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About an alternative S-variable condition for state-feedback design

Dimitri Peaucelle[1], Yohei Hosoe[2], Yoshio Ebihara[3]

[1] LAAS-CNRS, Univ. Toulouse, CNRS, Toulouse, France
[2] Kyoto University, Kyoto, Japan
[3] Kyushu University, Fukuoka, Japan

***Abstract*— The study of recent papers brought our attention to an alternative matrix inequality condition for state-feedback design. This condition falls in the category of S-variable results. At the difference of previous conditions it does not involve Schur complement or duality arguments and thus simplifies significantly the mathematical derivations. We provide in this paper a detailed description of the core elements of this new to us result. We then expose some derivations for pole location and time-response performances of uncertain systems with constant or time-varying uncertainties, as well as for some non-linear systems.**
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I. Introduction

In this paper we continue our study of the S-variable approach as we defined it in [9], which has also many other names such as the descriptor system approach [11], dilated LMI approach [12], [10], enhanced LMI characterization [1], extended LMI characterization [21] or mentioned via Finsler's lemma core mathematical tool [15]. As in the seminal papers [13], [18] we consider D-stability which corresponds to pole location of linear systems in regions of the complex plane and allows to specify constraints on the times responses of systems such as exponential convergence, damping and frequencies of the oscillating components. As in [11], [26], [3] we exploit the particularity that S-variable approach is well suited for the study of descriptor systems, which provides convex solutions even for rationaly-dependent uncertain or non-linear systems [16], [22], [25], [7], [19].

For this given framework there are many results for the analysis of uncertain and non-linear systems but fewer for the design of controllers. Even the simplest case of state-feedback design appears to be complicated. As in the case of non-descriptor systems, most linearizing change of variables results are build upon analysis conditions of a dual system and use a Lyapunov certificate X taken to be the inverse of the Lyapunov matrix $P = X^{-1}$ for the original system. Obtaining such dual system for descriptor systems reveals to be possible but not trivial and only for special cases [23], [20]. Yet, we recently noticed that there exists an alternative strategy. It is exploited in [14] and we also found out that it was exploited earlier in [3]. The result also imposes a modification on the Lyapunov certificate $Q = V^{-T}PV^{-1}$ but does not need the whole machinery of system duality.

The outline of the paper is as follows. First as preliminaries we defined the mathematical set-up and recall the core features of most exploited technique for state-feedback design which involves (explicitly or sometimes implicitly) the use of a dual or transposed system. We then move to the central contribution of the paper which describes the alternative linearizing change of variables, its drawbacks but also some extensions to time-varying uncertain systems as well as to some non-linear systems. Finally we illustrate the result on an academic example.

II. Preliminaries

A. Notation

I and 0 stand for the identity and zero matrices of appropriate size given by the context. The notation $\star$ is used to denote terms in matrix inequalities that may be deduced by symmetry. $\overline{A}$ is the conjugate of the matrix A and A^* is the conjugate transpose. $\{A\}^{\mathcal{H}}$ stands for the Hermitian matrix $\{A\}^{\mathcal{H}} = A + A^*$. The $e_i = (0 \cdots 1 \cdots 0)^T$ vectors form the standard basis of $\mathbb{R}^n$. $A \succ B$ is the matrix inequality stating that $A - B$ is symmetric positive definite. The terminology *"congruence operation of A on B"* is used to denote $A^T BA$. If A is full column rank, and $B \succ C$, the congruence operation of A on $B \succ C$ gives the valid matrix inequality: $A^T BA \succ A^T CA$. A matrix inequality of the type $N(X) \succ 0$ is said to be a linear matrix inequality (LMI for short), if $N(X)$ is affine in the decision variables X. LMIs are convex and solutions can be found by efficient semi-definite programming tools. Decision variables are highlighted using the blue color. $\Xi_{\bar{v}} = \{\xi_{v=1\dots\bar{v}} \geq 0,\ \sum_{v=1}^{\bar{v}} \xi_v = 1\}$ is the unitary simplex in $\mathbb{R}^{\bar{v}}$. The elements ξ of unitary simplexes are used to describe polytopic type uncertainties. In the following, uncertainties are highlighted using the red color.

B. Problem statement

We consider state-feedback ($u = Kx$) design for descriptor uncertain systems defined by

$$E_x(\theta)\delta[x] + E_\pi(\theta)\pi = A(\theta)x + B(\theta)u \qquad (1)$$

where u is the control input vector, x is the state, π is an internal vector of signals implicitly dependent of x and u,

$\delta[x]$ can be either $\delta[x](t) = \dot{x}(t)$ in case of continuous-time systems or $\delta[x](t) = x(t+1)$ in case of discrete-time systems, and where the matrices are affine in uncertainties θ that lie in a polytope defined as the convex hull of a finite number of vertices:

$$\Theta = \{\theta = \sum_{v=1}^{\bar{v}} \xi_v \theta^{[v]} \ : \ \xi \in \Xi_{\bar{v}}\}. \qquad (2)$$

For simplicity of notations we denote $E_x^{[v]} = E_x(\theta^{[v]})$, $E_\pi^{[v]} = E_\pi(\theta^{[v]})$, $A^{[v]} = A(\theta^{[v]})$ and $B^{[v]} = B(\theta^{[v]})$.

In this paper, we assume regular impulse free systems and more specifically that the matrix

$$\left[\begin{array}{cc} E_\pi(\theta) & E_x(\theta) \end{array}\right] \in \mathbb{R}^{n\times(n_x+n_\pi)}$$

is square ($n = n_x + n_\pi$) non-singular for all uncertainties θ. The assumption implies that the system can also be written in the following non-descriptor form $\delta[x] = \hat{A}_1(\theta)x + \hat{B}_1(\theta)u$ where

$$\left[\begin{array}{cc} E_x(\theta) & E_\pi(\theta) \end{array}\right]^{-1} \left[\begin{array}{cc} A(\theta) & B(\theta) \end{array}\right] = \left[\begin{array}{cc} \hat{A}_1(\theta) & \hat{B}_1(\theta) \\ \hat{A}_2(\theta) & \hat{B}_2(\theta) \end{array}\right]. \qquad (3)$$

The non-descriptor representation has the disadvantage of being rationally-dependent on the uncertain parameters θ. As discussed in papers such as [16], [22], it is always possible, at the expense of introducing the artificial internal vector π but without modifying the state x, to build an affine descriptor representation (1) whatever initial rationally-dependent representation. The assumption that (1) is affine polytopic in the parameters is hence not restrictive compared to linear systems rational in the parameters. The same technique is also used in [24], [25], [7] to handle non-linear systems that are rational in the states.

The poles of the uncertain closed-loop system are defined as the complex valued scalars $\lambda(\theta)$ such that there exists nonzero vectors $(x_\lambda^*(\theta), \pi_\lambda^*(\theta))^*$ solution to

$$\lambda(\theta)E_x(\theta)x_\lambda(\theta) + E_\pi(\theta)\pi_\lambda(\theta) = (A(\theta) + B(\theta)K)x_\lambda(\theta). \qquad (4)$$

With the considered regular impulse free assumption, the poles of the descriptor model coincide exactly with those of the non-descriptor model $\delta[x] = (\hat{A}_1(\theta) + \hat{B}_1(\theta)K)x$.

We aim at finding state-feedback gains such that the poles lie in open sets satisfying a quadratic inequality

$$\mathcal{D}_R = \left\{ \lambda \in \mathbb{C} : \left(\begin{array}{c} 1 \\ \lambda \end{array}\right)^* R \left(\begin{array}{c} 1 \\ \lambda \end{array}\right) = \begin{array}{c} R_{11} \\ +\{R_{12}\lambda\}^{\mathcal{H}} \\ +R_{22}\bar{\lambda}\lambda \end{array} < 0 \right\} \qquad (5)$$

where $R = R^*$. In case $\delta[x](t) = \dot{x}(t)$, $R_{11} = R_{22} = 0$ and $R_{12} = 1$ the property is exactly the stability of continuous-time systems. In case $\delta[x](t) = x(t+1)$, $R_{11} = -1$, $R_{22} = 1$ and $R_{12} = 0$ the property is exactly the stability of discrete-time systems. For this reason the property that all poles lie in the region $\mathcal{D}_R$ is called $\mathcal{D}_R$-stability. When $R_{22} > 0$ the region is an open disc of $\mathbb{C}$. When $R_{22} < 0$ the region is the exterior of a closed disk. When $R_{22} = 0$ the region is an open half-plane. The region $\mathcal{D}_{\overline{R}}$ is the symmetric of $\mathcal{D}_R$ with respect to the real axis.

C. Existing results for robust state-feedback

Theorem 1: Given a matrix A_S whose eigenvalues are all in $\mathcal{D}_R$, if there exist $X^{[v=1\dots\bar{v}]} \succ 0$, S and T solution to the following $v \in \{1\cdots\bar{v}\}$ LMIs

$$\overline{R} \otimes X^{[v]} \prec \left\{ \left[\begin{array}{c} A^{[v]}S + B^{[v]}T \\ -S \end{array}\right] \left[\begin{array}{c} A_S \\ -I \end{array}\right]^* \right\}^{\mathcal{H}} \qquad (6)$$

then $K = TS^{-1}$ is a state-feedback gain such that the uncertain system $\delta[x] = (A(\theta) + B(\theta)K)x$ is robustly $\mathcal{D}_R$-stable.

This well established result [9] has proved to be very powerful but has some limitations. In particular it is hard to extend it for descriptor systems. To have an idea why this is the case, let us summarize the main feature of this result.

Because of the convexity of the semi-definite cone as the LMIs (6) hold on vertices, they also hold on any convex linear combination $\theta \in \Theta$ of these

$$\overline{R} \otimes X(\theta) \prec \left\{ \left[\begin{array}{c} A(\theta)S + B(\theta)T \\ -S \end{array}\right] \left[\begin{array}{c} A_S \\ -I \end{array}\right]^* \right\}^{\mathcal{H}} \qquad (7)$$

where $X(\theta) = \sum_{v=1}^{\bar{v}} \xi_v X^{[v]} \succ 0$. Let $\overline{\lambda}(\theta)$ be any eigenvalue of $(A(\theta) + B(\theta)K)^*$ and $x_d(\theta)$ be the associated eigenvector such that $(A(\theta) + B(\theta)K)^* x_d(\theta) = \overline{\lambda}(\theta)x_d(\theta)$. By congruence of $\left(\begin{array}{cc} x_d(\theta)^* & \lambda(\theta)x_d(\theta)^* \end{array}\right)$ on (7) one gets

$$\left(\begin{array}{cc} 1 & \lambda(\theta) \end{array}\right) \overline{R} \left(\begin{array}{c} 1 \\ \overline{\lambda}(\theta) \end{array}\right) (x_d^*(\theta)X(\theta)x_d(\theta)) < 0$$

which proves that the eigenvalues $\overline{\lambda}(\theta)$ are in $\mathcal{D}_{\overline{R}}$ and hence the eigenvalues $\lambda(\theta)$ of $A(\theta) + B(\theta)K$ are in $\mathcal{D}_R$, thus proving robust $\mathcal{D}_R$-stability.

Note that the proof relies strongly on the fact that the conjugate of poles of the closed-loop system are the eigenvalues of the conjugate transpose of the matrices defining the system, and they lie in the region defined by the conjugate matrix $\overline{R}$. This property is not achievable (at least not easily) for descriptor system. In the special case of periodic systems with lifting it has been achieved in [23] and also obtained in [20] for a specific type of descriptor multi-affine representations of non-descriptor systems rational in the uncertainties. But there are no simple extensions of this state-feedback result for general descriptor systems such as the ones we consider.

Before getting to the main result of the paper, let us recall that the formulation in Theorem 1 has attracted much attention because it can be related to a previous result known as quadratic stability [2]. Quadratic stability exists only for convex regions $\mathcal{D}_R$ such as disks ($R_{22} > 0$) and half-planes ($R_{22} = 0$). In the following Theorem we denote $\hat{A}^{[v]} = A^{[v]}X + B^{[v]}Y$.

Theorem 2: In case $R_{22} \geq 0$, if there exist $X \succ 0$ and Y solution to the following $v \in \{1\cdots\bar{v}\}$ LMIs

$$\left[\begin{array}{cc} R_{11}X + \{R_{12}\hat{A}^{[v]}\}^{\mathcal{H}} & \star \\ R_{22}\hat{A}^{[v]} & -R_{22}X \end{array}\right] \prec 0 \ , \text{ if } R_{22} > 0 \qquad (8)$$

$$R_{11}X + \{R_{12}\hat{A}^{[v]}\}^{\mathcal{H}} \prec 0 \ , \ \text{if } R_{22} = 0 \tag{9}$$

then $K = YX^{-1}$ is a state-feedback gain such that $\delta[x] = (A(\theta) + B(\theta)K)x$ is robustly $\mathcal{D}_R$-stable.

As studied in details in [9], if $R_{22} > 0$, the choice $A_S = -\frac{R_{12}^*}{R_{22}}I$ (all eigenvalues of A_S are the at the center of the disc $\mathcal{D}_R$) guarantees that LMIs of Theorem 1 are less conservative than those of Theorem 2. Moreover the proof relies on the choice of $X^{[v]} = X$. This comparison to quadratic stability explains why the attention of the control community has concentrated on conditions of Theorem 1 as soon as the seminal papers [8], [17].

III. New S-variable results for state-feedback

A. Linearizing change of variables

First let us remind that Theorem 1 is related to this analysis condition:

$$\overline{R} \otimes X(\theta) \prec \left\{ \begin{bmatrix} A_{cl}(\theta) \\ -I \end{bmatrix} \begin{bmatrix} S_1 & S_2 \end{bmatrix} \right\}^{\mathcal{H}}$$

that coincides with (7) when taking $A_{cl}(\theta) = A(\theta)+B(\theta)K$, $S_1 = SA_S^*$, $S_2 = -S$. A counterpart of that analysis condition, which does not need to consider the transpose conjugate system, is the following

$$R \otimes P(\theta) \prec \left\{ \begin{bmatrix} \tilde{S}_1 \\ \tilde{S}_2 \end{bmatrix} \begin{bmatrix} A_{cl}(\theta) & -I \end{bmatrix} \right\}^{\mathcal{H}}. \tag{10}$$

This counterpart has proved to be very efficient to deal with descriptor system analysis (see for example Chapter 2 of [9]) but, at our best knowledge, was almost never used for state-feedback design because of the seemingly impossibility of performing a linearizing change of variables when the $\tilde{S}_2$ matrix multiplies $A_{cl}(\theta) = A(\theta)+B(\theta)K$ from the left-hand side. As discovered in [14] and can also be seen earlier in [3], this difficulty can be avoided and the result is extremely simple. Take $V = -\tilde{S}_2^{-*}$, by congruence of $\text{diag}(V^*, V^*)$ on (10) one gets

$$R \otimes Q(\theta) \prec \left\{ \begin{bmatrix} A_V^* \\ -I \end{bmatrix} \begin{bmatrix} A_{cl}(\theta)V & -V \end{bmatrix} \right\}^{\mathcal{H}}. \tag{11}$$

where $A_V^* = V^*\tilde{S}_1$ and $Q(\theta) = V^*P(\theta)V$. This formulation allows to recover a simple linearizing change of variables $A_{cl}(\theta)V = A(\theta)V + B(\theta)W$ with $W = KV$. With this fact we may now formulate the central result for robust state-feedback of descriptor systems

Theorem 3: Given a system $E_{xV}\delta[x] + E_{\pi V}\pi = A_V x$ of same dimensions as (1) and having its poles in $\mathcal{D}_R$, if there exist $Q^{[v=1\ldots\bar{v}]} \succ 0$, V, W and U solution to the following $v \in \{1 \cdots \bar{v}\}$ LMIs

$$\begin{bmatrix} R \otimes Q^{[v]} & 0 \\ 0 & 0 \end{bmatrix} \prec \left\{ \begin{bmatrix} A_V^* \\ -E_{xV}^* \\ -E_{\pi V}^* \end{bmatrix} M^{[v]} \right\}^{\mathcal{H}} \tag{12}$$

where

$$M^{[v]} = \begin{bmatrix} (A^{[v]}V + B^{[v]}W) & -E_x^{[v]}V & -E_\pi^{[v]}U \end{bmatrix}$$

then $K = WV^{-1}$ is a state-feedback gain robustly $\mathcal{D}_R$-stabilyzing the system (1).

Proof First we state the reason of the assumptions on the $E_{xV}\delta[x] + E_{\pi V}\pi = A_V x$ system. Let λ_V be any pole and (x_V, π_V) be the associated vectors. By congruence of $\begin{pmatrix} x_V^* & \bar{\lambda}_V x_V^* & \pi_V^* \end{pmatrix}$ on (12) one gets

$$\begin{pmatrix} 1 & \bar{\lambda}_V \end{pmatrix} R \begin{pmatrix} 1 \\ \lambda_V \end{pmatrix} (x_V^* Q^{[v]} x_V) < 0.$$

Since $Q^{[v]} \succ 0$ it implies that λ_V is in $\mathcal{D}_R$.

Now let us consider the robust state-feedback property. Because of the convexity of the semi-definite cone as the LMIs (12) hold on vertices, they also hold on any convex linear combination $\theta \in \Theta$ of these

$$\begin{bmatrix} R \otimes Q(\theta) & 0 \\ 0 & 0 \end{bmatrix} \prec \left\{ \begin{bmatrix} A_V^* \\ -E_{xV}^* \\ -E_{\pi V}^* \end{bmatrix} M(\theta) \right\}^{\mathcal{H}} \tag{13}$$

where

$$M(\theta) = \begin{bmatrix} (A(\theta)V + B(\theta)W) & -E_x(\theta)V & -E_\pi(\theta)U \end{bmatrix}$$

and $Q(\theta) = \sum_{v=1}^{\bar{v}} \xi_v Q^{[v]} \succ 0$. Let $\lambda(\theta)$ be any pole of the system and let $(x_\lambda(\theta), \pi_\lambda(\theta))$ be the associated vectors such that (4) holds. By congruence of $\begin{bmatrix} x_\lambda(\theta)^* V^{-*} & \bar{\lambda}(\theta) x_\lambda(\theta)^* V^{-*} & \pi_\lambda(\theta)^* U^{-*} \end{bmatrix}$ on (13) one gets

$$\begin{pmatrix} 1 & \bar{\lambda}(\theta) \end{pmatrix} R \begin{pmatrix} 1 \\ \lambda(\theta) \end{pmatrix} (x_\lambda^*(\theta) P(\theta) x_\lambda(\theta)) < 0$$

where $P(\theta) = V^{-*}Q(\theta)V^{-1} \succ 0$, which proves that $\lambda(\theta)$ is in $\mathcal{D}_R$. It holds for all poles and all $\theta \in \Theta_{\bar{v}}$. ■

Following the same lines as the proof above the following analysis theorem holds for a given $K = K_o$.

Theorem 4: If there exist $P^{[v=1\ldots\bar{v}]} \succ 0$ and S solution to the following $v \in \{1 \cdots \bar{v}\}$ LMIs

$$\begin{bmatrix} R \otimes P^{[v]} & 0 \\ 0 & 0 \end{bmatrix} \prec \left\{ SM_c^{[v]} \right\}^{\mathcal{H}} \tag{14}$$

where

$$M_c^{[v]} = \begin{bmatrix} (A^{[v]} + B^{[v]}K_o) & -E_x^{[v]} & -E_\pi^{[v]} \end{bmatrix}$$

then the closed loop of (1) with $u = K_o x$ is robustly $\mathcal{D}_R$-stable.

Moreover, with classical S-variable arguments [9], the LMIs are necessary and sufficient for systems without uncertainties.

Theorem 5: Given a value $\theta = \theta_o$ and a state-feedback $K = K_o$ the system (1) with $u = K_o x$ is $\mathcal{D}_R$-stable if and only if there exist $P_o \succ 0$ and S solution to the following LMI

$$\begin{bmatrix} R \otimes P_o & 0 \\ 0 & 0 \end{bmatrix} \prec \{SM_c(\theta_o)\}^{\mathcal{H}} \tag{15}$$

where

$$M_c(\theta_o) = \begin{bmatrix} (A(\theta_o) + B(\theta_o)K_o) & -E_x(\theta_o) & -E_\pi(\theta_o) \end{bmatrix}.$$

The proof of necessity relies on Finsler's lemma and one admissible choice of the S-variable is

$$S^* = \tau \begin{bmatrix} (A(\theta_o)+B(\theta_o)K_o) & -E_x(\theta_o) & -E_\pi(\theta_o) \end{bmatrix}$$

for some scalar $\tau > 0$.

B. Choice of the E_{xV}, $E_{\pi V}$, A_V matrices

Although not written in blue color, the matrices E_{xV}, $E_{\pi V}$ and A_V are decision variables of the problem. Once these matrices are selected, the design problem becomes convex and solvable by semi-definite programming tools. Searching simultaneously for these matrices and the other decision variables makes the problem a Bilinear Matrix Inequality which is known to be in general non-convex and there is no direct method that would find a feasible solution for sure when such solution exists. Hence, we provide some appropriate heuristics choices.

The first one is to choose the virtual system among non-descriptor representations:

$$\begin{bmatrix} E_{xV} & E_{\pi V} & A_V \end{bmatrix} = \begin{bmatrix} I & 0 & A_{1V} \\ 0 & I & A_{2V} \end{bmatrix}.$$

In that case the system boils down to $\delta[x] = A_{1V}x$ and $\pi = A_{2V}x$. It is trivially $\mathcal{D}_R$ stable whatever A_{2V} if A_{1V} has its eigenvalues in $\mathcal{D}_R$. An easy choice is $A_{1V} = \lambda I$ with $\lambda \in \mathcal{D}_R$. Moreover, if $R_{22} > 0$, the choice $A_{1V} = -\frac{R_{12}^*}{R_{22}} I$ is appropriate and is less conservative than quadratic stability for non-descriptor systems of Theorem 2. Indeed, for the choice of $Q^{[v]} = X$, $V = R_{22}X$ and $W = R_{22}Y$, the conditions (12) for non-descriptor systems read as

$$R \otimes X \prec \begin{bmatrix} \{-R_{12}\hat{A}^{[v]}\}^{\mathcal{H}} & \star \\ R_{12}^* X - R_{22}\hat{A}^{[v]} & 2R_{22}X \end{bmatrix}.$$

After congruence of $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ this inequality is exactly (8).

Another heuristic choice is to choose some 'nominal' value of the uncertainty $\theta_o \in \Theta_{\bar{v}}$, which may be the center of the simplex ($\theta_{ov} = \frac{1}{\bar{v}}$), or which corresponds to the most probable value of the uncertain parameters. For that value one can compute the non-descriptor model of the plant $\delta[x] = \hat{A}_1(\theta_o)x + \hat{B}_1(\theta_o)u$ and K_o a $\mathcal{D}_R$ stabilizing state-feedback for it. K_o can be designed, for example, with help of Theorem 2. Then, Theorem 5 guarantees the existence of a solution to

$$\begin{bmatrix} R \otimes Q_o & 0 \\ 0 & 0 \end{bmatrix} \prec \left\{ \begin{bmatrix} A_V^* \\ -E_{xV}^* \\ -E_{\pi V}^* \end{bmatrix} M_c(\theta_o) \right\}^{\mathcal{H}}. \quad (16)$$

The heuristic is to solve this LMI for the choice of nominal parameters θ_o and nominal state-feedback gain K_o. The LMIs are guaranteed to be feasible. Alternatively, by Finsler lemma, one can choose $E_{xV} = E_x(\theta_o)$, $E_{\pi V} = E_\pi(\theta_o)$, $A_V = A(\theta_o)+B(\theta_o)K_o$ without conservatism. Such choices of E_{xV}, $E_{\pi V}$, A_V matrices ensure that (13) is feasible at least for $\theta = \theta_o$.

C. Some extensions

1) Full information feedback: It is easy to notice that if one modifies the Theorem with the following matrix

$$M^{[v]} = \begin{bmatrix} (A^{[v]}V + B^{[v]}W_x) & -E_x^{[v]}V & (B^{[v]}W_\pi - E_\pi^{[v]}U) \end{bmatrix}$$

where W_π is an additional decision variable, then it results in the design of a full-information feedback $u = K_x x + K_\pi \pi$ where $K_x = W_x V^{-1}$ and $K_\pi = W_\pi U^{-1}$.

2) Pole location in unions of regions: As exposed in details in [9], the S-variable approach allows to solve multi performance design problems without imposing the conservative Lyapunov Shaping Paradigm [5]. The conditions provide additional degrees of freedom by the search of distinct certificates $Q_p(\theta)$ for each performance. In case of pole location in intersection of quadratic regions of the complex plane the extension of Theorem 3 reads as:

Theorem 6: Given $p = 1 \ldots \bar{p}$ systems $E_{xVp}\delta[x] + E_{\pi Vp}\pi = A_{Vp}x$ of same dimensions as (1) and having their poles respectively in $\mathcal{D}_{R_p}$, if there exist $Q_{p=1\ldots\bar{p}}^{[v=1\ldots\bar{v}]} \succ 0$, V, W and U solution to the following ($p \in \{1\cdots\bar{p}\}, v \in \{1\cdots\bar{v}\}$) LMIs

$$\begin{bmatrix} R_p \otimes Q_p^{[v]} & 0 \\ 0 & 0 \end{bmatrix} \prec \left\{ \begin{bmatrix} A_{Vp}^* \\ -E_{xVp}^* \\ -E_{\pi Vp}^* \end{bmatrix} M^{[v]} \right\}^{\mathcal{H}} \quad (17)$$

then $K = WV^{-1}$ is a state-feedback gain robustly $\mathcal{D}_{Rp}$-stabilyzing the system (1) for all $p \in \{1\cdots\bar{p}\}$.

3) Dynamic performance of time-varying systems: As proved in [6] the pole location problem has, for some special choices of R_p, extensions for continuous linear time-varying systems. The matrix inequality conditions can provide information on minimal and maximal decay rate, on the damping ratio and on the natural frequencies of the time responses. The only difference compared to Theorem 6 is that the LTV case with no information on the derivative of the parameters requires to search for parameter-independent Q_p certificates.

Theorem 7: Given $p = 1\ldots\bar{p}$ systems $E_{xVp}\dot{x} + E_{\pi Vp}\pi = A_{Vp}x$ of same dimensions as (1) and having their poles respectively in $\mathcal{D}_{R_p}$, if there exist $Q_{p=1\ldots\bar{p}} \succ 0$, V, W and U solution to the following ($p \in \{1\cdots\bar{p}\}, v \in \{1\cdots\bar{v}\}$) LMIs

$$\begin{bmatrix} R_p \otimes Q_p & 0 \\ 0 & 0 \end{bmatrix} \prec \left\{ \begin{bmatrix} A_{Vp}^* \\ -E_{xVp}^* \\ -E_{\pi Vp}^* \end{bmatrix} M^{[v]} \right\}^{\mathcal{H}} \quad (18)$$

then $K = WV^{-1}$ is a state-feedback gain robustly $\mathcal{D}_{Rp}$-stabilyzing the time-varying system (1) for all $p \in \{1\cdots\bar{p}\}$.

In case bounds on the time derivatives of the uncertain parameters are known the results can be readily extended to parameter-dependent certificates $Q_p(\theta(t))$ at the expense of introducing derivatives of the certificates in the formulas. See [6] for details.

4) Dynamic performance of non-linear systems: One special case of time-varying parameters is when parameters are state-dependent. To keep notations simple, we shall consider in this paragraph that $\theta_i(t) = x_i(t)$ with $i \in \mathcal{I}$ defining a subset of states with cardinality $q \le n_x$ and we assume the polytope Θ is a hyper-rectangle with 2^q vertices such that the components of the state are in symmetric intervals around the zero equilibrium: $\Theta = \{\forall i \in \mathcal{I} |x_i| \le \mathrm{x}_i\}$. The additional difficulty is to prove that the state remains in the polytopic set. This is obtained if initial conditions are in the level set of a Lyapunov function contained in the polytope Θ. For the next theorem we assume that the $\mathcal{D}_{R_1}$-stability condition is with $R_1 = \begin{bmatrix} 2\alpha & 1 \\ 1 & 0 \end{bmatrix}$, $\alpha > 0$ thus imposing exponential stability of the closed-loop. The corresponding Lyapunov certificate $P_1 = V^{-*}Q_1V^{-1}$ is used to describe the ellipsoidal level set of initial conditions. Condition (19) coming from [4] guarantees the level set to be inside the polytopic set Θ.

Theorem 8: Given $p = 1 \dots \bar{p}$ systems $E_{xVp}\dot{x} + E_{\pi Vp}\pi = A_{Vp}x$ of same dimensions as (1) and having their poles respectively in $\mathcal{D}_{R_p}$, if there exist $Q_{p=1\dots\bar{p}} \succ 0$, V, W and U solution to the $(p \in \{1 \cdots \bar{p}\}, v \in \{1 \cdots \bar{v}\})$ LMIs (18) and the additional $i \in \mathcal{I}$ LMIs

$$\begin{bmatrix} \mathrm{x}_i^2 Q_1 & V^* e_i \\ e_i^T V & 1 \end{bmatrix} \succeq 0 \tag{19}$$

then $K = WV^{-1}$ is a state-feedback gain robustly $\mathcal{D}_{Rp}$-stabilyzing the nonlinear system (1) for all $p \in \{1 \cdots \bar{p}\}$ and all initial conditions satisfying $x^T(0)V^{-*}Q_1V^{-1}x(0) \le 1$.

IV. Numerical Example

The following example borrowed from [7]

$$\dot{x}_1 = -\frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 - x_2 \qquad \dot{x}_2 = -u$$

admits a descriptor representation (1) with $\pi = x_1^2$ and

$$\begin{bmatrix} E_x & E_\pi(x) & -A(x) & -B \end{bmatrix} = \left[\begin{array}{cc|c|cc|c} 1 & 0 & \frac{1}{2}x_1 & \frac{3}{2}x_1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -x_1 & 0 & 0 \end{array}\right].$$

Notice that this representation is smaller than the one proposed in [7] which is profitable when building matrix inequality conditions. We aim at finding a state-feedback that locally guarantees exponential stability of the type $\|x(t)\| \le \beta_1 e^{-\alpha t}$ and such that the oscillatory type responses $x(t) = x_a(t)\cos(\omega t + \psi) - x_b(t)\sin(\omega t + \psi)$ satisfy the damping property $\|x(t)\| \le \beta_2 e^{-\omega \tan(\phi) t}$. This corresponds to the choice of two $\mathcal{D}_R$-stability constraints with

$$R_1 = \begin{bmatrix} 2\alpha & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad R_2 = \begin{bmatrix} 0 & e^{-j\phi} \\ e^{j\phi} & 0 \end{bmatrix}$$

We consider the case $\alpha = 1$, $\phi = \pi/4$ and $\mathrm{x}_1 = 0.6$.

We first design two state-feedback gains for the linearized system around the equilibrium $x_1 = x_2 = 0$. The first gain $K_1 = \begin{bmatrix} -8.7816 & 4.8575 \end{bmatrix}$ ensures closed-loop exponential stability. The second $K_2 = \begin{bmatrix} -1.4152 & 4.0848 \end{bmatrix}$ ensures the damping property. For each controller we separately solve the LMIs (16) to get candidate matrices E_{xVp}, $E_{\pi Vp}$, A_{Vp} for $p = 1, 2$. Having build these, we apply Theorem 8 that provides us with a state-feedback gain

$$K = \begin{bmatrix} -9.0863 & 5.7317 \end{bmatrix} \tag{20}$$

that solves the considered problem. Finally, we apply the analysis result of Theorem 4, combined with a condition of the type (19) to get an estimate of the initial condition set for which the properties hold. The set is plotted in Figure 1 as well as some trajectories of the closed-loop system. The damping criterion leads to these trajectories with little overshoot.

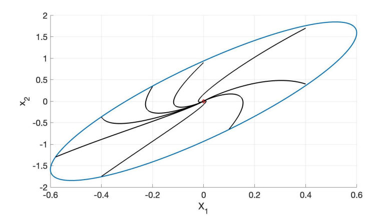

Fig. 1. Trajectories of the closed-loop system with state-feedback (20)

The procedure is then repeated considering full-information feedback. It results in a state-dependent state-feedback gain

$$K(x_1) = \begin{bmatrix} -9.0863 + 0.1580x_1 & 5.7317 \end{bmatrix}.$$

The trajectories are almost identical as above.

To test the sensitivity of the result to the heuristic choice of virtual systems in Theorem 8, we performed the procedure again at the difference that at first step we search for a state-feedback that satisfies simultaneously both performances for the linearized system. We found by applying the Lyapunov Shaping Paradigm the state-feedback gain $K_1 = K_2 = \begin{bmatrix} -16.5144 & 13.4166 \end{bmatrix}$ which is used as above to get candidate matrices E_{xVp}, $E_{\pi Vp}$, A_{Vp} by solving separately the LMIs (16) for $p = 1, 2$. For that heuristic choice of the vitual systems Theorem 8 gives:

$$K = \begin{bmatrix} -37.4696 & 15.9656 \end{bmatrix} \tag{21}$$

As upper, we apply the analysis result of Theorem 4 to get an estimate of the initial condition set of Figure 2. The result is better in the sense of a larger set of initial conditions.

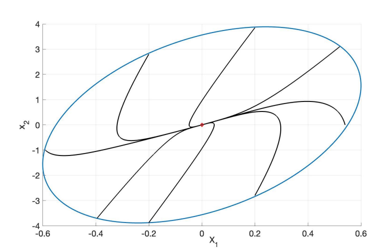

Fig. 2. Trajectories of the closed-loop system with state-feedback (21)

Figure 3 shows the decreasing nature of the quadratic Lyapunov function along four different trajectories of figure 2. The dashed line is the guaranteed upper bound on the exponential convergence of the function.

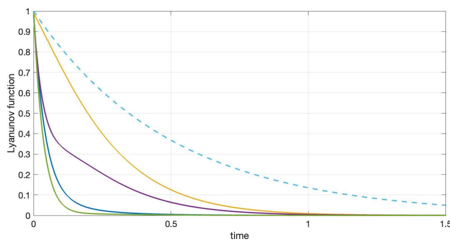

Fig. 3. Lyapunov function along four trajectories with state-feedback (21)

The same system may be lifted by adding an additional state $x_3 = x_1^2$ with its dynamics $\dot{x}_3 = 2x_1\dot{x}_1$. The lifted system with the additional exogenous function $\pi_2 = x_1^3$ admits a descriptor representation

$$\begin{bmatrix} E_x^l & E_\pi^l(x) & -A^l(x) & -B^l \end{bmatrix} = \left[\begin{array}{ccc|cc|ccc|c} 1 & 0 & 0 & \frac{1}{2}x_1 & 0 & \frac{3}{2}x_1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3x_1 & x_1 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_1 & 0 & 1 & 0 \end{array}\right]$$

The results of this conference paper do not apply since the $\begin{bmatrix} E_x^l & E_\pi^l \end{bmatrix}$ is not invertible but extensions are possible as described in [9]. When applied to this lifted system, the extended conditions may provide Lyapunov certificates that are quadratic in the three 'states' x_1, x_2 and $x_3 = x_1^2$. It hence allows to go beyond ellipsoidal Lyapunov level sets. It also allows to build full-information controllers of higher order in x_1.

V. Conclusions

An alternative methodology for dealing with state-feedback is studied. The main advantage is that it allows to deal in a rather simple way with affine descriptor systems and hence with many systems that can easily be converted to this form. In this paper we considered only a sub case with invertibility assumptions on the descriptor part of the models. Further work shall consider the general case without this assumption. Our attention will also go to further numerical testing of the method, both concerning the heuristic choices of the virtual system involved in the main theorem and on more involved examples.

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