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Extended Mixed Filtering based on Zonotopic and Gaussian Uncertainties for Discrete-Time Nonlinear Systems*

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Abstract— **In this paper, we propose a mixed filter for discrete-time nonlinear dynamical systems whose uncertainties are composed of both deterministic and stochastic terms. In practice, such mixed composition of uncertainties may appear when assimilating measured data and approximating models. The unknown-but-bounded terms are here represented by zonotopes, which are efficient representations of centrally symmetric convex polytopes. In turn, the stochastic terms are represented by Gaussian random vectors (GRVs) which address confidence regions with high probability. The proposed state estimator is based on linearized models, using the quasi-linear parameter-varying (LPV) approach. The effectiveness of our proposal is illustrated in two case studies.**

I. INTRODUCTION

Nonlinear state estimators are often designed under approximate techniques such as the unscented transformation [1], analytical linearization around reference trajectories [2], [3], [4], analytical linearization around points belonging to compact sets [5], [6], mean value extension [6], [7], and quasi-LPV approach [8], [9]. Depending on the context, one technique may be more suitable than others. For instance, the quasi-LPV approach could replace the analytical linearization when nonlinear functions do not fulfill with continuity issues [1], [2]. Also, according to the type of uncertainty present in a dynamical system, the approximation may yield guaranteed (set-based context), likely (stochastic context), or combined (mixed context) solutions.

There exist different forms of combining uncertainties in a state estimator. A possibility has been exploited in [10], [11], [12], and [13], where the classical Kalman filter [14] has been extended to include interval parameters in conditional distributions or in system matrices. An alternative is to assume mixed uncertainties as in [15], [16] and [17] for linear systems, in which a direct sum among elements and realizations from unknown-but-bounded and stochastic uncertainties, respectively, is executed. Specifically, in [15], the authors have mixed GRVs with ellipsoids, [16] has mixed GRVs with zonotopes, whereas the authors in [17] have mixed GRVs with constrained zonotopes (an extension of zonotopes) to incorporate trajectory constraints. The use of zonotopes in [16] has allowed to efficiently mitigate the conservatism caused by ellipsoids during Minkowski sums.

The state estimators can be categorized as predictors, filters, and smoothers, according to the assimilated measurements [2], [18]. Specifically, the predictors cannot incorporate current and future measurements to yield the state estimates. This limitation does not invalidate the application of predictors for control loops and fault diagnosis, for instance, but the filters may achieve better accuracy and precision [18]. According to the prior classification, we denote here the mixed algorithms of [8] and [16] as predictors. In [16], the author has proposed a zonotopic and Gaussian Kalman estimator for linear systems (here referred as ZGKP), which has been extended to nonlinear systems in [8] through the quasi-LPV approach (here called EZGKP). Motivated by [16], in [19], we have proposed another zonotopic and Gaussian Kalman estimator for linear systems (called ZGKF).

In this paper, we propose an extended zonotopic and Gaussian Kalman filter, denoted as EZGKF, for nonlinear dynamical systems. This estimator is an extension of the EZGKP [8] to a filter version using, as initial motivation, the linear structure of the ZGKF, and posteriorly, the quasi-LPV approach. In comparison with the predictor version, the EZGKF brings up specific challenges whose solution methodology may imply computational benefits. Moreover, both accuracy and precision can be significantly enhanced. In short, the contributions of this paper are: (i) the new extended zonotopic and Gaussian Kalman filter (EZGKF) for nonlinear dynamical systems whose uncertainties are composed of both deterministic and stochastic terms; and (ii) the worst-case complexity analysis for both EZGKF and EZGKP [8].

II. PRELIMINARIES

The letters c, z, and g denote center, zonotope, and GRV, respectively. Capital letters denote matrices, while lowercase letters denote vectors.

A. Random Vectors

Let X be an n-dimensional random vector. The mean and covariance matrix of X are given by $\bar{x} = \mathrm{E}[X]$ and $P^{\mathrm{xx}} = \mathrm{cov}(X, X) \triangleq \mathrm{E}[(X - \bar{x})(X - \bar{x})^\top]$, respectively, where $\mathrm{E}[\cdot]$ is the expected value operator and $(\cdot)^\top$ is the matrix transpose. A GRV X is characterized by its Gaussian probability density function $p(x)$, which is completely defined by the mean $\bar{x}$ and covariance matrix P^{xx}. Therefore, a GRV X can be abbreviated by $X \sim \mathcal{N}(\bar{x}, P^{\mathrm{xx}})$.

A random variable with chi-square distribution for n degrees of freedom is defined as

$$X^{\chi} \triangleq (X - \bar{x})^\top (P^{\mathrm{xx}})^{-1} (X - \bar{x}), \quad (1)$$

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whose realization x^χ satisfies $x^\chi \geq 0$. The cumulative distribution function $c(x)$ of a given probability density function $p(x)$ is defined as

$$c(x) \triangleq \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} p(\theta_1, \ldots, \theta_n) d\theta_1 \ldots d\theta_n, \quad (2)$$

such that $0 \leq c(x) \leq 1,\ \forall x \in \mathbb{R}^n$.

The affine transformation and sum operations of uncorrelated GRVs are computed as

$$LX + m \sim \mathcal{N}(L\bar{x} + m, LP^{\mathrm{xx}}L^\top), \quad (3)$$
$$X + W \sim \mathcal{N}(\bar{x} + \bar{w}, P^{\mathrm{xx}} + P^{\mathrm{ww}}), \quad (4)$$

where $L \in \mathbb{R}^{b\times n}$, $m \in \mathbb{R}^b$, and $W \sim \mathcal{N}(\bar{w}, P^{\mathrm{ww}})$.

The following lemma allows to overestimate the product between a covariance matrix M and a parameter $\zeta \in [-1, 1]$ by an upper bounded covariance matrix $\overline{M}$.

Lemma 1 ([8]): Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $M = V\Lambda V^{-1}$ be its spectral decomposition with $V^\top V = \mathrm{I}_n$ and real diagonal Λ, where I_n is the $(n \times n)$-dimensional identity matrix. Let $\overline{M} = V|\Lambda|V^\top$. Then, $\overline{M} \succeq 0_{n\times n}$ and, $\forall\zeta \in [-1, 1]$, $\zeta M \preceq \overline{M}$, with $0_{n\times n}$ being a zero matrix.

B. Sets

A box is an n-dimensional interval vector defined as $[x] \triangleq \{x \in \mathbb{R}^n : x_i^{\mathrm{L}} \leq x_i \leq x_i^{\mathrm{U}},\ i = 1, \ldots, n\}$, with x_i^{L} and x_i^{U} being the known lower and upper bounds. A unitary box composed of n unitary intervals is denoted as $\mathcal{B}^n \triangleq [-1, 1]^n$. An interval matrix is the set defined as

$$[A] \triangleq \{A \in \mathbb{R}^{m\times n} : A_{i,j}^{\mathrm{L}} \leq A_{i,j} \leq A_{i,j}^{\mathrm{U}}\}, \quad (5)$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Given an interval matrix $[A]$, $\mathrm{mid}_{i,j}([A]) \triangleq \frac{1}{2}(A_{i,j}^{\mathrm{L}} + A_{i,j}^{\mathrm{U}})$ is its (i, j)th midpoint, while $\mathrm{rad}_{i,j}([A]) \triangleq \frac{1}{2}(A_{i,j}^{\mathrm{U}} - A_{i,j}^{\mathrm{L}})$ is its (i, j)th radius for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Usual operations involving interval vectors and matrices are defined in [20], such as sum, subtraction, product, and division.

Let $G^{\mathrm{x}} \in \mathbb{R}^{n\times n_g}$ and $\bar{x} \in \mathbb{R}^n$ be the generator matrix and the center, respectively. The zonotope $\mathcal{X}$ of order n_g is defined as

$$\mathcal{X} \triangleq \{G^{\mathrm{x}}, \bar{x}\} = G^{\mathrm{x}}\mathcal{B}^{n_g} \oplus \bar{x} = \{(G^{\mathrm{x}}\xi + \bar{x}) : \xi \in \mathcal{B}^{n_g}\}, \quad (6)$$

where $\oplus$ represents the Minkowski sum (elementwise) and $\xi \in \mathbb{R}^{n_g}$ is an element of the unitary box $\mathcal{B}^{n_g}$.

If the generator matrix G^{x} is square and diagonal, then the zonotope $\mathcal{X}$ can represent a box $[x]$ as $\mathcal{X} = \{\mathrm{diag}(\mathrm{rad}([x])), \mathrm{mid}([x])\}$, where the operator $\mathrm{diag}(\cdot)$ returns a diagonal matrix from a vector, or a vector with the diagonal elements from a square matrix. The generator matrix of a zonotope $\mathcal{X}$ can also be an interval matrix as in (5). Therefore, (6) is extended to the family of zonotopes [7]

$$[\mathcal{X}] \triangleq \{[G^{\mathrm{x}}], \bar{x}\} = \{(G^{\mathrm{x}}\xi + \bar{x}) : G^{\mathrm{x}} \in [G^{\mathrm{x}}],\ \xi \in \mathcal{B}^{n_g}\}, \quad (7)$$

with $[G^{\mathrm{x}}] \subset \mathbb{R}^{n\times n_g}$ being an interval generator matrix.

The affine transformation, Minkowski sum, and interval hull operations involving zonotopes can be, respectively, computed as [21]

$$L\mathcal{X} \oplus b = \{LG^{\mathrm{x}}, L\bar{x} + b\}, \quad (8)$$
$$\mathcal{X} \oplus \mathcal{W} = \{\begin{bmatrix} G^{\mathrm{x}} & G^{\mathrm{w}} \end{bmatrix}, \bar{x} + \bar{w}\}, \quad (9)$$
$$\square\mathcal{X} = [\bar{x} - r^{\mathrm{z}}, \bar{x} + r^{\mathrm{z}}], \quad (10)$$

where $\mathcal{X} = \{G^{\mathrm{x}}, \bar{x}\} \subset \mathbb{R}^n$, $\mathcal{W} = \{G^{\mathrm{w}}, \bar{w}\} \subset \mathbb{R}^n$, $L \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $r^{\mathrm{z}} = |G^{\mathrm{x}}|1_{n_g\times 1}$, $|\cdot|$ is the elementwise absolute value operator, and $1_{n_g\times 1}$ is a one matrix.

In order to relate the family of zonotopes (7) to the standard format (6), and thereby, to enable the use of usual operations as (8)-(10), we have the zonotope inclusion $\diamond\mathcal{X} \supset [\mathcal{X}]$ [7], given by

$$\diamond\mathcal{X} \triangleq \begin{bmatrix} \mathrm{mid}([G^{\mathrm{x}}]) & L \end{bmatrix} \begin{bmatrix} \mathcal{B}^{n_g} \\ \mathcal{B}^n \end{bmatrix} \oplus \bar{x}, \quad (11)$$

where $L \in \mathbb{R}^{n\times n}$ is a diagonal matrix such as $L_{i,i} = \sum_{j=1}^{n_g} \mathrm{rad}_{i,j}([G^{\mathrm{x}}]),\ i = 1, \ldots, n$..

C. Mixed Vectors

Definition 1 ([16]): Consider the zonotope $\mathcal{X}_k = \{G_k^{\mathrm{x}}, 0_{n\times 1}\}$ and the GRV $X_k \sim \mathcal{N}(0_{n\times 1}, P_k^{\mathrm{xx}})$. Given the elements z_k^{x} and realizations g_k^{x}, from $\mathcal{X}_k$ and X_k, respectively, the mixed vector x_k is defined as

$$x_k \triangleq c_k^{\mathrm{x}} + z_k^{\mathrm{x}} + g_k^{\mathrm{x}}, \quad (12)$$

where $c_k^{\mathrm{x}} \in \mathbb{R}^n$ is the mixed center of x_k.

By analyzing the definition of the chi-square random variable X^χ in (1), we can use the following confidence ellipsoid definition.

Definition 2 ([16]): Consider the GRV $X \sim \mathcal{N}(\bar{x}, P^{\mathrm{xx}})$, where $P^{\mathrm{xx}} \succ 0_{n\times n}$, and the significance level, or type I error, $\alpha \in [0, 1]$. The confidence ellipsoid is defined as

$$\mathcal{E} \triangleq \{x \in \mathbb{R}^n : (x - \bar{x})^\top(\varsigma P^{\mathrm{xx}})^{-1}(x - \bar{x}) \leq 1\}, \quad (13)$$

where $\varsigma \geq 0$ is the greatest value for the chi-square random variable with n degrees of freedom (1), taken from the cumulative distribution function (2) with $(1 - \alpha)$ confidence level, such that the probability $p(x \in \mathcal{E}) = (1-\alpha)$ is satisfied.

Thereby, we present the *confidence box*, in order to merge two bounded sets. This result will be used to sketch states and approximate nonlinear models.

Lemma 2 ([16]): Let the mixed center c^{x}, the zonotope $\mathcal{X}$, and the GRV X with covariance P^{xx} be characterizations of the mixed vector x. Let also the confidence ellipsoid $\mathcal{E}$ be defined as (13). Then, the confidence box $\mathcal{I}^\alpha$ is given by

$$\mathcal{I}^\alpha = c^{\mathrm{x}} \oplus \square\mathcal{X} \oplus \square\mathcal{E}, \quad (14)$$

where $\square\mathcal{X}$ is the interval hull of $\mathcal{X}$ given by (10),

$$\square\mathcal{E} = [-r^{\mathrm{g}}, r^{\mathrm{g}}] \quad (15)$$

is the interval hull of $\mathcal{E}$ with $r^{\mathrm{g}} = \sqrt{\varsigma\mathrm{diag}(P^{\mathrm{xx}})}$, and $\varsigma \geq 0$ is defined in (13).

III. PROBLEM STATEMENT

Consider that a given discrete-time nonlinear dynamical system be transformed into a quasi-LPV system as follows

$$x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + w_{k-1}, \quad (16)$$
$$y_k = C_k x_k + v_k, \quad (17)$$

where $A_{k-1} \in \mathbb{R}^{n\times n}$, $B_{k-1} \in \mathbb{R}^{n\times p}$, and $C_k \in \mathbb{R}^{m\times n}$ are the system matrices, $u_{k-1} \in \mathbb{R}^p$ is the deterministic input vector, $y_k \in \mathbb{R}^m$ is the measured output vector, $w_{k-1} \in \mathbb{R}^n$ is the process noise, $v_k \in \mathbb{R}^m$ is the measurement noise, and $x_k \in \mathbb{R}^n$ is the state vector to be estimated over $k \in \mathbb{Z}_+$. The matrices B_{k-1} and C_k are assumed to be deterministic, whereas $A_{k-1} = A_{0,k-1} + \Delta_{k-1}$ is determined by the quasi-LPV approach using the states x_{k-1} and possible exogenous signals, with $\Delta_{k-1} = \sum_{i=1}^{n_\delta} A_{i,k-1}\delta_{i,k-1}$ being the approximated uncertainty around $A_{0,k-1}$ and $\delta_{i,k-1} \in [-1, 1]$ the scheduling parameter containing state information. The matrices $A_{0,k-1}, \ldots, A_{n_\delta,k-1}$ are obtained from A_{k-1}.

Remark 1: In (16), B_{k-1} is deterministic because any nonlinearity with respect to x_{k-1} is placed in A_{k-1}. Some manners of obtaining a quasi-LPV representation have been illustrated in [8].

The initial state x_0 and the noises w_{k-1} and v_k can be decomposed as $x_0 = c_0^{\mathrm{x}} + z_0^{\mathrm{x}} + g_0^{\mathrm{x}}$, $w_{k-1} = z_{k-1}^{\mathrm{w}} + g_{k-1}^{\mathrm{w}}$, and $v_k = z_k^{\mathrm{v}} + g_k^{\mathrm{v}}$. The terms z_0^{x}, z_{k-1}^{w}, and z_k^{v} are obtained from the known zero-center zonotopes $\hat{\mathcal{X}}_0$, $\mathcal{W}_{k-1}$, and $\mathcal{V}_k$, respectively, with generator matrices $\hat{G}_0^{\mathrm{x}} \in \mathbb{R}^{n\times n_g}$, $G_{k-1}^{\mathrm{w}} \in \mathbb{R}^{n\times n_g^{\mathrm{w}}}$, and $G_k^{\mathrm{v}} \in \mathbb{R}^{m\times n_g^{\mathrm{v}}}$, such that $G_k^{\mathrm{v}}(G_k^{\mathrm{v}})^\top \succ 0_{m\times m}$. The realizations g_0^{x}, g_{k-1}^{w}, and g_k^{v} are obtained from random vectors, which are approximated by the known, uncorrelated, zero-mean GRVs $\hat{X}_0$, W_{k-1}, and V_k, respectively, with covariance $\hat{P}_0^{\mathrm{xx}} \succ 0_{n\times n}$, $Q_{k-1} \succeq 0_{n\times n}$, and $R_k \succ 0_{m\times m}$. Each GRV is uncorrelated to itself for different time instants.

Remark 2: The mixed decomposition of x_0, w_{k-1} and v_k incorporates the specific cases, that is, purely stochastic or set-based. In practice, uncertainties are set instead of specific values since realizations are unknown. Then, the state vector x_k is as stochastic as set based, and its time evolution can be investigated using (12). The models (16) and (17) describe a linearized dynamic based on real values, which can be mathematically manipulated to characterize the corresponding sets and random vectors, separately. This individual procedure is already known in the literature [2], [22].

According to (16) and (17), we define the state-estimation problem investigated here.

Problem 1: Consider the dynamical system represented by (16)-(17), for which the input u_{k-1} and the measurement y_k are known. Given the center estimate $\hat{c}_0^{\mathrm{x}}$, zonotopes $\hat{\mathcal{X}}_0$, $\mathcal{W}_{k-1}$, and $\mathcal{V}_k$, and GRVs $\hat{X}_0$, W_{k-1}, and V_k, the goal is to determine the mixed state estimates given by $\hat{c}_k^{\mathrm{x}}$, $\hat{\mathcal{X}}_k$, and $\hat{X}_k$, for $k \in \mathbb{Z}_+$, based on the minimum-variance criterion. To achieve that, we define the cost function

$$J_k^{\mathrm{zg}}(K_k) \triangleq (1-\eta)\mathrm{tr}(\hat{G}_k^{\mathrm{x}}(\hat{G}_k^{\mathrm{x}})^\top) + \eta\mathrm{tr}(\hat{P}_k^{\mathrm{xx}}), \tag{18}$$

where $\eta \in [0, 1]$ is a known weight parameter, K_k is the gain matrix that minimizes J_k^{zg}, and $\mathrm{tr}(\cdot)$ is the matrix trace.

Problem 1 employs the weighted-sum approach to equivalently express the solutions K_k of a biobjective optimization problem in terms of a monobjective formulation.

IV. EZGKF ALGORITHM

Consider the quasi-LPV model described in (16)-(17). If the matrix A_{k-1} was known, the optimal estimator design for Problem 1 would be reached by defining the mixed state estimates $\hat{c}_k^{\mathrm{x}}$, $\hat{\mathcal{X}}_k$, and $\hat{X}_k$, as in [19]. However, since A_{k-1} depends on the scheduling parameter δ_{k-1}, the optimal design requires modifications to yield explicit solutions. We desire here to keep both the gain matrix $K_k \in \mathbb{R}^{n\times m}$ and the center estimate $\hat{c}_k^{\mathrm{x}} \in \mathbb{R}^n$ without altering the solution enclosure. Therefore, we propose the following suboptimal estimator:

$$\hat{c}_k^{\mathrm{x}} = (\mathrm{I}_n - K_kC_k)\hat{c}_{k|k-1}^{\mathrm{x}} + K_ky_k, \tag{19}$$

$$\hat{z}_k^{\mathrm{x}} = (\mathrm{I}_n - K_kC_k)\hat{z}_{k|k-1}^{\mathrm{x}} - K_kz_k^{\mathrm{v}}, \tag{20}$$

$$\hat{g}_k^{\mathrm{x}} = (\mathrm{I}_n - K_kC_k)\hat{g}_{k|k-1}^{\mathrm{x}} - K_kg_k^{\mathrm{v}}, \tag{21}$$

where

$$\hat{c}_{k|k-1}^{\mathrm{x}} \triangleq A_{0,k-1}\hat{c}_{k-1}^{\mathrm{x}} + B_{k-1}u_{k-1}, \tag{22}$$

$$\hat{z}_{k|k-1}^{\mathrm{x}} \triangleq A_{k-1}\hat{z}_{k-1}^{\mathrm{x}} + \Delta_{k-1}\hat{c}_{k-1}^{\mathrm{x}} + z_{k-1}^{\mathrm{w}}, \tag{23}$$

$$\hat{g}_{k|k-1}^{\mathrm{x}} \triangleq A_{k-1}\hat{g}_{k-1}^{\mathrm{x}} + g^{\mathrm{w}_{k-1}}. \tag{24}$$

Our main task is to determine the forecast estimates $\hat{\mathcal{X}}_{k|k-1}$ and $\hat{X}_{k|k-1}$ that enclose the worst case related to $\hat{z}_{k|k-1}^{\mathrm{x}}$ and $\hat{g}_{k|k-1}^{\mathrm{x}}$. These results are addressed next by two propositions.

Proposition 1: Let $\hat{\mathcal{X}}_{k-1} = \{\hat{G}_{k-1}^{\mathrm{x}}, 0_{n\times 1}\} \ni \hat{z}_{k-1}^{\mathrm{x}}$, $\mathcal{W}_{k-1} = \{G_{k-1}^{\mathrm{w}}, 0_{n\times 1}\} \ni z_{k-1}^{\mathrm{w}}$, and $\mathcal{B}^{n_\delta} \ni \delta_{k-1}$ be the known sets. Then, the elements $\hat{z}_{k|k-1}^{\mathrm{x}}$ of (23) are enclosed by the zonotope $\hat{\mathcal{X}}_{k|k-1} = \{\hat{G}_{k|k-1}^{\mathrm{x}}, 0_{n\times 1}\}$ with generator matrix

$$\hat{G}_{k|k-1}^{\mathrm{x}} = \begin{bmatrix} A_{0,k-1}\hat{G}_{k-1}^{\mathrm{x}} & G_{k-1}^{\mathrm{w}} & \Phi \end{bmatrix}, \tag{25}$$

where $\Phi = \begin{bmatrix} A_{1,k-1}\begin{bmatrix}\hat{c}_{k-1}^{\mathrm{x}} & \hat{G}_{k-1}^{\mathrm{x}}\end{bmatrix} & \cdots & A_{n_\delta,k-1}\begin{bmatrix}\hat{c}_{k-1}^{\mathrm{x}} & \hat{G}_{k-1}^{\mathrm{x}}\end{bmatrix}\end{bmatrix}$.

Proof: By making explicit the matrices $A_{0,k-1}$ and $A_{i,k-1}$ in (23), we obtain $\hat{z}_{k|k-1}^{\mathrm{x}} = A_{0,k-1}\hat{z}_{k-1}^{\mathrm{x}} + z_{k-1}^{\mathrm{w}} + \sum_{i=1}^{n_\delta} A_{i,k-1}(\hat{c}_{k-1}^{\mathrm{x}} + \hat{z}_{k-1}^{\mathrm{x}})\delta_{i,k-1}$. Replacing the elements $\hat{z}_{k-1}^{\mathrm{x}}$, z_{k-1}^{w}, and $\delta_{i,k-1}$ by their corresponding set yields $\hat{\mathcal{X}}_{k|k-1} = A_{0,k-1}\hat{\mathcal{X}}_{k-1} \oplus \mathcal{W}_{k-1} \oplus A_{1,k-1}(\hat{c}_{k-1}^{\mathrm{x}} \oplus \hat{\mathcal{X}}_{k-1})[-1, 1] \oplus \cdots \oplus A_{n_\delta,k-1}(\hat{c}_{k-1}^{\mathrm{x}} \oplus \hat{\mathcal{X}}_{k-1})[-1, 1]$. Since $\hat{c}_{k-1}^{\mathrm{x}}[-1, 1]$ contributes with a new generator and $\mathcal{B}^{n_g}[-1, 1] = \mathcal{B}^{n_g}$, with n_g being the order of $\hat{\mathcal{X}}_{k-1}$, the usual operations (8) and (9) imply the zero-center zonotope $\hat{\mathcal{X}}_{k|k-1}$ with generator matrix given by (25). ■

Proposition 2: Let the GRVs $\hat{X}_{k-1} \sim \mathcal{N}(0_{n\times 1}, \hat{P}_{k-1}^{\mathrm{xx}})$ and $W_{k-1} \sim \mathcal{N}(0_{n\times 1}, Q_{k-1})$ be known representations for the uncorrelated realizations $\hat{g}_{k-1}^{\mathrm{x}}$ and g_{k-1}^{w}, respectively, and $\mathcal{B}^{n_\delta} \ni \delta_{k-1}$. Then, the realizations $\hat{g}_{k|k-1}^{\mathrm{x}}$ of (24) are represented by the GRV $\hat{X}_{k|k-1} \sim \mathcal{N}(0_{n\times 1}, \hat{P}_{k|k-1}^{\mathrm{xx}})$ with covariance matrix

$$\hat{P}_{k|k-1}^{\mathrm{xx}} = A_{0,k-1}\hat{P}_{k-1}^{\mathrm{xx}}A_{0,k-1}^\top + Q_{k-1} + \Omega_{ii} + \Omega_{ij}, \tag{26}$$

where

$$\Omega_{ii} \triangleq \sum_{i=1}^{n_\delta}\Big(A_{0,k-1}\hat{P}_{k-1}^{\mathrm{xx}}A_{i,k-1}^\top + A_{i,k-1}\hat{P}_{k-1}^{\mathrm{xx}}A_{0,k-1}^\top + A_{i,k-1}\hat{P}_{k-1}^{\mathrm{xx}}A_{i,k-1}^\top\Big),$$

$$\Omega_{ij} \triangleq \sum_{i=1}^{n_\delta-1}\sum_{j=i+1}^{n_\delta}(A_{i,k-1}\hat{P}_{k-1}^{\mathrm{xx}}A_{j,k-1}^\top + A_{j,k-1}\hat{P}_{k-1}^{\mathrm{xx}}A_{i,k-1}^\top).$$

Proof: Since the realizations $\hat{g}_{k-1}^{\mathrm{x}}$ and g_{k-1}^{w} are uncorrelated in (24), the usual operations (3) and (4) leads to the GRV $X_{k|k-1} \sim \mathcal{N}(0_{n\times 1}, P_{k|k-1}^{\mathrm{xx}})$ with covariance $P_{k|k-1}^{\mathrm{xx}} = A_{k-1}\hat{P}_{k-1}^{\mathrm{xx}}A_{k-1}^\top + Q_{k-1}$. However, the matrix A_{k-1} is unknown and we are interested in reaching a covariance matrix such that $\hat{P}_{k|k-1}^{\mathrm{xx}} \succeq P_{k|k-1}^{\mathrm{xx}}$. To achieve that, we first make explicit the matrices $A_{0,k-1}$ and $A_{i,k-1}$ in $P_{k|k-1}^{\mathrm{xx}}$, yielding $P_{k|k-1}^{\mathrm{xx}} = \sum_{i=1}^{n_\delta}\sum_{j=1}^{n_\delta}\delta_{i,k-1}\delta_{j,k-1}A_{i,k-1}\hat{P}_{k-1}^{\mathrm{xx}}A_{j,k-1}^\top + A_{0,k-1}\hat{P}_{k-1}^{\mathrm{xx}}A_{0,k-1}^\top + Q_{k-1} + \sum_{j=1}^{n_\delta}\delta_{j,k-1}A_{0,k-1}\hat{P}_{k-1}^{\mathrm{xx}}A_{j,k-1}^\top + \sum_{i=1}^{n_\delta}\delta_{i,k-1}A_{i,k-1}\hat{P}_{k-1}^{\mathrm{xx}}A_{0,k-1}^\top$. As the summed matrices are square, the former summation can be split up as $\sum_{i=1}^{n_\delta}\sum_{j=1}^{n_\delta}\delta_{i,k-1}\delta_{j,k-1}A_{i,k-1}\hat{P}^{\mathrm{xx}}_{k-1}A^\top_{j,k-1} = \sum_{i<j}\delta_{i,k-1}\delta_{j,k-1}(A_{i,k-1}\hat{P}^{\mathrm{xx}}_{k-1}A^\top_{j,k-1} + A_{j,k-1}\hat{P}^{\mathrm{xx}}_{k-1}A^\top_{i,k-1}) + \sum_{i=1}^{n_\delta}\delta^2_{i,k-1}A_{i,k-1}\hat{P}^{\mathrm{xx}}_{k-1}A^\top_{i,k-1}$. After gathering the single summations in a unique sum, and thereby, applying Lemma 1, we obtain the upper bounded covariance matrix (26), completing the proof. ■

Remark 3: Proposition 1 could be formulated with (11) by assuming an interval matrix $[A_{k-1}]$ instead of an affine matrix given by $A_{0,k-1},\ldots,A_{n_\delta,k-1}$. In this case, we first manipulate the original zonotopic parcel $z^{\mathrm{x}}_{k|k-1} = A_{k-1}\hat{z}^{\mathrm{x}}_{k-1} + z^{\mathrm{w}}_{k-1}$ to obtain the corresponding zonotope $\{\Psi, 0_{n\times1}\} = \diamond\{[A_{k-1}]\hat{\mathcal{X}}_{k-1}\}\oplus\mathcal{W}_{k-1}$. Second, the uncertainty $\Delta_{k-1}\hat{c}^{\mathrm{x}}_{k-1}$ is shifted to the prior zonotope, yielding $\hat{\mathcal{X}}_{k|k-1}$ with generator matrix $\hat{G}^{\mathrm{x}}_{k|k-1} = \begin{bmatrix}\Psi & \mathrm{diag}(\mathrm{rad}([A_{k-1}]\hat{c}^{\mathrm{x}}_{k-1}))\end{bmatrix}$. In doing so, the order of $\hat{\mathcal{X}}_{k|k-1}$ changes from $[n_g + n^{\mathrm{w}}_g + n_\delta(1+n_g)]$ to $[n_g + n^{\mathrm{w}}_g + 2n]$, with n_g being the order of $\hat{\mathcal{X}}_{k-1}$.

Thanks to Propositions 1 and 2, we achieve forecast estimates with the standard definitions of zonotopes and GRVs. Therefore, the elements $\hat{z}^{\mathrm{x}}_k$ in (20) and the realizations $\hat{g}^{\mathrm{x}}_k$ in (21) are, respectively, characterized by the zonotope $\hat{\mathcal{X}}_k = \{\hat{G}^{\mathrm{x}}_k, 0_{n\times1}\}$ and GRV $\hat{X}_k \sim \mathcal{N}(0_{n\times1}, \hat{P}^{\mathrm{xx}}_k)$, where

$$\hat{G}^{\mathrm{x}}_k = \left[(\mathrm{I}_n - K_kC_k)\hat{G}^{\mathrm{x}}_{k|k-1} \quad -K_kG^{\mathrm{v}}_k\right], \tag{27}$$

$$\hat{P}^{\mathrm{xx}}_k = (\mathrm{I}_n - K_kC_k)\hat{P}^{\mathrm{xx}}_{k|k-1}(\mathrm{I}_n - K_kC_k)^\top + K_kR_kK^\top_k, \tag{28}$$

with $\hat{G}^{\mathrm{x}}_{k|k-1}$ and $\hat{P}^{\mathrm{xx}}_{k|k-1}$ being given by (25) and (26), respectively. In this case, the uncorrelation between $\hat{g}^{\mathrm{x}}_{k|k-1}$ and g^{v}_k was exploited.

Recall that both zonotope $\hat{\mathcal{X}}_k$ and GRV $\hat{X}_k$ are obtained by assuming a known matrix K_k. In order to optimally yield this design matrix, and thereby, solve Problem 1, we propose the following theorem.

Theorem 1: Let $\hat{G}^{\mathrm{x}}_k$ in (27) and $\hat{P}^{\mathrm{xx}}_k$ in (28) be the instances of the cost function J^{zg}_k in (18), as well as the known weight parameter $\eta \in [0,1]$. Then, minimizing J^{zg}_k results in the optimal gain

$$\check{K}_k = P^{\mathrm{zg}}_{k|k-1}C^\top_k(S^{\mathrm{zg}}_{k|k-1})^{-1}, \tag{29}$$

where

$$P^{\mathrm{zg}}_{k|k-1} = (1-\eta)(\hat{G}^{\mathrm{x}}_{k|k-1}(\hat{G}^{\mathrm{x}}_{k|k-1})^\top) + \eta\hat{P}^{\mathrm{xx}}_{k|k-1}, \tag{30}$$

$$R^{\mathrm{zg}}_k = (1-\eta)(G^{\mathrm{v}}_k(G^{\mathrm{v}}_k)^\top) + \eta R_k, \tag{31}$$

$$S^{\mathrm{zg}}_{k|k-1} = C_kP^{\mathrm{zg}}_{k|k-1}C^\top_k + R^{\mathrm{zg}}_k, \tag{32}$$

and $\hat{G}^{\mathrm{x}}_{k|k-1}$ and $\hat{P}^{\mathrm{xx}}_{k|k-1}$ are given by (25) and (26), respectively.

Proof: Since $\mathrm{tr}(M+N) = \mathrm{tr}(M)+\mathrm{tr}(N)$ and $\mathrm{tr}(M) = \mathrm{tr}(M^\top)$ [23], the function J^{zg}_k in (18) can be rewritten as

$$J^{\mathrm{zg}}_k = \mathrm{tr}(P^{\mathrm{zg}}_{k|k-1}) - 2\mathrm{tr}(P^{\mathrm{zg}}_{k|k-1}C^\top_kK^\top_k) + \mathrm{tr}(K_kS^{\mathrm{zg}}_{k|k-1}K^\top_k),$$

where $P^{\mathrm{zg}}_{k|k-1}$ and $S^{\mathrm{zg}}_{k|k-1}$ are given by (30) and (32), respectively. Given that both $S^{\mathrm{zg}}_{k|k-1}$ and $P^{\mathrm{zg}}_{k|k-1}$ are symmetric, $\partial\mathrm{tr}(MK^\top N)/\partial K = M^\top N^\top$ and $\partial\mathrm{tr}(MKNK^\top O)/\partial K = NK^\top OM + N^\top K^\top M^\top O^\top$ [23], the procedure $\partial J^{\mathrm{zg}}_k(K_k)/\partial K_k = 0$ yields $P^{\mathrm{zg}}_{k|k-1}C^\top_k = K_kS^{\mathrm{zg}}_{k|k-1}$. Finally, the fact of $R^{\mathrm{zg}}_k \succ 0_{m\times m}$ is enough to guarantee that $S^{\mathrm{zg}}_{k|k-1} \succ 0_{m\times m}$, whose inversion yields the optimal gain $\check{K}_k$ in (29). ■

Recall that the number of generators of $\hat{G}^{\mathrm{x}}_k$ in (27) is larger than the amount of generators of the initial matrix $\hat{G}^{\mathrm{x}}_{k-1}$. Iteratively, this growth may imply undesired numerical issues such as the increase of both storage and processing time. In order to mitigate the computational burden, some order reduction should be applied to either $\hat{G}^{\mathrm{x}}_{k-1}$ or $\hat{G}^{\mathrm{x}}_k$ to fix its order in a desired value φ_g.

In Algorithm 1, we show how to execute one iteration of the proposed EZGKF. Since the confidence box $\mathcal{I}^\alpha_k$ given by (14) has specific applications, it is also addressed as output.

Algorithm 1: $[\hat{c}^{\mathrm{x}}_k, \hat{G}^{\mathrm{x}}_k, \hat{P}^{\mathrm{xx}}_k, \mathcal{I}^\alpha_k] = \mathrm{EZGKF}\big(\hat{c}^{\mathrm{x}}_{k-1}, \hat{G}^{\mathrm{x}}_{k-1}, \hat{P}^{\mathrm{xx}}_{k-1}, A_{0,k-1}, A_{1,k-1},\ldots,A_{n_\delta,k-1}, B_{k-1}, u_{k-1}, G^{\mathrm{w}}_{k-1}, Q_{k-1}, y_k, C_k, G^{\mathrm{v}}_k, R_k, \alpha, \eta, \varphi_g\big)$

1 Apply the order-reduction algorithm of [24] to $\hat{G}^{\mathrm{x}}_{k-1}$ to fix its order in φ_g
2 Obtain the forecast estimates $\hat{c}^{\mathrm{x}}_{k|k-1}$ in (22), $\hat{G}^{\mathrm{x}}_{k|k-1}$ in (25), and $\hat{P}^{\mathrm{xx}}_{k|k-1}$ in (26)
3 Determine the covariance matrices $P^{\mathrm{zg}}_{k|k-1}$, R^{zg}_k, and $S^{\mathrm{zg}}_{k|k-1}$ using (30)-(32) to compute the gain matrix $\check{K}_k$ in (29)
4 Calculate the current state estimates $\hat{c}^{\mathrm{x}}_k$ in (19), $\hat{G}^{\mathrm{x}}_k$ in (27), and $\hat{P}^{\mathrm{xx}}_k$ in (28)
5 Compute the confidence box $\mathcal{I}^\alpha_k$ using (14)

A. Complexity Analysis

In Table I, we show the worst-case complexity order of the proposed EZGKF. For completeness, we also derived the total complexity order of EZGKP [8] for one iteration considering decoupled process and measurement noises, yielding

$$O(\mathrm{EZGKP}) = O\Big(n_g\log(n_g) + n(n_g + p + n^{\mathrm{w}}_g + mn_\delta n^{\mathrm{v}}_g) + n^2(m + n_\delta\varphi_g + nn^2_\delta + n_\delta n^{\mathrm{v}}_g) + m^2(n + m + n^{\mathrm{v}}_g)\Big).$$

By comparing the complexity orders, we note that

$$O(\mathrm{EZGKF}) - O(\mathrm{EZGKP}) = nmn^{\mathrm{v}}_g(1-n_\delta) + n^2(n^{\mathrm{w}}_g - n_\delta n^{\mathrm{v}}_g) - nn^{\mathrm{w}}_g,$$

whose negative result will denote that EZGKF is faster than EZGKP. Since $n_\delta \geq 1$, EZGKF may achieve a better computational performance than EZGKP.

Remark 4: For the cases in which w_k and v_k are correlated, the strategy of [25] can be applied to the proposed EZGKF to remove the correlation that appears between the estimation error $e_k = \hat{z}^{\mathrm{x}}_k + \hat{g}^{\mathrm{x}}_k$ and the process noise $w_k = z^{\mathrm{w}}_k + g^{\mathrm{w}}_k$. In doing so, the current orders n^{w}_g and n^{v}_g should be replaced by $n^{\mathrm{v}}_{g,k-1}$ and $n^{\mathrm{v}}_{g,k}$, respectively. For the EZGKP [8], we should define $n^{\mathrm{w}}_g = n^{\mathrm{v}}_g$. These are the unique modifications occurred in the total worst-case complexity orders derived here. Then, the derived complexity difference between EZGKP and EZGKF does not change under $n^{\mathrm{v}}_{g,k-1} = n^{\mathrm{v}}_{g,k} = n^{\mathrm{v}}_g$.

TABLE I
COMPLEXITY ORDER $O(\cdot)$ OF EACH STEP FROM EZGKF.

Step	$O(\cdot)$
1	$n_g(n+\log(n_g))$
2	$np + n_\delta n^2\varphi_g + n_\delta^2 n^3$
3	$n^2(n_g^{\mathrm{w}} + n_\delta\varphi_g + m) + m^2(n_g^{\mathrm{v}} + n + m)$
4	$n^2(n + m + n_g^{\mathrm{w}} + n_\delta\varphi_g) + nm(m + n_g^{\mathrm{v}})$
5	$n(n_g^{\mathrm{w}} + n_g^{\mathrm{v}} + n_\delta\varphi_g)$
Total	$n_g\log(n_g) + n(n_g + p + mn_g^{\mathrm{v}})$ $+n^2(m + n_\delta\varphi_g + nn_\delta^2 + n_g^{\mathrm{w}}) + m^2(n + m + n_g^{\mathrm{v}})$

V. NUMERICAL RESULTS

Next, the proposed EZGKF is experimented over two numerical examples, which contain specific discussions about the advantages of EZGKF over EZGKP [8]. For comparison purposes, we compute three performance indexes, namely: (i) the *mean processing time* (T^{CPU}), given by $T^{\mathrm{CPU}} \triangleq \dfrac{1}{m_s}\dfrac{1}{k_f}\sum_{j=1}^{m_s}\sum_{k=1}^{k_f} t_{k,j}$, where $m_s \in \mathbb{Z}_+$ is the number of Monte Carlo simulations, $k_f \in \mathbb{Z}_+$ is the time horizon, and $t_{k,j}$ is the time interval to execute the kth iteration of an algorithm in the jth Monte Carlo simulation; (ii) the *root mean square error* of the ith state (RMSE_i), given by

$$\mathrm{RMSE}_i \triangleq \frac{1}{m_s}\sum_{j=1}^{m_s}\sqrt{\frac{1}{(k_f - k_0 + 1)}\sum_{k=k_0}^{k_f}(\hat{c}^{\mathrm{x}}_{i,k,j} - x_{i,k,j})^2},$$

where $i = 1,\dots,n$ and with $k_0 \in \mathbb{Z}_+$ being defined to disregard the initialization effect; and (iii) the *average largest radius ratio of box* ($r^{\square}$), given by $r^{\square} \triangleq \frac{1}{m_s}\frac{1}{k_f}\sum_{j=1}^{m_s}\sum_{k=1}^{k_f}\max_i \mathrm{rad}(\mathcal{I}^{\alpha}_{k,j})$, where $\mathcal{I}^{\alpha}$ is given by (14).

The following computer configuration was used: 8 GB RAM 1333 MHz, Windows 10 Pro, and AMD FX-6300 CPU 3.50 GHz. All implementations were executed in MATLAB 9.11 with INTLAB 12 [26] and MPT3 [27].

A. Van der Pol Oscillator

This subsection points out a relevant application of mixed algorithms, since it attributes a real meaning to represent the uncertainties based on the available knowledge about them. Consider the Euler-discretized Van der Pol oscillator given by [28] $x_k = f(x_{k-1}) + \begin{bmatrix}0\\1\end{bmatrix} z^{\mathrm{w}}_{k-1}$, where $f(x_{k-1}) = \begin{bmatrix} x_{1,k-1} + T_s x_{2,k-1} \\ -T_s x_{1,k-1} + (T_s + 1 - T_s x_{1,k-1}^2)x_{2,k-1}\end{bmatrix}$, with T_s being the sampling time, $x_k = \begin{bmatrix} x_{1,k} & x_{2,k}\end{bmatrix}^\top \in \mathbb{R}^2$, and $z^{\mathrm{w}}_{k-1} \in \mathbb{R}$ being an uncertain input whose bounds are known. The measurement model is given by $y_k = \begin{bmatrix}1 & 1\end{bmatrix} x_k + g^{\mathrm{v}}_k$, with $g^{\mathrm{v}}_k \in \mathbb{R}$ being a noise term whose values are described by the GRV $V \sim \mathcal{N}(0, 0.04)$. To simulate the system, we set $x_0 = 1_{2\times 1}$, $T_s = 0.1$ s, $k_f = 500$, $m_s = 100$, and

$$z^{\mathrm{w}}_k = \begin{cases} T_s\sin(2kT_s), & \text{if } kT_s < 10 \text{ s or } kT_s \geq 30 \text{ s},\\ T_s(\sin(2kT_s) + 0.5), & \text{if } 10 \text{ s} \leq kT_s < 20 \text{ s},\\ T_s(\sin(2kT_s) - 0.5), & \text{if } 20 \text{ s} \leq kT_s < 30 \text{ s}.\end{cases}$$

To estimate states, we set $\hat{c}^{\mathrm{x}}_0 = 0_{2\times 1}$, $\hat{\mathcal{X}}_0 = \{0.5\mathrm{I}_2, 0_{2\times 1}\}$, $\hat{X}_0 \sim \mathcal{N}(0_{2\times 1}, \frac{1}{4\varsigma_x}\mathrm{I}_2)$, $\mathcal{W} = \{\begin{bmatrix}0 & 0\\ 1.5T_s & 0\end{bmatrix}, 0_{2\times 1}\}$, $\varsigma_x = 11.829$, $\varphi_g = 100$, $k_0 = 20$, $\eta = 0.5$, and $\alpha = 0.0027$ such that $x_0 \in \mathcal{I}^{\alpha}_0$. To obtain $f(x_{k-1}) = A_{k-1}x_{k-1}$, we choose $A_{k-1} = \begin{bmatrix} 1 & T_s \\ -T_s & (T_s + 1 - T_s z_{k-1})\end{bmatrix}$, where $z_{k-1} = x^2_{1,k-1}$. By defining the confidence interval $[z_{k-1}] = (\mathcal{I}^{\alpha}_{1,k-1})^2$ with $\alpha = 0.0027$, the partial matrices $A_{0,k-1}$ and $A_{1,k-1}$ are finally given by $A_{0,k-1} = \begin{bmatrix} 1 & T_s \\ -T_s & T_s + 1 - T_s\mathrm{mid}([z_{k-1}])\end{bmatrix}$ and $A_{1,k-1} = \begin{bmatrix} 0 & 0 \\ 0 & -T_s\mathrm{rad}([z_{k-1}])\end{bmatrix}$.

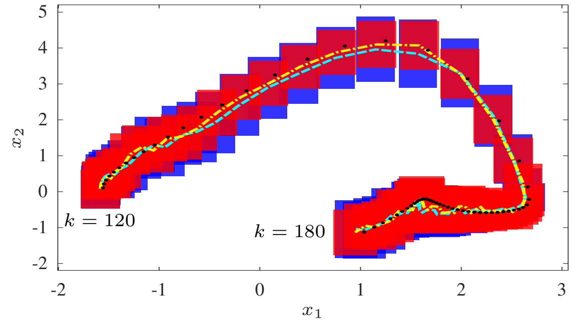

Fig. 1. State estimation for the case study of Subsection V-A. The true states (black $\cdot$) are approximately involved by blue (EZGKP) and red (EZGKF) confidence boxes, and punctually estimated by cyan (EZGKP) and green (EZGKF) centers for the discrete time horizon $k \in \{120, \dots, 180\}$.

TABLE II
PERFORMANCE INDEXES FOR THE CASE STUDY OF SUBSECTION V-A.

Indexes	EZGKP [8]	EZGKF
T^{CPU}	3.1 ms	3.1 ms
RMSE_1	0.047 (100%)	0.045 (↓**4%**)
RMSE_2	0.173 (100%)	0.130 (↓**25%**)
$r^{\square}$	0.733 (100%)	0.611 (↓**17%**)

For comparison purposes, Fig. 1 already illustrates an enhancement of both accuracy and precision when employing EZGKF rather than EZGKP. Then, we expect to attain better centers and confidence domains with EZGKF. This expectancy is corroborated by Table II through smaller RMSE and $r^{\square}$.

Table II also shows that no significant difference of T^{CPU} is noted between the algorithms. This issue accords with the performance analysis made in Subsection IV-A, which points out no difference of complexity between the EZGKP and EZGKF algorithms for this example. The reached value of T^{CPU} is also much smaller than the sampling time, enabling the direct use of such algorithms in real-time applications.

B. LPV Application

Now, we illustrate a case in which Remark 3 can be interesting to compose the EZGKF; this modification is here called EZGKF-M. Based on [7], consider the discrete-time LPV dynamical system given by

$$x_k = \begin{bmatrix} 0 & -0.5 \\ 1 & 1 + \delta_{k-1}\end{bmatrix} x_{k-1} + w_{k-1},$$
$$y_k = \begin{bmatrix} -2 & 1\end{bmatrix} x_k + v_k,$$

where $\delta_{k-1} \in [-0.3, 0.3]$. We consider that $w_{k-1} = z^{\mathrm{w}}_{k-1} + g^{\mathrm{w}}_{k-1}$ and $v_k = z^{\mathrm{v}}_k + g^{\mathrm{v}}_k$, whose characterizations are given by $\mathcal{W} = \{0.5L, 0_{2\times 1}\}$, with $L = 0.02\begin{bmatrix} -6 & 0 \\ 1 & 0\end{bmatrix}$, $W \sim \mathcal{N}(0_{2\times 1}, \frac{1}{36}LL^\top)$, $\mathcal{V} = \{0.1, 0\}$, and $V \sim \mathcal{N}(0, \frac{1}{9}0.01)$.

TABLE III
PERFORMANCE INDEXES FOR THE CASE STUDY OF SUBSECTION V-B.

Indexes	EZGKP [8]	EZGKF	EZGKF-M
T^{CPU}	1.6 ms (100%)	1.6 ms	1.2 ms (↓**25%**)
RMSE_1	0.0437 (100%)	0.0348 (↓20%)	0.0328 (↓**25%**)
RMSE_2	0.0513 (100%)	0.0449 (↓12%)	0.0419 (↓**18%**)
$r^{\square}$	0.646 (100%)	0.600 (↓7%)	0.386 (↓**40%**)

The system is simulated with $x_0 = \begin{bmatrix}0.1 & 0.1\end{bmatrix}^\top$, $k_f = 40$, and $m_s = 100$. The terms g_{k-1}^{w} and g_k^{v} are taken from GRVs, while the terms z_{k-1}^{w} and z_k^{v} are taken from uniform distributions. To estimate the states, we additionally set the following parameters: $\hat{c}_0^{\mathrm{x}} = 0_{2\times1}$, $\hat{G}_0^{\mathrm{x}} = 0.1\mathrm{I}_2$, $\hat{P}_0^{\mathrm{xx}} = \frac{0.04}{47.316}\mathrm{I}_2$, $\varphi_g = 4000$, $k_0 = 4$, $\eta = 0.5$, $\alpha = 0.0027$, and matrices $A_{0,k-1} = \begin{bmatrix}0 & -0.5\\ 1 & 1\end{bmatrix}$ and $A_{1,k-1} = \begin{bmatrix}0 & 0\\ 0 & 0.3\end{bmatrix}$.

The results of state estimation are summarized in Table III. As expected, EZGKF-M implies the smallest performance criteria. Remark 3 is here a good option because the state matrix A_{k-1} is really interval; unlike Subsection V-A where the state matrix is affine.

VI. CONCLUSIONS

This paper presented a new mixed filter for nonlinear systems called EZGKF. By employing the quasi-LPV approach, no analytical linearization is required. The proposed EZGKF is a filter version of the nonlinear predictor EZGKP [8]. As previously investigated in the literature [2], [29], the precision and the accuracy of filters may be better than those of predictors because both past and current measurements are incorporated by the state estimator. According to the explicit calculus, we derived the worst-case computational complexity of the proposed EZGKF and shown that it may imply smaller computational cost than the EZGKP, making our proposal more appealing. The benefits of using the proposed EZGKF instead of the EZGKP are illustrated in a two-state practical example and in an LPV application. As future work, we intend to analyze the case where the output matrix is uncertain, enabling then the linearization of the output equations via the quasi-LPV approach.

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