. 2023-YBGY-380).

[1]Xuehui Ma and Fucai Qian are with the School of Automation and Information Engineering, Xi'an University of Technology, Xi'an, China xuehui.yx@gmail.com, qianfc@xaut.edu.cn.

[2]Yutong Chen is with the Department of Electronic Engineering, Tsinghua University, Beijing, China yt-chen21@mails.tsinghua.edu.cn.

[3]Shiliang Zhang is with the Department of Informatics, University of Oslo, Oslo, Norway shilianz@ifi.uio.no.

[4]Yushuai Li is with the Department of Computer Science, Aalborg University, Aalborg, Denmark yushuaili@ieee.org.

[5]Zhiyong Sun is with the Department of Electrical Engineering, Eindhoven University of Technology, Netherlands z.sun@tue.nl.

to represent and handle unobservable system states and unknown process and observation noises in LQC, which is prohibitive for Kalman filter-based estimation used in LQG that merely deals with Gaussian noises; (ii) we integrate ellipsoidal set-membership filter to learn and reduce the uncertainty boundaries of the unobservable system states; and (iii) we derive a robust control law based on ellipsoidal set estimation and robust optimization, and we consider both the noise and the state estimation error in the control law derivation. We demonstrate how our method improves the LQC performance in comparison with robust control with fixed feasible sets [2].

This paper is organized as follows. We formulate the linear quadratic control problem with noises bounded by ellipsoidal sets in Section II. Section III shows the ellipsoidal set learning for system state estimation, and Section IV presents the optimal robust control law derivation. In Section V, numerical simulations are conducted to demonstrate the effectiveness of our approach. We conclude our work in Section VI.

II. Problem Statement

This work considers the control of the following discrete-time linear dynamic system

$$\boldsymbol{x}(k+1) = \boldsymbol{A}(k)\boldsymbol{x}(k) + \boldsymbol{B}(k)\boldsymbol{u}(k) + \boldsymbol{w}(k), \quad k = 0, 1, \cdots, N-1, \tag{1}$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$ is the state vector and is not observable, $\boldsymbol{u}(k) \in \mathbb{R}^r$ denotes the control vector, matrices $\boldsymbol{A}(k) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B}(k) \in \mathbb{R}^{n\times r}$ are known, $\boldsymbol{w}(k) \in \mathbb{R}^n$ is the process noise. We describe the system observation $\boldsymbol{y}(k)$ as

$$\boldsymbol{y}(k) = \boldsymbol{C}(k)\boldsymbol{x}(k) + \boldsymbol{v}(k), \quad k = 1, 2, \cdots, N, \tag{2}$$

where $\boldsymbol{C}(k) \in \mathbb{R}^{m\times n}$ is a known matrix, $\boldsymbol{v}(k) \in \mathbb{R}^r$ is the observation noise.

We assume the process and observation noises are bounded in the following ellipsoidal sets $\mathcal{W}(k)$ and $\mathcal{V}(k)$, respectively

$$\mathcal{W}(k) \triangleq \left\{\boldsymbol{w}(k) \in \mathbb{R}^n \left| \boldsymbol{w}^T(k)\boldsymbol{P}_w^{-1}(k)\boldsymbol{w}(k) \le 1 \right.\right\}, \tag{3}$$

$$\mathcal{V}(k) \triangleq \left\{\boldsymbol{v}(k) \in \mathbb{R}^m \left| \boldsymbol{v}^T(k)\boldsymbol{P}_v^{-1}(k)\boldsymbol{v}(k) \le 1 \right.\right\}, \tag{4}$$

where $\boldsymbol{P}_w$ and $\boldsymbol{P}_v$ are the matrices defining the shape, size, and orientation of the ellipsoids. Beyond that no information about the noises is available. We also assume the initial state vector $\boldsymbol{x}(0)$ is unknown yet bounded by the following ellipsoid

$$\mathcal{X}(0) \triangleq \left\{\boldsymbol{x}(0) \in \mathbb{R}^n \left| \left[\boldsymbol{x}(0) - \bar{\boldsymbol{x}}_0\right]^T \boldsymbol{P}_0^{-1} \left[\boldsymbol{x}(0) - \bar{\boldsymbol{x}}_0\right] \le 1 \right.\right\}, \tag{5}$$

where $\bar{\boldsymbol{x}}_0$ is the center of the ellipsoidal set, and $\boldsymbol{P}_0^{-1}$ is the ellipsoid shape matrix.

The objective in this control is to gain a control sequence $\{u(0), u(1), \cdots, u(N-1)\}$, which minimizes the performance index below

$$J = \boldsymbol{x}^T(N)\boldsymbol{Q}(N)\boldsymbol{x}(N) + \sum_{k=0}^{N-1} \left\{\boldsymbol{x}^T(k)\boldsymbol{Q}(k)\boldsymbol{x}(k) + \boldsymbol{u}^T(k)\boldsymbol{R}(k)\boldsymbol{u}(k)\right\}, \tag{6}$$

where $\boldsymbol{Q}(k) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{R}(k) \in \mathbb{R}^{r\times r}$ are positive semi-definite and positive definite symmetric matrices, respectively.

Our hypothesis is that the process noises, observation noises, and the system state are of unknown distributions, and each of them is bounded in an ellipsoidal set. In that way, the control will be different from LQG where the noises and states possess Gaussian distributional property and the resulted solution is orientated by Gaussian associated approaches. Specifically, instead of solving the minimization of expectation of cost function J as is in LQG, we consider the minimization of the worst-case of J under the ellipsoidal-set-bounded noises and system states. We formulate the control objective as the solving of the following constrained optimization problem

$$\begin{aligned}
(\mathcal{P}): \quad & \min_{\{u(0),\cdots,u(N-1)\}} \max_{\{\boldsymbol{w}(k),\boldsymbol{v}(k)\}} J \\
s.t. \quad & \boldsymbol{x}(k+1) = \boldsymbol{A}(k)\boldsymbol{x}(k) + \boldsymbol{B}(k)\boldsymbol{u}(k) + \boldsymbol{w}(k), \\
& \boldsymbol{y}(k) = \boldsymbol{C}(k)\boldsymbol{x}(k) + \boldsymbol{v}(k), \\
& \boldsymbol{w}^T(k)\boldsymbol{P}_w^{-1}(k)\boldsymbol{w}(k) \le 1, \\
& \boldsymbol{v}^T(k)\boldsymbol{P}_v^{-1}(k)\boldsymbol{v}(k) \le 1, \\
& k = 0, 1, \cdots, N-1.
\end{aligned} \tag{7}$$

In problem $(\mathcal{P})$, the cost J is a function of system state $\boldsymbol{x}(k)$, which is unobservable and bounded in an ellipsoidal set. In addressing the unavailability of $\boldsymbol{x}(k)$, this study estimates $\boldsymbol{x}(k)$ based on the observations and control signals. Such an estimation is achieved through ellipsoidal set learning as elaborated in Section III. Since the estimation error of system state can negatively affect the control, we take into account the estimation error in deriving the robust control law based on the ellipsoidal sets, as detailed in Section IV.

III. Ellipsoidal Set Learning

This section details the learning of ellipsoidal sets for the unobservable system states $\boldsymbol{x}(k)$. As shown in (5), the initial state is confined in an ellipsoidal set, which is sufficiently large to cover all possible worst cases. Here we utilize the system observations to learn and reduce the size of the ellipsoidal sets iteratively, thus mitigating uncertainties in the control. Following the main technique in [8], we describe the ellipsoidal set learning in Theorem 1.

Theorem 1. *[8] Consider a linear system described by (1) and (2), where the noises $\boldsymbol{w}(k)$ and $\boldsymbol{v}(k)$ are sequences of uncertain variables confined in ellipsoidal sets (3) and (4), respectively. Let the initial system state be confined in the ellipsoidal set (5). Then the ellipsoidal set for the state $\boldsymbol{x}(k)$ given observation $\boldsymbol{y}(k)$ is*

$$\mathcal{X}(k|k) \triangleq \{\boldsymbol{x}(k) \in \mathbb{R}^n | \left[\boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k|k)\right]^T \boldsymbol{P}^{-1}(k|k) \left[\boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k|k)\right] \le 1\}. \tag{8}$$

Define the scalar parameters $p_k \in (0, \infty)$ and $q_k \in [0, \infty)$, and then we update $\hat{\boldsymbol{x}}(k|k)$ and $\boldsymbol{P}(k|k)$ recursively by two periods:

(i) time update

$$\hat{\boldsymbol{x}}(k|k-1)=\boldsymbol{A}(k-1)\hat{\boldsymbol{x}}(k-1|k-1)+\boldsymbol{B}(k-1)\boldsymbol{u}(k-1), \tag{9}$$

$$\boldsymbol{P}(k|k-1)=(p_k^{-1}+1)\boldsymbol{A}(k-1)\boldsymbol{P}(k-1|k-1) \boldsymbol{A}^T(k-1)+(p_k+1)\boldsymbol{P}_w(k), \tag{10}$$

and (ii) observation update

$$\boldsymbol{K}(k)=\boldsymbol{P}(k|k-1)\boldsymbol{C}^T(k)\left[\boldsymbol{C}(k)\boldsymbol{P}(k|k-1)\boldsymbol{C}^T(k) +q_k^{-1}\boldsymbol{P}_v(k)\right]^{-1}, \tag{11}$$

$$\hat{\boldsymbol{x}}(k|k)=\hat{\boldsymbol{x}}(k|k-1)+\boldsymbol{K}(k)\left[\boldsymbol{y}(k)-\boldsymbol{C}(k)\hat{\boldsymbol{x}}(k|k-1)\right], \tag{12}$$

$$\boldsymbol{P}(k|k)=\{\left[\boldsymbol{I}-\boldsymbol{K}(k)\boldsymbol{C}(k)\right]\boldsymbol{P}(k|k-1)\left[\boldsymbol{I}-\boldsymbol{K}(k)\boldsymbol{C}(k)\right]^T +q_k^{-1}\boldsymbol{K}(k)\boldsymbol{P}_v(k)\boldsymbol{K}^T(k)\}\beta_k, \tag{13}$$

$$\boldsymbol{\epsilon}(k)=\boldsymbol{y}(k)-\boldsymbol{C}(k)\hat{\boldsymbol{x}}(k|k-1), \tag{14}$$

$$\beta_k=1+q_k-\boldsymbol{\epsilon}^T(k)\left[q_k^{-1}\boldsymbol{P}_v(k)\boldsymbol{C}(k)\boldsymbol{P}(k|k-1) \boldsymbol{C}^T(k)\right]^{-1}\boldsymbol{\epsilon}(k). \tag{15}$$

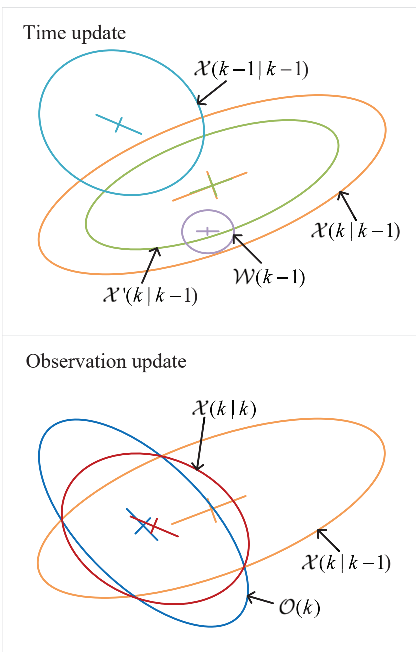

Fig. 1. Geometrical description: learning progress of ellipsoidal set for the system state at the k instant. Ellipsoids in this figure denote boundaries of feasible sets, which we call ellipsoidal sets. The crosses are centres of the ellipsoids, or ellipsoidal sets. The dark red ellipsoid during the observation update represents the set $\mathcal{X}(k|k)$, where the position of the ellipsoid is determined by the ellipsoid center $\hat{\boldsymbol{x}}(k|k)$, and the ellipsoid shape is determined by the matrix $\boldsymbol{P}(k|k)$ cf. Equation 8.

Remark 1. Fig. 1 geometrically explains the ellipsoidal set learning process in Theorem 1, taking an example of ellipsoidal set for 2-dimensional system states. During the time update period, the ellipsoidal set $\mathcal{X}(k-1|k-1)$ is linearly transformed to $\mathcal{X}'(k|k-1)$, where the centre is $\hat{\boldsymbol{x}}(k|k-1)$ obtained by (9), and the shape matrix is calculated as $\boldsymbol{A}(k-1)\boldsymbol{P}(k-1|k-1)\boldsymbol{A}^T(k-1)$. The ellipsoidal set $\mathcal{X}(k|k-1)$ is gained as the optimal ellipsoid containing the vector sum of two ellipsoids $\mathcal{X}'(k|k-1)$ and $\mathcal{W}(k-1)$, where the shape matrix $\boldsymbol{P}(k|k-1)$ is obtained by (10). During the observation update period, we use the information of system observation $\boldsymbol{y}(k)$ to update the system state represented by the ellipsoid $\mathcal{X}(k|k-1)$. Particularly, we calculate the ellipsoidal set $\mathcal{X}(k|k)$ as the the optimal ellipsoid that contains the intersection of two sets $\mathcal{O}(k)$ and $\mathcal{X}(k|k-1)$, where $\mathcal{O}(k)=\{\boldsymbol{x}(k):(\boldsymbol{y}(k)-\boldsymbol{C}(k)\boldsymbol{x}(k))^T\boldsymbol{P}_v^{-1}(k)(\boldsymbol{y}(k)-\boldsymbol{C}(k)\boldsymbol{x}(k))\}$ is an ellipsoidal set which confines the state values consistent with the current observation $\boldsymbol{y}(k)$. The centre $\hat{\boldsymbol{x}}(k|k)$ and the shape matrix $\boldsymbol{P}(k|k)$ of the updated ellipsoidal set can be obtained by (12) and (13), respectively.

Remark 2. In Theorem 1, the scalar parameters p_k and q_k can be obtained by different criteria, and here we choose the minimum trace criterion [8]. Under this criterion, p_k satisfies

$$p_k=\left(\frac{\operatorname{tr}(\boldsymbol{A}(k-1)\boldsymbol{P}(k-1|k-1)\boldsymbol{A}^T(k-1))}{\operatorname{tr}(\boldsymbol{P}_w(k))}\right)^{1/2}, \tag{16}$$

and the parameter q_k satisfies

$$\frac{\sum_{j=1}^n \operatorname{diag}_j(\boldsymbol{V}^{-1}\boldsymbol{P}(k|k-1)\boldsymbol{V})\frac{\lambda_q(j)}{(1+q_k\lambda_q(j))^2}}{\sum_{j=1}^n \operatorname{diag}_j(\boldsymbol{V}^{-1}\boldsymbol{P}(k|k-1)\boldsymbol{V})\frac{1}{(1+q_k\lambda_q(j))}}=\frac{\beta_k'}{\beta_k}, \tag{17}$$

where $\boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1}=\boldsymbol{P}(k|k-1)\boldsymbol{C}^T(k)\boldsymbol{P}_v^{-1}(k)\boldsymbol{C}(k)$, and $\boldsymbol{D}$ is a diagonal matrix containing eigenvalues $\lambda_q(j)$ of matrix $\boldsymbol{P}(k|k-1)\boldsymbol{C}^T(k)\boldsymbol{P}_v^{-1}(k)\boldsymbol{C}(k)$ on its diagonal, and $\boldsymbol{V}$ is a matrix with the corresponding eigenvectors as its columns. diag_j denotes the j-th diagonal element of the matrix, and n is the number of the eigenvalues. β_k' is the partial derivative of β_k with respect to q_k, given by

$$\beta_k'=1-\boldsymbol{\epsilon}^T(k)\left[q_k^{-1}\boldsymbol{P}_v(k)\boldsymbol{C}(k)\boldsymbol{P}(k|k-1)\boldsymbol{C}^T(k)\right]^{-1} q_k^{-2}\boldsymbol{P}_v(k)\left[q_k^{-1}\boldsymbol{P}_v(k)\boldsymbol{C}(k)\boldsymbol{P}(k|k-1)\boldsymbol{C}^T(k)\right]^{-1}\boldsymbol{\epsilon}(k), \tag{18}$$

Note that (17) does not have a solution if

$$\left(1-\boldsymbol{\epsilon}^T(k)\boldsymbol{P}_v^{-1}(k)\boldsymbol{\epsilon}(k)\right)\operatorname{tr}\left(\boldsymbol{P}(k|k-1)\right)- \operatorname{tr}\left(\boldsymbol{P}(k|k-1)\boldsymbol{C}^T(k)\boldsymbol{P}_v^{-1}(k)\boldsymbol{C}(k)\boldsymbol{P}(k|k-1)\right)>0, \tag{19}$$

and then $q_k=0$ is optimal.

IV. Robust Optimal Control Based on Ellipsoidal Set Learning

This section derives the control law upon the estimated system state that are iteratively learned by ellipsoidal set learning. As such, our approach differs from existing solutions to LQC control using fixed estimated value for system state [2]. Further, we add a constraint in the control problem (7), i.e., the state estimation error induced by the ellipsoidal set learning. Taking the estimation error into consideration as a constraint and penalizing this constraint in the control law derivation enables us to improve the control performance of our approach. Our control law derivation based on learned system state is described in the following.

With the ellipsoidal set learning algorithm in Theorem 1, we obtain the updated system state sets at the instant k as

$$\left\{\boldsymbol{x}(k) \in \mathbb{R}^n | \left[\boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k|k)\right]^T \boldsymbol{P}^{-1}(k|k) \left[\boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k|k)\right] \leq 1\right\}. \tag{20}$$

We use the center of the updated ellipsoidal set $\hat{\boldsymbol{x}}(k|k)$ to represent the value of the system state, and then, the state learning error is defined as

$$\boldsymbol{\eta}(k) = \boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k|k). \tag{21}$$

Substitute (21) into (20), and we can obtain the boundary of the state estimation error described by the ellipsoidal set below

$$\left\{\boldsymbol{\eta}(k) \in \mathbb{R}^n \left| \boldsymbol{\eta}^T(k)\boldsymbol{P}^{-1}(k|k)\boldsymbol{\eta}(k) \leq 1\right.\right\}. \tag{22}$$

Integrate the constraint (22) into the problem ($\mathcal{P}$) in (7), and then we reformulate this problem to the following truncated robust control problem from instant t to N:

$$\begin{aligned}
(\mathcal{P}_t): \quad & \min_{\{\boldsymbol{u}(k)\}|_{k=t}^{N-1}} \max_{\{\boldsymbol{w}(k),\boldsymbol{v}(k)\}|_{k=t}^{N-1}} J(t) \\
s.t. \quad & \boldsymbol{x}(k+1) = \boldsymbol{A}(k)\boldsymbol{x}(k) + \boldsymbol{B}(k)\boldsymbol{u}(k) + \boldsymbol{w}(k), \\
& \boldsymbol{y}(k) = \boldsymbol{C}(k)\boldsymbol{x}(k) + \boldsymbol{v}(k), \\
& \boldsymbol{\eta}^T(k)\boldsymbol{P}^{-1}(k|k)\boldsymbol{\eta}(k) \leq 1, \\
& \boldsymbol{w}^T(k)\boldsymbol{P}_w^{-1}(k)\boldsymbol{w}(k) \leq 1, \\
& \boldsymbol{v}^T(k)\boldsymbol{P}_v^{-1}(k)\boldsymbol{v}(k) \leq 1, \\
& k = t, t+1, \cdots, N-1,
\end{aligned} \tag{23}$$

where the cost function $J(t)$ is defined as

$$J(t) = \sum_{k=t}^{N} \boldsymbol{x}^T(k)\boldsymbol{Q}(k)\boldsymbol{x}(k) + \sum_{k=t}^{N-1} \boldsymbol{u}^T(k)\boldsymbol{R}(k)\boldsymbol{u}(k). \tag{24}$$

Note that the first component $\boldsymbol{u}(t)$ in the solved control sequences $\{\boldsymbol{u}(k)\}|_{k=t}^{N-1}$ is the control law at the instant t. We detail the derivation of the control law by solving the problem ($\mathcal{P}_t$) in (23) in the following.

Lemma 1. *The system state vector at the instant k can be expressed as*

$$\boldsymbol{x}(k) = \tilde{\boldsymbol{A}}_t^{k-1}\boldsymbol{x}(t) + \tilde{\boldsymbol{B}}_t^{k-1}\boldsymbol{U}_t + \tilde{\boldsymbol{C}}_t^{k-1}\boldsymbol{W}_t \tag{25}$$

for any $k = t, t+1, \cdots, N-1$, where $\tilde{\boldsymbol{A}}_t^{k-1} \in \mathbb{R}^{n\times n}$, $\tilde{\boldsymbol{B}}_t^{k-1} \in \mathbb{R}^{n\times(N-t)\cdot r}$, $\tilde{\boldsymbol{C}}_t^{k-1} \in \mathbb{R}^{n\times(N-t)\cdot n}$, $\boldsymbol{U}_t \in \mathbb{R}^{(N-t)\cdot r}$, $\boldsymbol{W}_t \in \mathbb{R}^{(N-t)\cdot n}$ are respectively defined as

$$\tilde{\boldsymbol{A}}_t^{k-1} = \left\{ \begin{array}{cc} \prod_{i=t}^{k-1} \boldsymbol{A}(i) & for \quad k \geqslant t+1 \\ 1 & for \quad else \end{array} \right. , \tag{26}$$

$$\tilde{\boldsymbol{B}}_t^{k-1} = \left[\begin{array}{cccc} \tilde{\boldsymbol{A}}_{t+1}^{k-1}\boldsymbol{B}(t) & \cdots & \boldsymbol{B}(k-1) & \boldsymbol{0}_{n\times(N-k)\cdot r} \end{array}\right], \tag{27}$$

$$\tilde{\boldsymbol{C}}_t^{k-1} = \left[\begin{array}{ccccc} \tilde{\boldsymbol{A}}_{t+1}^{k-1} & \tilde{\boldsymbol{A}}_{t+2}^{k-1} & \cdots & \boldsymbol{I} & \boldsymbol{0}_{n\times(N-k)\cdot n} \end{array}\right], \tag{28}$$

$$\boldsymbol{U}_t = \left[\begin{array}{cccc} \boldsymbol{u}^T(t) & \boldsymbol{u}^T(t+1) & \cdots & \boldsymbol{u}^T(N-1) \end{array}\right]^T, \tag{29}$$

$$\boldsymbol{W}_t = \left[\begin{array}{cccc} \boldsymbol{w}^T(t) & \boldsymbol{w}^T(t+1) & \cdots & \boldsymbol{w}^T(N-1) \end{array}\right]^T. \tag{30}$$

Proof. The proof is immediate with the system dynamic (1). □

Lemma 2. *The cost function $J(t)$ in (24) can be expressed as*

$$\begin{aligned} J(t) =& \boldsymbol{x}^T(t)\mathscr{A}_t\boldsymbol{x}(t) + \boldsymbol{U}_t^T\mathscr{B}_t\boldsymbol{U}_t + 2\boldsymbol{b}_t^T\boldsymbol{U}_t \\ &+ \boldsymbol{W}_t^T\mathscr{C}_t\boldsymbol{W}_t + 2\boldsymbol{c}_t^T\boldsymbol{W}_t + 2\boldsymbol{U}_t^T\mathscr{D}_t\boldsymbol{W}_t, \end{aligned} \tag{31}$$

where $\mathscr{A}_t \in \mathbb{R}^{n\times n}$, $\mathscr{B}_t \in \mathbb{R}^{(N-t)\cdot r\times(N-t)\cdot r}$, $\mathscr{C}_t \in \mathbb{R}^{(N-t)\cdot n\times(N-t)\cdot n}$, $\mathscr{D}_t \in \mathbb{R}^{(N-t)\cdot r\times(N-t)\cdot n}$, $\boldsymbol{b}_t \in \mathbb{R}^{(N-t)\cdot r}$, $\boldsymbol{c}_t \in \mathbb{R}^{(N-t)\cdot n}$ are respectively defined as

$$\mathscr{A}_t = \sum_{k=t}^{N} (\tilde{\boldsymbol{A}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{A}}_t^{k-1}, \tag{32}$$

$$\begin{aligned} \mathscr{B}_t = & \sum_{k=t+1}^{N} (\tilde{\boldsymbol{B}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{B}}_t^{k-1} \\ & + \operatorname{diag}(\boldsymbol{R}(t), \boldsymbol{R}(t+1), \cdots, \boldsymbol{R}(N-1)), \end{aligned} \tag{33}$$

$$\mathscr{C}_t = \sum_{k=t+1}^{N} (\tilde{\boldsymbol{C}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{C}}_t^{k-1}, \tag{34}$$

$$\mathscr{D}_t = \sum_{k=t+1}^{N} (\tilde{\boldsymbol{B}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{C}}_t^{k-1}, \tag{35}$$

$$\boldsymbol{b}_t = \left(\sum_{k=t+1}^{N} (\tilde{\boldsymbol{B}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{A}}_t^{k-1}\right)\boldsymbol{x}(t), \tag{36}$$

$$\boldsymbol{c}_t = \left(\sum_{k=t+1}^{N} (\tilde{\boldsymbol{C}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{A}}_t^{k-1}\right)\boldsymbol{x}(t). \tag{37}$$

Proof. By substituting Equation (25) into the first part of cost function $J(t)$, we obtain

$$\begin{aligned} J(t) =& \sum_{k=t}^{N} \boldsymbol{x}^T(k)\boldsymbol{Q}(k)\boldsymbol{x}(k) + \sum_{k=t}^{N-1} \boldsymbol{u}^T(k)\boldsymbol{R}(k)\boldsymbol{u}(k) \\
=& \boldsymbol{x}^T(t)\left(\sum_{k=t}^{N}(\tilde{\boldsymbol{A}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{A}}_t^{k-1}\right)\boldsymbol{x}(t) \\
&+ \left(\sum_{k=t+1}^{N} 2\boldsymbol{x}^T(t)(\tilde{\boldsymbol{A}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{B}}_t^{k-1}\right)\boldsymbol{U}_t \\
&+ \left(\sum_{k=t+1}^{N} 2\boldsymbol{x}^T(t)(\tilde{\boldsymbol{A}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{C}}_t^{k-1}\right)\boldsymbol{W}_t \\
&+ \boldsymbol{U}_t^T\left(\sum_{k=t+1}^{N} (\tilde{\boldsymbol{B}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{B}}_t^{k-1}\right)\boldsymbol{U}_t \\
&+ \boldsymbol{U}_t^T\left(\sum_{k=t+1}^{N} 2(\tilde{\boldsymbol{B}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{C}}_t^{k-1}\right)\boldsymbol{W}_t \\
&+ \boldsymbol{W}_t^T\left(\sum_{k=t+1}^{N} (\tilde{\boldsymbol{C}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{C}}_t^{k-1}\right)\boldsymbol{W}_t \\
&+ \boldsymbol{U}_t^T\operatorname{diag}(\boldsymbol{R}(t), \boldsymbol{R}(t+1), \cdots, \boldsymbol{R}(N-1))\boldsymbol{U}_t. \end{aligned} \tag{38}$$

The lemma then follows based on the definition in (32), (33), (34), (35), (36), and (37). □

Lemma 3. *With estimation error of the state given by (21), the cost function $J(t)$ in (31) can be rewritten in the form*

$$\begin{aligned}J(t) =&\hat{\boldsymbol{x}}^T(t|t)\mathscr{A}_t\hat{\boldsymbol{x}}(t|t) + \boldsymbol{U}_t^T\mathscr{B}_t\boldsymbol{U}_t + 2\hat{\boldsymbol{b}}_t^T\boldsymbol{U}_t + \boldsymbol{\xi}_t^T\hat{\mathscr{C}}_t\boldsymbol{\xi}_t \\ &+ 2\hat{\boldsymbol{c}}_t^T\boldsymbol{\xi}_t + 2\boldsymbol{U}_t^T\hat{\mathscr{D}}_t\boldsymbol{\xi}_t,\end{aligned} \tag{39}$$

where $\hat{\boldsymbol{b}}_t \in \mathbb{R}^{(N-t)\cdot r}$, $\hat{\mathscr{C}}_t \in \mathbb{R}^{(N-t+1)\cdot n\times(N-t+1)\cdot n}$, $\hat{\boldsymbol{c}}_t \in \mathbb{R}^{(N-t)\cdot n+1}$, $\hat{\mathscr{D}}_t \in \mathbb{R}^{(N-t)\cdot r\times((N-t)\cdot r+n)}$, $\boldsymbol{\xi}_t \in \mathbb{R}^{(N-t)\cdot n+1}$ are respectively defined as

$$\hat{\boldsymbol{b}}_t = \left(\sum_{k=t+1}^{N}(\tilde{\boldsymbol{B}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{A}}_t^{k-1}\right)\hat{\boldsymbol{x}}(t|t), \tag{40}$$

$$\hat{\mathscr{C}}_t = \begin{bmatrix} \mathscr{A}_t & \boldsymbol{S}_c^T \\ \boldsymbol{S}_c & \mathscr{C}_t \end{bmatrix}, \boldsymbol{S}_c = \sum_{k=t+1}^{N}(\tilde{\boldsymbol{C}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{A}}_t^{k-1}, \tag{41}$$

$$\hat{\boldsymbol{c}}_t = \begin{bmatrix} \boldsymbol{0} \\ \left(\sum_{k=t+1}^{N}(\tilde{\boldsymbol{C}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{A}}_t^{k-1}\right)\hat{\boldsymbol{x}}(t|t) \end{bmatrix}, \tag{42}$$

$$\hat{\mathscr{D}}_t = \begin{bmatrix} \sum_{k=t+1}^{N}(\tilde{\boldsymbol{B}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{A}}_t^{k-1} & \mathscr{D}_t \end{bmatrix}, \tag{43}$$

$$\boldsymbol{\xi}_t^T = \begin{bmatrix} \boldsymbol{\eta}^T(t) & \boldsymbol{W}_t^T \end{bmatrix}. \tag{44}$$

Proof. The lemma follows by recalling the cost function in (31), and substituting $\boldsymbol{x}(t) = \hat{\boldsymbol{x}}(t|t) + \boldsymbol{\eta}(t)$ yields to

$$\begin{aligned}&J(t) = (\hat{\boldsymbol{x}}(t|t)+\boldsymbol{\eta}(t))^T(t)\mathscr{A}_t(\hat{\boldsymbol{x}}(t|t)+\boldsymbol{\eta}(t)) + \boldsymbol{U}_t^T\mathscr{B}_t\boldsymbol{U}_t \\ &+ 2\left(\left(\sum_{k=t+1}^{N}(\tilde{\boldsymbol{B}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{A}}_t^{k-1}\right)(\hat{\boldsymbol{x}}(t|t)+\boldsymbol{\eta}(t))\right)^T\boldsymbol{U}_t \\ &+ 2\left(\left(\sum_{k=t+1}^{N}(\tilde{\boldsymbol{C}}_t^{k-1})^T\boldsymbol{Q}(k)\tilde{\boldsymbol{A}}_t^{k-1}\right)(\hat{\boldsymbol{x}}(t|t)+\boldsymbol{\eta}(t))\right)^T\boldsymbol{W}_t \\ &+ \boldsymbol{W}_t^T\mathscr{C}_t\boldsymbol{W}_t + 2\boldsymbol{U}_t^T\mathscr{D}_t\boldsymbol{W}_t.\end{aligned} \tag{45}$$

By collecting all the terms in (45), we obtain (39). ☐

Theorem 2. *The problem $(\mathcal{P}_t)$ in (23) is equivalent to the following formulation:*

$$\begin{aligned}(\mathcal{P}_t): \quad &\min_{\boldsymbol{z}_t} \quad \rho_t \\ s.t. \quad &\begin{bmatrix} \boldsymbol{I} & \boldsymbol{z}_t & \boldsymbol{F}_t \\ \boldsymbol{z}_t^T & \rho_t - \tau_1 - \tau_2(N-t) & -\boldsymbol{h}_t^T \\ \boldsymbol{F}_t^T & -\boldsymbol{h}_t & \boldsymbol{G}_t \end{bmatrix} \geq 0,\end{aligned} \tag{46}$$

where ρ_t, $\boldsymbol{z}_t$, τ_1, τ_2 are decision variables. The vector $\boldsymbol{h}_t$ and the matrix $\boldsymbol{F}_t$, $\boldsymbol{M}^{\boldsymbol{\eta}}(t)$, $\boldsymbol{M}_t^W$, $\boldsymbol{G}_t$ are respectively defined as

$$\boldsymbol{h}_t = \hat{\boldsymbol{c}}_t - \hat{\mathscr{D}}_t^T\mathscr{B}_t^{-1}\hat{\boldsymbol{b}}_t, \tag{47}$$

$$\boldsymbol{F}_t = \mathscr{B}_t^{-1/2}\hat{\mathscr{D}}_t. \tag{48}$$

$$\boldsymbol{M}^{\boldsymbol{\eta}}(t) = \begin{bmatrix} \boldsymbol{P}^{-1}(t) & 0 \\ 0 & 0 \end{bmatrix}, \tag{49}$$

$$\boldsymbol{M}_t^W = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{P}_t^W \end{bmatrix}, \tag{50}$$

$$\boldsymbol{G}_t = -\hat{\mathscr{C}}_t + \tau_1\boldsymbol{M}^{\boldsymbol{\eta}}(t) + \tau_2\boldsymbol{M}_t^W + \boldsymbol{F}_t^T\boldsymbol{F}_t\ , \tag{51}$$

where $\boldsymbol{P}_t^W = \mathrm{diag}(\boldsymbol{P}_w^{-1}(t), \cdots, \boldsymbol{P}_w^{-1}(N-1))$. The corresponding control vector $\boldsymbol{U}_t$ is calculated by

$$\boldsymbol{U}_t = \mathscr{B}_t^{-1/2}\boldsymbol{z}_t - \mathscr{B}_t^{-1}\hat{\boldsymbol{b}}_t, \tag{52}$$

and the control law at the instant t is $\boldsymbol{U}_t$'s first element:

$$\boldsymbol{u}(t) = \boldsymbol{U}_t(0). \tag{53}$$

Proof. According to the definition of $\mathscr{B}_t$ in (33), $\mathscr{B}_t$ is positive definite since $\boldsymbol{Q}(k) \succeq 0$ and $\boldsymbol{R}(k) \succ 0$. Therefore, $\mathscr{B}_t^{-1}$ exists and is also symmetric positive definite. Then we can obtain $(\mathscr{B}_t^{-1/2})^T = \mathscr{B}_t^{-1/2}$. Substituting the control vector in (52) into the cost function (39) yields

$$\begin{aligned}&J(t) = \hat{\boldsymbol{x}}^T(t|t)\mathscr{A}_t\hat{\boldsymbol{x}}(t|t) + \boldsymbol{z}_t^T\boldsymbol{z}_t - \hat{\boldsymbol{b}}_t^T\mathscr{B}_t^{-1}\hat{\boldsymbol{b}}_t + \boldsymbol{\xi}_t^T\hat{\mathscr{C}}_t\boldsymbol{\xi}_t \\ &+ 2\left(\hat{\boldsymbol{c}}_t^T - \hat{\boldsymbol{b}}_t^T(\mathscr{B}_t^{-1})^T\hat{\mathscr{D}}_t\right)\boldsymbol{\xi}_t + 2\boldsymbol{z}_t^T(\mathscr{B}_t^{-1/2})^T\hat{\mathscr{D}}_t\boldsymbol{\xi}_t\ .\end{aligned} \tag{54}$$

With the definitions in (47) and (48), $J(t)$ can be rewritten as

$$J(t) = \boldsymbol{z}_t^T\boldsymbol{z}_t + 2\boldsymbol{h}_t^T\boldsymbol{\xi}_t + 2\boldsymbol{z}_t^T\boldsymbol{F}_t\boldsymbol{\xi}_t + \boldsymbol{\xi}_t^T\hat{\mathscr{C}}_t\boldsymbol{\xi}_t + const. \tag{55}$$

Therefore, the problem $(\mathcal{P}_t)$ in (23) can be reformed as

$$\begin{aligned}(\mathcal{P}_t): \quad &\min_{\boldsymbol{z}_t}\max_{\{\boldsymbol{w}(k),\boldsymbol{\eta}(t)\}} \boldsymbol{z}_t^T\boldsymbol{z}_t + 2\boldsymbol{h}^T\boldsymbol{\xi} + 2\boldsymbol{z}_t^T\boldsymbol{F}\boldsymbol{\xi} + \boldsymbol{\xi}^T\hat{\mathscr{C}}\boldsymbol{\xi} \\ s.t. \quad &\boldsymbol{w}^T(k)\boldsymbol{P}_w^{-1}(k)\boldsymbol{w}(k) \leq 1, k = t, t+1, \cdots, N-1, \\ &\boldsymbol{\eta}^T(t)\boldsymbol{P}^{-1}(t)\boldsymbol{\eta}(t) \leq 1.\end{aligned} \tag{56}$$

We introduce an auxiliary variable ρ_t and rewrite (56) as

$$\begin{aligned}(\mathcal{P}_t): \quad &\min_{\boldsymbol{z}_t} \quad \rho_t \\ s.t. \quad &\rho_t - \boldsymbol{z}_t^T\boldsymbol{z}_t - 2\boldsymbol{h}_t^T\boldsymbol{\xi}_t - 2\boldsymbol{z}_t^T\boldsymbol{F}_t\boldsymbol{\xi}_t - \boldsymbol{\xi}_t^T\hat{\mathscr{C}}_t\boldsymbol{\xi}_t \geq 0, \\ &\boldsymbol{W}_t^T\boldsymbol{P}_t^W\boldsymbol{W}_t \leq N-t, \\ &\boldsymbol{P}_t^W = \mathrm{diag}(\boldsymbol{P}_w^{-1}(t), \cdots, \boldsymbol{P}_w^{-1}(N-1)), \\ &\boldsymbol{\eta}^T(t)\boldsymbol{P}^{-1}(t)\boldsymbol{\eta}(t) \leq 1.\end{aligned} \tag{57}$$

The first constraint in problem (57) can be reformulated as

$$\begin{bmatrix} 1 \\ \boldsymbol{\xi}_t \end{bmatrix}^T \begin{bmatrix} \rho_t - \boldsymbol{z}_t^T\boldsymbol{z}_t & -\boldsymbol{h}_t^T - \boldsymbol{z}_t^T\boldsymbol{F}_t \\ -\boldsymbol{h}_t - \boldsymbol{F}_t^T\boldsymbol{z}_t & -\hat{\mathscr{C}}_t \end{bmatrix} \begin{bmatrix} 1 \\ \boldsymbol{\xi}_t \end{bmatrix} \geq 0. \tag{58}$$

The second constraint is rewritten as

$$\begin{bmatrix} \boldsymbol{\eta}(t) \\ \boldsymbol{W}_t \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{P}_t^W \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}(t) \\ \boldsymbol{W}_t \end{bmatrix} \leq N-t\ . \tag{59}$$

With the definition of matrix $\boldsymbol{M}_t^W$ in (50), this constraint can be reformed as

$$\begin{bmatrix} 1 \\ \boldsymbol{\xi}_t \end{bmatrix}^T \begin{bmatrix} N-t & 0 \\ 0 & -\boldsymbol{M}_t^W \end{bmatrix} \begin{bmatrix} 1 \\ \boldsymbol{\xi}_t \end{bmatrix} \geq 0\ . \tag{60}$$

The third constraint can be rewritten as

$$\begin{bmatrix} \boldsymbol{\eta}(t) \\ \boldsymbol{W}_t \end{bmatrix}^T \begin{bmatrix} \boldsymbol{P}^{-1}(t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}(t) \\ \boldsymbol{W}_t \end{bmatrix} \leq 1\ . \tag{61}$$

With the definition of matrix $\boldsymbol{M}^{\boldsymbol{\eta}}(t)$ in (49), this constraint can be reformed as

$$\begin{bmatrix} 1 \\ \boldsymbol{\xi}_t \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -\boldsymbol{M}^{\boldsymbol{\eta}}(t) \end{bmatrix} \begin{bmatrix} 1 \\ \boldsymbol{\xi}_t \end{bmatrix} \geq 0. \tag{62}$$

According to the S-procedure [15], for all $\boldsymbol{\xi}_t$ that satisfies the constraints in (60) and (62), the constraint in (58) also holds if there exist $\tau_1 \geq 0$ and $\tau_2 \geq 0$ such that

$$\begin{bmatrix} \rho_t - \boldsymbol{z}_t^T \boldsymbol{z}_t & -\boldsymbol{h}_t^T - \boldsymbol{z}_t^T \boldsymbol{F}_t \\ -\boldsymbol{h}_t - \boldsymbol{F}_t^T \boldsymbol{z}_t & -\hat{\mathscr{C}}_t \end{bmatrix} - \tau_1 \begin{bmatrix} 1 & 0 \\ 0 & -\boldsymbol{M}^{\boldsymbol{\eta}}(t) \end{bmatrix} - \tau_2 \begin{bmatrix} N-t & 0 \\ 0 & -\boldsymbol{M}_t^W \end{bmatrix} \geq 0. \tag{63}$$

We collect all terms in (63) and reform them as

$$\begin{bmatrix} \rho_t - \tau_1 - \tau_2(N-t) & -\boldsymbol{h}_t^T \\ -\boldsymbol{h}_t & \boldsymbol{G}_t \end{bmatrix} - \begin{bmatrix} \boldsymbol{z}_t & \boldsymbol{F}_t \end{bmatrix}^T \begin{bmatrix} \boldsymbol{z}_t & \boldsymbol{F}_t \end{bmatrix} \geq 0. \tag{64}$$

According to the Schur complement theorem [15], (64) can be rewritten as (46), and then the proof is completed. □

Note that after the reformulation, now $(\mathcal{P}_t)$ in Theorem 2 is a typical semidefinite programming (SDP) problem that can be solved efficiently by several available algorithms.

V. Simulations and Results

This section demonstrates the performance of our approach by numerical simulations. We also compare our methods with the LQC control law designed by Bertsimas and Brown [2].

The simulation considers the control problem $(\mathcal{P}_t)$ in (23) with the following particulars

$$\boldsymbol{A}(k) = (1 + 0.05\sin(k)) \begin{bmatrix} 0.6 & 0.7 \\ 0.25 & 0.5 \end{bmatrix}, \tag{65}$$

$$\boldsymbol{B}(k) = \begin{bmatrix} 1 \\ 0.3 \end{bmatrix}, \boldsymbol{C}(k) = \begin{bmatrix} 0.2 & 1 \end{bmatrix}, \tag{66}$$

$$\boldsymbol{P}_w(k) = \begin{bmatrix} (0.1\arctan(k))^2 & 0 \\ 0 & (0.1\arctan(k))^2 \end{bmatrix}, \tag{67}$$

$$\boldsymbol{P}_v(k) = (0.1\arctan(k))^2, \tag{68}$$

$$\boldsymbol{Q}(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{R}(k) = 1, \quad N = 30. \tag{69}$$

Assume the initial state is confined in an ellipsoidal set, where the ellipsoid center $\hat{\boldsymbol{x}}(0)$ and the shape matrix $\boldsymbol{P}(0)$ are set as

$$\hat{\boldsymbol{x}}(0) = \begin{bmatrix} \hat{x}_1(0) \\ \hat{x}_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}, \boldsymbol{P}(0) = \begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}. \tag{70}$$

Based on the above simulation configurations, we use the toolbox YALMIP to solve the SDP problem as explained in Theorem 2. We implement the proposed method in MATLAB.

Fig. 2 shows the learning progress of system state x_1 and x_2 represented by an ellipsoidal set in one simulation. We observe that the ellipsoids representing the boundary of the learned state shrink over time, indicating the mitigated uncertainty of state estimation by the ellipsoidal-set learning. The center of the estimated ellipsoidal set asymptotically converges to the true value of system state after 3 steps. Fig. 3 compares performances of the control under the proposed robust control and that with fixed feasible set. We run 100 simulations under both of the methods, and check whether the system states are concentrated around zero, which is the anticipated control result. Fig. 3 manifests more concentrated system states around zero under the proposed approach, implying an enhanced control performance in the system with the existence of unknown and bounded noises.

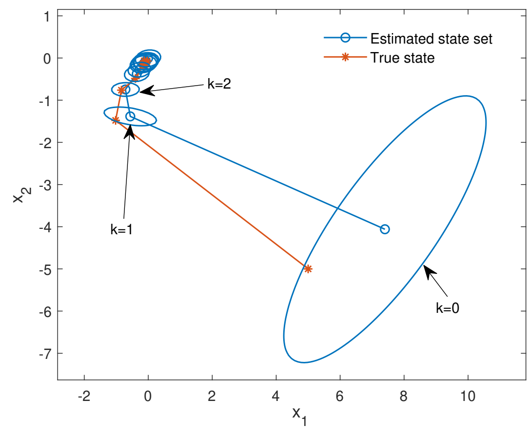

Fig. 2. The learning progress of ellipsoidal set for the system state.

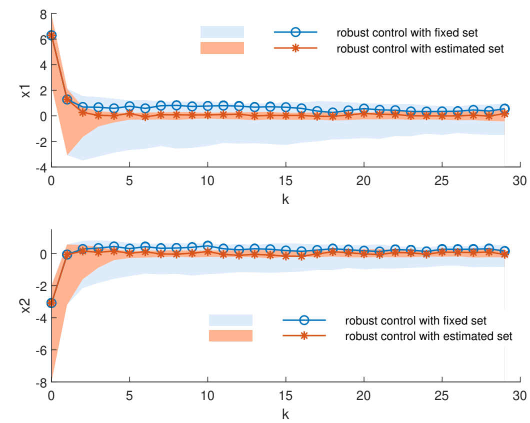

Fig. 3. Control performances under robust control with ellipsoidal and fixed feasible set. The red and blue areas show range of the system state during 100 simulations under the control with ellipsoidal set and fixed set, respectively. The circle-marked line presents the result of an one-time simulation.

TABLE I
Comparison between robust control performance with fixed set and estimated set.

T	J_E	J_{C_1}	J_{C_2}	Ratio
5	1.98	18.95	12.36	53.32%
10	0.99	9.47	6.18	53.24%
15	0.65	6.32	4.12	53.40%
20	0.50	4.74	3.09	53.40%
30	0.34	3.16	2.06	53.40%

Table I presents the average result of 100 simulations. In this table, T denotes the control horizon, J_E is the state estimation performance index defined as $\sum_{k=1}^{N}\sum_{i=1}^{n}(\boldsymbol{x}_i(k)-\hat{\boldsymbol{x}}_i(k))^2$ for the proposed method. J_{C_1} and J_{C_2} are the control performance indices for robust control with fixed set and ellipsoidal set, respectively, and they are both calculated by $\sum_{k=1}^{N}\sum_{i=1}^{n}(\boldsymbol{x}_i(k))^2$. $Ratio$ represents the performance improvement defined as $(J_{C_1}-J_{C_2})/J_{C_2}$. Table I shows that the state estimation improves over time, and the proposed method brings ca. 53% improvement compared with robust control with fixed feasible set.

VI. Conclusion

We presented a robust control method for linear systems with unknown noises and unobservable system states, which are represented by ellipsoidal sets. We integrated this ellipsoidal representation in the control, and adopted ellipsoidal set-membership filter to iteratively learn and reduce the ellipsoidal set boundaries. This leads to a narrowed and concentrated range of worst cases feeding to the robust control. Upon the learned ellipsoidal sets, we derived a robust optimal control law by solving a semidefinite programming (SDP) problem, which guarantees high-performing control in the worst cases while avoiding over-conservativeness. We envisage future research to reduce computation complexity, since SDP in our control law derivation can be computationally expensive as the control problem expands e.g., with a large number of system states.

References

[1] Karl J Åström. *Introduction to Stochastic Control Theory*. Courier Corporation, 2012.

[2] Dimitris Bertsimas and David B Brown. Constrained stochastic LQC: a tractable approach. *IEEE Transactions on Automatic Control*, 52(10):1826–1841, 2007.

[3] Duan Li, Fucai Qian, and Peilin Fu. Optimal nominal dual control for discrete-time linear-quadratic gaussian problems with unknown parameters. *Automatica*, 44(1):119–127, 2008.

[4] Xuehui Ma, Fucai Qian, Shiliang Zhang, and Li Wu. Adaptive quantile control for stochastic system. *ISA Transactions*, 123:110–121, 2022.

[5] Yasmina Becis-Aubry, Mohamed Boutayeb, and Mohamed Darouach. State estimation in the presence of bounded disturbances. *Automatica*, 44(7):1867–1873, 2008.

[6] Matthias Lorenzen, Mark Cannon, and Frank Allgöwer. Robust MPC with recursive model update. *Automatica*, 103:461–471, 2019.

[7] Anilkumar Parsi, Andrea Iannelli, and Roy S. Smith. An explicit dual control approach for constrained reference tracking of uncertain linear systems. *IEEE Transactions on Automatic Control*, 68(5):2652–2666, 2023.

[8] DG Maksarov and JP Norton. State bounding with ellipsoidal set description of the uncertainty. *International Journal of Control*, 65(5):847–866, 1996.

[9] Yuan Wang, Jian Huang, Dongrui Wu, Zhi-Hong Guan, and Yan-Wu Wang. Set-membership filtering with incomplete observations. *Information Sciences*, 517:37–51, 2020.

[10] Elena Arcari, Andrea Iannelli, Andrea Carron, and Melanie N. Zeilinger. Stochastic MPC with robustness to bounded parametric uncertainty. *IEEE Transactions on Automatic Control*, pages 1–14, 2023.

[11] Xuehui Ma, Shiliang Zhang, Yushuai Li, Fucai Qian, and Tingwen Huang. Adaptive robust tracking control with active learning for linear systems with ellipsoidal bounded uncertainties. *arXiv preprint arXiv:2308.03727*, 2023.

[12] Fucai Qian, Jianjun Gao, and Duan Li. Adaptive robust tracking for uncertain system. In *49th IEEE Conference on Decision and Control (CDC)*, pages 4293–4298. IEEE, 2010.

[13] Anilkumar Parsi, Andrea Iannelli, and Roy S Smith. Scalable tube model predictive control of uncertain linear systems using ellipsoidal sets. *International Journal of Robust and Nonlinear Control*, 2022.

[14] Andrea Iannelli, Mohammad Khosravi, and Roy S Smith. Structured exploration in the finite horizon linear quadratic dual control problem. *IFAC-PapersOnLine*, 53(2):959–964, 2020.

[15] Stephen Boyd, Laurent El Ghaoui, Eric Feron, and Venkataramanan Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.