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Disagreement, limit cycles and Zeno solutions in continuous opinion dynamics with binary actions

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***Abstract*— This paper studies a mathematical model of opinion dynamics on social networks, which features continuous opinions and binary actions. The binary actions are a suitable quantization of the opinions, which evolve in continuous time. The model thus takes the form of a differential equation with discontinuous right-hand side: we explore the asymptotic behavior of its Caratheodory solutions, which turns out to be unexpectedly rich. By considering specific classes of graphs, namely lines and rings, we not only find attractive extended equilibria where the opinions are not in agreement, but also limit cycles and solutions that exhibit the Zeno phenomenon, whereby switching points accumulate in finite time.**

I. Introduction

In the last twenty years the control community has developed a strong interest in studying differential models that describe, or are at least inspired by, the evolution of opinions and beliefs in human groups and social networks [1]. A broad variety of models have been studied, as illustrated by several surveys [2], [3], [4], [5], [6] and multiple results have been obtained, which connect the topology of the social network with the asymptotic behavior of the dynamics. A key question has been to determine whether or not in the long run the dynamics reaches a consensus, whereby the opinions of all individuals are in agreement [7]. Problems of identification, estimation, optimization and control have also been considered [8], [9], [10], [11].

In this paper, we focus on an apparently simple dynamics of continuous-time opinion evolution, in which the individuals hold scalar opinions and take binary actions. Our dynamics of interest reads as

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}\big(q(x_j) - x_i\big), \tag{1}$$

where $i \in \{1, \dots, n\}$ is the index of the individual, $x_i \in [0,1]$ is the opinion of individual i, the coefficients $a_{ij} \in \{0,1\}$ are the entries of the adjacency matrix of a strongly connected directed graph, and the quantizer $q : [0,1] \to \{0,1\}$ denotes rounding to the closest integer, that is, $q(z) = \lfloor z + \frac{1}{2} \rfloor$, so that $q(\frac{1}{2}) = 1$. We shall refer to x as the vector of the *opinions* and to $q(x)$ as the vector of the *actions* (the quantizer operating componentwise on vectors).

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The aim of model (1) is not to have a general or realistic description of social interactions, but to emphasize the effect of quantization on the asymptotic behaviour, which can actually be quite disrupting. This way of including quantization in opinion dynamics can represent situations in which social influence is mediated through binary (or, more generally, discrete) choices by the individuals. Dynamical models with continuous opinions and discrete actions have attracted significant attention in the last decade. These models are often referred to as CODA models (Continuous Opinions Discrete Actions) and have been proposed by [12] and later studied by several authors [13], [14], [15], [16]. It is of note that some authors have instead studied models where opinions and actions co-evolve through distinct, though closely interrelated, dynamics [17]. Consistently with this nomenclature, we shall refer to dynamics (1) as to a COBA model (Continuous Opinions Binary Actions).

Some basic facts about COBA dynamics (1) can be deduced from more general results about the CODA model studied in [15], which has the same expression as (1) but opinions in $\mathbb{R}$. These results, which we shall recall below, include the existence of complete solutions and the convergence of solutions to consensus (that is, a state where all opinions are equal) for some specific graphs, such as the complete graph.

Obtaining a more complete picture of its convergence properties, however, has proved more elusive. One difficulty is that solutions to (1) can converge to points that are not equilibria, and which we refer to as *extended equilibria*: this pathological behavior is allowed by the right-hand side of (1) being discontinuous. Several questions about the long-time behavior of solutions have remained open and some will be addressed in this paper. First, a key open question is whether the dynamics is always convergent: we shall answer it negatively by showing examples of limit cycles. Second, it remains open whether it is possible to obtain results of global convergence to non-trivial opinion profiles: we shall answer this question positively.

In this paper, we advance the knowledge about the binary actions case by proving several new results about COBA dynamics (1). These contributions include

1) proving that, on any graph, each extended equilibrium has a basin of attraction of positive measure (Proposition 1);
2) proving convergence from any initial condition on an undirected line graphs: the limit point is an extended equilibrium but needs not be a consensus (Proposition 2);

3) showing the existence of limit cycles (Proposition 3) and of Zeno solutions (Proposition 4) on directed ring graphs. Curiously, the constructions of these cyclic and Zeno solutions rely on the properties of the golden ratio and of the Fibonacci sequence.

These results highlight the richness of the qualitative behavior of COBA dynamics (1) and the complexity of its study.

The rest of this paper is organized as follows. In Section II we collect relevant preliminaries and the facts about the COBA dynamics (1) that can be directly derived from the literature. The following sections present our contributions: Section III contains the proof that every extended equilibrium has sizable basin of attraction; Section IV contains the proof that the COBA model is convergent on line graphs; Section V contains the proof that the COBA model has a limit cycle on directed ring graphs with six nodes; Section VI contains the proof that the COBA model has Zeno solutions on the directed ring with three nodes. Finally, conclusions are drawn in Section VII.

II. Known facts about COBA dynamics

Several facts about (1) can be promptly deduced from the results in [15], which considered the more general dynamics that has the same form of (1) and initial conditions in $\mathbb{R}^n$. We state these facts here for completeness.

First of all, it should be observed that the right-hand side of equation (1) is discontinuous. Hence, solutions to (1) in the classical sense may fail to exist for some initial condition or may fail to be complete (i.e., defined on $[0, +\infty)$). In fact, more general notions of solutions are available in the literature about non-smooth systems and can be useful to study opinion dynamics with discontinuous right-hand side [18], [19]. For the system at hand, we know from [15, Theorem 2.1] that a complete Caratheodory solution to (1) exists from every initial condition. Hence, for this reason in this paper we only consider solutions in the Caratheodory sense.

Definition 1 (Caratheodory solution): Given a differential equation

$$\dot{x}(t) = f(x(t)), \tag{2}$$

an initial condition $x_0 \in \mathbb{R}^n$ and $I = (t_0, T) \subset \mathbb{R}$ where $T \leq +\infty$, a Caratheodory solution is an absolutely continuous function $\varphi : I \to \mathbb{R}^n$ such that $\varphi(t_0) = x_0$ and

$$\dot{\varphi}(t) = f(\varphi(t)),$$

for almost all $t \geq t_0$. Equivalently, a Caratheodory solution is a function that satisfies

$$\varphi(t) = x_0 + \int_{t_0}^{t} f(\varphi(\tau))d\tau.$$

Definition 2 (Caratheodory equilibrium): Point $x^* \in \mathbb{R}^n$ is a Caratheodory equilibrium if there exists a Caratheodory solution φ such that $\varphi(t) = x^*$ for all t.

In order to emphasize that the right-hand side of (1) is piecewise affine, it is convenient to introduce the following sets.

Definition 3 (Quantization cube): Given $k \in \mathbb{Z}^n$, we define $S_k \subset \mathbb{R}^n$ as

$$S_k = \{x \in \mathbb{R}^n | k_i - \frac{1}{2} \leq x_i < k_i + \frac{1}{2}, i = 1, \ldots, n\}.$$

S_k is the set of vectors whose componentwise quantization is constant and equal to k: for this reason it is called quantization cube. We say that a solution to (1) undergoes a *switch* when it crosses the boundary of any set S_k.

The situation in which the vector field restricted to S_k has an equilibrium point on the border of S_k is particularly interesting and deserves to be described with a definition.

Definition 4 (Extended equilibrium): Let $k \in \mathbb{Z}^n$ and f_k be

$$(f_k)_i(x) = \sum_{j=1}^{n} a_{ij}(k_j - x_i).$$

An extended equilibrium is a point $x^* \in \mathbb{R}^n$ such that there exists $k^* \in \mathbb{Z}^n$ such that $f_{k^*}(x^*) = 0$ and $x^* \in \overline{S_{k^*}}$.

Notice that all (Caratheodory) equilibria are extended equilibria, but extended equilibria need not be (Caratheodory) equilibria. Indeed, $f_{k^*}(x^*) = 0$ does not imply that $f(x^*) = 0$ (where we use f to denote the right-hand side of (1)).

In [15] it was found that on complete and complete bipartite graphs, convergence to consensus is achieved for all initial conditions. However, simulations suggest that, in other type of graphs, solutions usually converge to non-consensus extended equilibria, hence the importance of such points. Their properties are further studied in the next section.

III. Local convergence of extended equilibria

Our interest in extended equilibria is motivated by the fact that they always have an attraction region with positive measure (in fact, a quantization cube), as described by the following proposition.

Proposition 1 (Basin of attraction of extended equilibria): Given an extended equilibrium x^*, consider $k^* \in \mathbb{Z}^n$ for which $f_{k^*}(x^*) = 0$. Then, for all $x_0 \in S_{k^*}$,

$$\varphi(t) \longrightarrow x^* \qquad \text{for } t \to +\infty.$$

Proof: Since $x_0 \in S_{k^*}$, we can write the coordinates of the solution starting in x_0 as

$$x_i = k_i^* + \Delta_i,$$

with $\Delta_i \in [-\frac{1}{2}, \frac{1}{2})$. We can do the same with the equilibrium point and write $x_i^* = k_i^* + \Delta_i^*$, where $\Delta_i^* \in [-\frac{1}{2}, \frac{1}{2}]$. The dynamics can thus be rewritten as

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}\big(q(x_j) - x_i\big) = \sum_{j \in N_j} k_j^* - d_i(k_i^* + \Delta_i),$$

where N_i is the neighborhood of node i and d_i is the cardinality of N_i, that is, the degree of node i. We can substitute the expression $k_i^* = x_i^* - \Delta_i^*$ to obtain

$$\dot{x}_i = \sum_{j \in N_j} k_j^* - d_i x_i^* - d_i(-\Delta_i^* + \Delta_i).$$

The first two terms are the derivative $f_{k^*}(x^*)$, which is null by definition of extended equilibrium. The end result is thus

$$\dot{x}_i = d_i(\Delta_i^* - \Delta_i),$$

where $d_i > 0$ because the graph is strongly connected. Each component moves towards the extended equilibrium and away from the frontier, as $\dot{x}_i > 0$ (x_i moves to the right) when $\Delta_i < \Delta_i^*$ (x_i is on the left of x_i^*) and vice versa. Thus, the solution cannot leave S_{k^*}, in which the dynamics reads

$$\dot{x} = k - Dx,$$

where k is a constant vector and $-D$ is a diagonal matrix with negative entries. As such, the equilibrium point is asymptotically stable, and in particular attracts all solutions starting in S_{k^*}. ∎

Even though extended equilibria are usually non-consensus points, we choose as example a particularly interesting extended equilibrium on the undirected ring graph with $n = 8$ nodes: a non-integer consensus point.

Example 1 (Hidden consensus): On an undirected ring graph, every node i interacts with $i-1$ and $i+1$ (modulo n). For x^* to be an extended equilibrium, it must hold that $x_i^* = \frac{q(x_{i-1})+q(x_{i+1})}{2}$. If we consider $S_{k^*} = S_{(0,0,1,1,0,0,1,1)}$, we have that $x^* = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, ...) \in \overline{S_{k^*}}$ is a non-integer consensus extended equilibrium. The peculiarity of this point is that nodes agree on the opinion $\frac{1}{2}$, yet they actions behave differently. In the solution shown in the left plot of Fig. 1, four opinions converge to $\frac{1}{2}$ from below, and as such are quantized to 0, and four from above, and consequently take action 1. From a social interpretation point of view, since only actions can be seen, not opinions, this situation would appear as a disagreement scenario. Thus, this type of consensus is undetectable.

The right plot in Fig. 1 shows several trajectories of the dynamics on the undirected ring, in order to further illustrate Proposition 1. The solutions starting in S_{k^*} are attracted by the "hidden consensus" $x^* \in \overline{S_{k^*}}$, where $x_1^* = x_2^* = \frac{1}{2}$ and $k_1^* = k_2^* = 0$. Solutions starting elsewhere have the first two components converging to (1,1) instead. Notice that $x^* \notin S_{k^*}$, hence the solution starting from x^* itself is not constant, but instead can be proved to converge to consensus.

IV. Dynamics on a Line

The dynamics (1) on a line graph with n nodes becomes

$$\begin{cases} \dot{x}_1 = q(x_2) - x_1, \\ \dot{x}_i = q(x_{i+1}) + q(x_{i-1}) - 2x_i, \\ \dot{x}_n = q(x_{n-1}) - x_n. \end{cases} \quad (3)$$

Proposition 2 (Convergence to extended equilibria): Dynamics (3) is convergent for all initial conditions and the convergence point is an extended equilibrium.

Proof: We start from the following consideration: if two adjacent nodes have the same quantized value at time T, so they will for all $t > T$. In fact, in a line graph we have $n-2$ nodes of degree 2, which require both their neighbours to have the opposite action in order to switch their action. Otherwise, if a neighbour has action 0 and the other has action 1, the considered node will asymptotically converge towards $\frac{1}{2}$, without ever leaving its quantization interval. Back to the initial statement, if $q(x_{i-1}) = q(x_i)$, then for x_i to switch action we would need x_{i-1} to switch action first, which cannot happen unless x_i does so first, hence $q(x_{i-1}) = q(x_i)$ for all $t > T$.

Now assume $q(x_{i-1}) = q(x_i) = 0$ for some i (the choice 0 is arbitrary, the same can be done for 1). The aim is to prove that in finite time we enter in a set S_k which we cannot leave, i.e. no component can switch action.

- If $q(x_{i+1}) = 0$, we repeat the reasoning starting from $q(x_i) = q(x_{i+1}) = 0$,
- if $q(x_{i+1}) = 1$, then
  - if $q(x_{i+2})$ eventually reaches 1, we repeat the reasoning starting from $q(x_{i+1}) = q(x_{i+2}) = 1$,
  - if $q(x_{i+2})$ stays at 0 for long enough, $q(x_{i+1})$ will drop to 0, and we repeat the reasoning starting from $q(x_{i+1}) = q(x_{i+2}) = 0$.

This process shows that starting from two adjacent nodes with the same action (hence constant action), in finite time we end up with more than two nodes with constant action. Eventually, the process reaches the border nodes, which have degree one and thus eventually reproduce their neighbour's action. At that point, we have finally reached a cube that cannot be left, in which the dynamics is linear, bounded and without cycles, hence it converges to a point in the closure of the cube.

If there are no two adjacent nodes with the same action to start the process, we are in a S_k with k of the type $(0, 1, 0, 1....)$. Such a cube does not contain an equilibrium, in fact every single component can switch actions. At the first switch, if $1 \leq m \leq n-1$ components switch at the same time, at least 2 adjacent nodes will have the same action: from this state, we can apply the reasoning described above. If instead all n components reach $\frac{1}{2}$ at the same time, we cannot enter cube $S_{k'}$ with $k'_i = 1 - k_i$, as the vector field in the latter domain pushes towards $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}...)$. Therefore, after reaching this point, the solution will enter another cube where two adjacent nodes share the same action. ∎

Remark 1 (Form of the extended equilibria): By the proof of Proposition 2, we see that extended equilibria lie on the closure of domains S_{k^*} such that their k^*'s are composed of alternating strings of 0s and 1s, each string having length of at least 2. The vector of opinions x^* is such that every component is equal to the average of its neighbours. For instance $k^* = (0, 0, 1, 1, 1, 0, 0)$ and $x^* = (0, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, 0)$.

Example 2 (Line with 4 nodes): On a line graph with $n = 4$ nodes, there are four possible extended equilibria:

1) $(0, 0, 0, 0) \in \overline{S_{(0,0,0,0)}}$,
2) $(0, \frac{1}{2}, \frac{1}{2}, 1) \in \overline{S_{(0,0,1,1)}}$,
3) $(1, \frac{1}{2}, \frac{1}{2}, 0) \in \overline{S_{(1,1,0,0)}}$,
4) $(1, 1, 1, 1) \in \overline{S_{(1,1,1,1)}}$.

Fig. 2 shows the trajectories of some solutions. The diagram shows components x_1 and x_2, starting from different initial

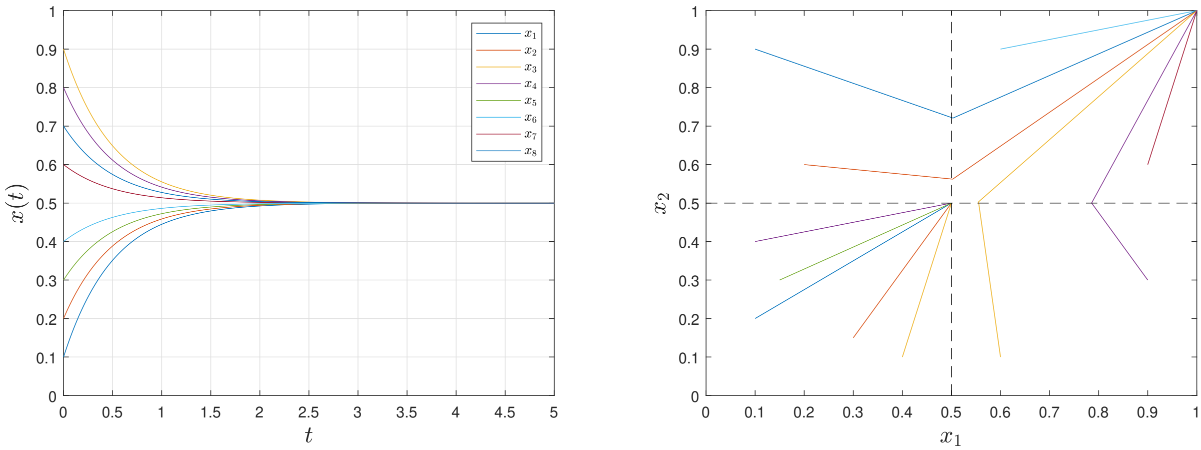

Fig. 1. Dynamics on the undirected ring. Left: time evolution of the "hidden consensus" solution in Example 1. Right: the first two components of trajectories that illustrate Proposition 1.

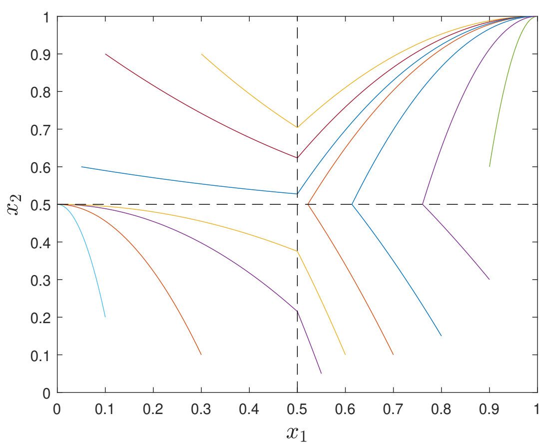

Fig. 2. Plot of x_1 and x_2 of solutions of the dynamics on a line graph with four nodes, where $q(x_3) = q(x_4) = 1$, presented in Example 2.

conditions. The components x_3 and x_4, which are not shown, have always the same initial conditions with quantized values $k_3 = k_4 = 1$, thus neither will leave that quantization interval. As per the components (x_1, x_2) shown in Fig. 2, we have different cases:

- $k_1 = 0,\ k_2 = 0$, we are in $S_{(0,0,1,1)}$ and thus $(x_1, x_2) \longrightarrow (0, \frac{1}{2})$: extended equilibrium 2;
- $k_1 = 0,\ k_2 = 1$, we are in $S_{(0,1,1,1)}$, only x_1 can change quantized value, we can only move to $S_{(1,1,1,1)}$ and converge to consensus (extended equilibrium 4);
- $k_1 = 1,\ k_2 = 1$, we are in $S_{(1,1,1,1)}$, convergence to equilibrium 4;
- $k_1 = 1,\ k_2 = 0$, we are in $S_{(1,0,1,1)}$, both x_1 and x_2 can change quantized value, and depending on which does it first we can converge to either extended equilibrium 2 or 4.

V. Limit cycle on a directed ring

In this section, we consider (1) on a directed ring, that is,

$$\begin{cases} \dot{x}_i = q(x_{i+1}) - x_i & i \in \{1, \dots, n-1\}, \\ \dot{x}_n = q(x_1) - x_n \end{cases} \tag{4}$$

and prove that for appropriately chosen initial conditions there exists a periodic trajectory.

Proposition 3 (Cycle): If $n = 6$, there exists a closed solution to (4).

Proof: We claim that the initial condition

$$x_0 = \left(\frac{\varphi}{2}, \frac{1}{2}, \frac{\varphi - 1}{2}, \frac{2-\varphi}{2}, \frac{1}{2}, \frac{3-\varphi}{2}\right)$$

with quantization $q(x_0) = (1, 0, 0, 0, 1, 1)$, where $\varphi = \frac{1+\sqrt{5}}{2}$, generates a closed solution. To show this, let us compute explicitly the solutions in $S_{q(x_0)}$. When restricted to $S_{q(x_0)}$, the differential equation (4) is linear and solved by

$$x_i(t) = (x_{0i} - q(x_{i+1}))e^{-t} + q(x_{i+1}),$$

that is, using the relation $\varphi^{-1} = \varphi - 1$,

$$x(t) = \left(\frac{\varphi}{2}e^{-t}, \frac{e^{-t}}{2}, \frac{\varphi^{-1}}{2}e^{-t}, 1 - \frac{\varphi}{2}e^{-t}, 1 - \frac{e^{-t}}{2}, 1 - \frac{\varphi^{-1}}{2}e^{-t}\right)$$

for $t \in (0, T)$ where T is the time at which one of the components reaches $\frac{1}{2}$ causing a switch of the dynamics. Such T can be found by setting $x_1(T) = \frac{1}{2}$, or $x_4(T) = \frac{1}{2}$, as they are the first components to reach $\frac{1}{2}$:

$$x_1(T) = \frac{1}{2} \Longrightarrow \frac{\varphi}{2}e^{-T} = \frac{1}{2} \Longrightarrow T = \ln(\varphi).$$

By computing $x_i(T)$ for all i, we obtain

$$x(T) = \left(\frac{1}{2}, \frac{\varphi^{-1}}{2}, 1 - \frac{\varphi}{2}, \frac{1}{2}, 1 - \frac{\varphi^{-1}}{2}, \frac{\varphi}{2}\right).$$

For the computation of $x_3(T)$, we used the relation $\varphi^{-2} = 2 - \varphi$. In fact, it holds that $\varphi^n = F_n\varphi + F_{n-1}$, where F_n is the n-th term of the Fibonacci sequence $F_n = F_{n-1} + F_{n-2},\ F_0 = 0,\ F_1 = 1$, which can be extended to $-\infty$ by

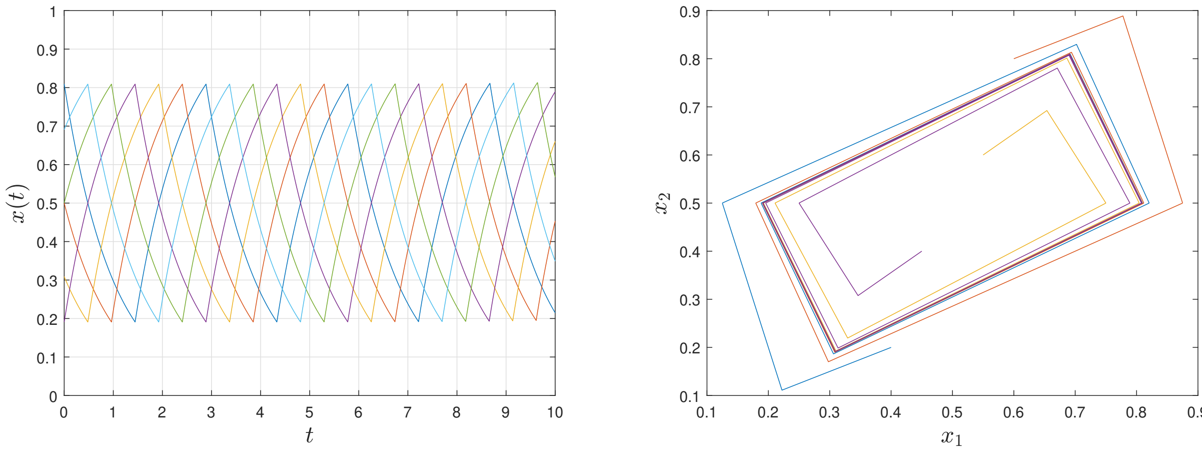

Fig. 3. The cycle constructed in Proposition 3. Left: time evolution of the periodic solution. Right: examples of different trajectories converging to the cycle.

writing $F_{n-2} = -F_{n-1} + F_n$, from which $\varphi^{-2} = F_{-2}\varphi + F_{-3} = -\varphi + 2$.

The proof is concluded by noticing that $x_i(T) = x_{i+1}(0)$ for all i. ■

The cycle solution that is constructed in the proof is illustrated in Fig. 3: simulations suggest that the cycle has non-neglegible basin of attraction.

VI. Zeno phenomenon on 3-node directed ring

Solutions to (1) may switch a finite or infinite number of times. When the switching times are infinite in number and their accumulation point is finite, we say that the solution exhibits the *Zeno phenomenon*. We show in this section that solutions to (1) can exhibit this pathological behavior, by proving an explicit construction for the directed ring with 3 nodes. The dynamics (1) applied on a directed ring with $n = 3$ becomes

$$\begin{cases} \dot{x}_1 = q(x_2) - x_1, \\ \dot{x}_2 = q(x_3) - x_2, \\ \dot{x}_3 = q(x_1) - x_3. \end{cases} \tag{5}$$

Proposition 4 (Zeno): There exist solutions to (5) that exhibit an accumulation of switching times in finite time.

Proof: Consider the initial condition $x_0 = (\frac{1}{2} + \delta, \frac{1}{2} - \epsilon, \frac{1}{2}) \in S_{(1,0,1)}$, where $0 < \delta < \frac{1}{2\varphi}$, $0 < \epsilon < \frac{1}{2}$, $\epsilon = \varphi\,\delta$, and $\varphi = \frac{1+\sqrt{5}}{2}$. Explicitly solving the equations in the neighborhood of x_0 yields

$$\begin{cases} x_1(t) = (\frac{1}{2} + \delta)e^{-t}, \\ x_2(t) = (-\frac{1}{2} - \epsilon)e^{-t} + 1, \\ x_3(t) = -\frac{1}{2}e^{-t} + 1. \end{cases}$$

The first component to reach $\frac{1}{2}$, causing a cube switch, is x_1, because $\delta < \epsilon$. The time $T_1 > 0$ at which this switch happens is such that $x_1(T_1) = \left(\frac{1}{2} + \delta\right)e^{-T_1} = \frac{1}{2}$, that is, $T_1 = \ln(1 + 2\delta)$. Computing $x_2(T_1)$ and $x_3(T_1)$, we obtain

$$\begin{cases} x_1(T_1) = \frac{1}{2}, \\ x_2(T_1) = \frac{1}{2} - \frac{\epsilon-\delta}{1+2\delta}, \\ x_3(T_1) = \frac{1}{2} + \frac{\delta}{1+2\delta}, \end{cases}$$

that, through the change of variables, $x_i' = 1 - x_{i+1}$ (where cyclically $n+1 = 1$) and dropping the prime notation results in

$$\begin{cases} x_1(T_1) = \frac{1}{2} + \delta', \\ x_2(T_1) = \frac{1}{2} - \epsilon', \\ x_3(T_1) = \frac{1}{2}, \end{cases}$$

with $\delta' = \frac{\epsilon-\delta}{1+2\delta} > 0$ and $\epsilon' = \frac{\delta}{1+2\delta} > 0$. Since $\epsilon = \varphi\delta$, we have $\delta' = \frac{(\varphi-1)\delta}{1+2\delta} = \frac{\varphi^{-1}\delta}{1+2\delta}$ where $\varphi^{-1} = \varphi - 1$.

Since the point we found is of the same structure of the starting point we can repeat the process, finding a sequence of points

$$\begin{cases} x_1(T_i) = \frac{1}{2} + \delta^{(i)}, \\ x_2(T_i) = \frac{1}{2} - \epsilon^{(i)}, \\ x_3(T_i) = \frac{1}{2}, \end{cases}$$

with $\delta^{(i)} = \frac{\varphi^{-1}\delta^{(i-1)}}{1+2\delta^{(i-1)}}$ and $\epsilon^{(i)} = \frac{\delta^{(i-1)}}{1+2\delta^{(i-1)}}$. Notice that $\epsilon^{(i)} = \varphi\delta^{(i)}$ for all i, so that we only need to study $\delta^{(i)}$. By induction on i, it is possible to obtain the expression

$$\delta^{(i)} = \frac{\varphi^{-i}\delta}{1 + 2\delta \sum_{j=1}^{i} \varphi^{-j+1}}.$$

When i diverges, $\delta^{(i)}$ converges to 0 and consequently $x_i(T_i)$ converges to $\frac{1}{2}$. By calculating the time $T = \sum_{i=1}^{\infty} T_i$ required to go through the infinite switches, we obtain

$$T = \sum_{i=1}^{\infty} T_i = \sum_{i=1}^{\infty} \ln(1 + 2\delta^{(i-1)}) < \sum_{i=1}^{\infty} \ln(1 + 2\delta\varphi^{-i+1})$$

$$\simeq \sum_{i=1}^{\infty} 2\delta\varphi^{-i+1} < +\infty.$$

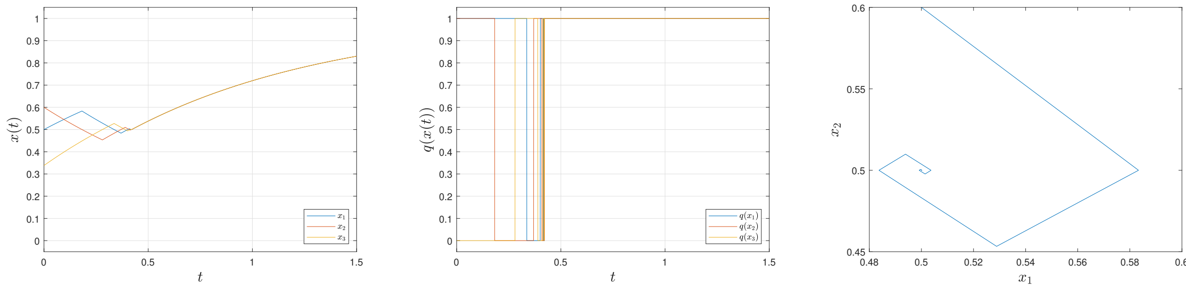

Fig. 4. Zeno solution from the proof of Proposition 4. Left plot: opinions as functions of time. Middle plot: actions as functions of time. Vertical segments represent switches: it is possible to see the accumulation point. Right plot: components x_1 and x_2 of trajectory (converging to $(\frac{1}{2}, \frac{1}{2})$) for $t \in (0, T)$.

The dynamics goes through infinite switches in a finite time, such phenomenon is referred to as Zeno point. The infinitely switching dynamics describe the solution in the interval $t \in [0, T)$, and proves that $\lim_{t \to T} x(t) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. ∎

The solution constructed in this proof, which converges to the Zeno point $\zeta = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, can be extended for $t > T$ by appending a Caratheodory solution that starts from initial condition ζ. One of these extended-beyond-Zeno solutions is shown in Fig. 4.

The Zeno solution is the only one that satisfies the condition for having infinite switches ($\delta^{(i)} < \epsilon^{(i)}$ for all i), thus no periodic solutions exist on the directed cycle with $n = 3$. Cases $n = 4$ and $n = 5$ are yet to be investigated.

VII. Conclusion

In this paper, we have shown how continuous dynamics with binary actions can produce rather complex and pathological behaviors, thereby illustrating that discretization of opinions can have disrupting effects on consensus. Indeed, we now know that Caratheodory solutions to (1) can converge to consensus, converge to non-consensus equilibria, converge to point that are *not* equilibria (called extended equilibria), converge to limit cycles, or even exhibit Zeno behaviours.

These insights leave multiple questions open and call for further work to reach a complete understanding of this dynamics. Indeed, with the important exception of Proposition 1, all constructions and convergence results have been obtained for very specific classes of graphs. For general graphs without a specific structure, simulations in [15] indicate that convergence to non-consensus equilibria is frequently observed. These evidences further strengthen our interest in more general conditions that determine the long-term qualitative behavior of the solutions.

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