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Full-Information Output Regulation of Linear Systems with Non-periodic Non-smooth Exogenous Signals

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Abstract— **In this paper, we address the problem of full-information output regulation for linear systems subject to non-periodic non-smooth exogenous signals generated by explicit-form models. We study the steady-state response of the system obtained by the interconnection of the linear system, the explicit-form exogenous system, and a time-varying state-feedback controller. We provide solutions in the form of regulator equations and then study the solvability conditions by separating the analysis in two cases, namely one without feedforward term and with continuous exogenous signals, *e.g.* triangular waves, and one with a feedforward term and discontinuous exogenous signals, *e.g.* square waves. We finally illustrate the results by means of two examples.**

I. Introduction

Output regulation is a fundamental problem in control theory. The problem consists in the design of a controller such that the closed-loop system is asymptotically stable and able to asymptotically track reference signals while rejecting disturbances. In this problem, both references and disturbances, called "exogenous signals", are assumed to be generated by a known signal generator named "exogenous system". Research into output regulation problems for linear systems dates back to 1970s, when Davison, Francis and Wonham studied the servomechanism problem [1], [2], [3], [4]. The scope of output regulation research was then expanded to nonlinear systems [5], [6], [7], [8], [9]. Recently, many studies focusing on output regulation of other classes of dynamical systems have been presented. To mention a few, multi-agent systems have gained great attention since 2010 [10], [11]; hybrid systems have been studied during the same time [12], [13], [14]; and regulation methods of linear stochastic systems have also been reported, see [15], [16].

However, in past output regulation studies non-periodic and non-smooth exogenous signals have rarely been considered. Non-smooth signals, such as sawtooths or pulse width modulation (PWM) signals, are commonly encountered in real-life applications. For example, discontinuous or non-differentiable signals, possibly non-periodic, appear in robotic manipulation, either as disturbances, as references, or due to the interconnection between agents and the environment [17], [18], [19]. This class of exogenous signals has not been considered by the regulation theory. On the one hand, many works have focused on smooth exogenous signals represented by a known, autonomous differential equation, which herein we call implicit generator [20]. This generator, in time-invariant settings, cannot model signals that show non-smoothness. Similarly, for these standard implicit generators even in time-varying settings, the existing papers have only considered periodic and smooth exogenous signals [21]. On the other hand, tracking and rejection of non-smooth signals have been studied in more general contexts. For instance, [22] studied linear output regulation problems with switching exogenous systems. The study only considered continuous non-smooth exogenous signals and solved the problem with a switching controller, while in our study we also consider discontinuous exogenous signals. Other studies that considered non-smooth exogenous signals are mostly based on hybrid linear systems, see [12], [13], [14], [23]. These studies solve the *periodic* non-smooth problem. However, output regulation with *non-periodic*, possibly discontinuous, exogenous signals still remains an open question.

In this paper, we address the linear output regulation problem for non-periodic non-smooth exogenous signals. These exogenous signals are modeled by a linear exogenous system represented in explicit form[1]. This class of exogenous systems can represent most signals with discontinuities or non-differentiabilities. This paper is focused on the full-information case[2] and considers a dynamic state feedback controller as the regulator. By interconnecting the linear system, the explicit-form exogenous system, and the regulator, we first characterize the steady-state response of the obtained closed-loop system. Then we provide solutions that rely on solving regulator equations and we determine the solvability conditions by analyzing two different configurations separately: systems without a feedforward path and systems with a feedforward path. The paper is concluded by two examples to illustrate our results.

The rest of the paper is organized as follows. Section II elaborates on the explicit-form exogenous system and formulates the full-information regulation problem. Section III introduces the proposed form of the feedback controller and analyzes the steady-state response of the closed-loop system. Section IV solves the posed problem by means of the regulator equations. The solvability is studied in Section V in which the feedforward case and non-feedforward case are considered separately, while Section VI provides illustrative numerical simulations. Section VII concludes the paper.

Notation. We use standard notation. $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers, $\mathbb{R}_{>0}$ denotes $\mathbb{R}_{\geq 0}\backslash\{0\}$, $\mathbb{C}_{<0}$ denotes the set of complex numbers with a strictly negative real part and $\mathbb{C}_{\geq 0}$ denotes $\mathbb{C}\backslash\mathbb{C}_{<0}$. The symbol I denotes

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[1]The terminology is taken from [24], [25].

[2]This means that the state of the linear system and the exogenous signal is available for feedback.

the identity matrix, $\sigma(A)$ denotes the spectrum of the matrix $A \in \mathbb{R}^{n\times n}$ and $\|A\|$ indicates its induced Euclidean matrix norm. The superscript $\top$ denotes the transposition operator.

II. Problem Formulation

In this section, we introduce the exogenous system in explicit form, we explain the class of exogenous signals we are interested in, and we formulate the full-information output regulation problem for linear systems with non-periodic non-smooth exogenous signals.

Consider a class of single-input, single-output, linear time-invariant systems in the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + E\omega(t), \\ e(t) &= Cx(t) + Du(t) + F\omega(t), \end{aligned} \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $E \in \mathbb{R}^{n\times \nu}$, $C \in \mathbb{R}^{1\times n}$, $D \in \mathbb{R}$, $F \in \mathbb{R}^{1\times \nu}$ with $\mathrm{Im}F \subseteq \mathrm{Im}C$, $x(t) \in \mathbb{R}^n$ the state, $u(t) \in \mathbb{R}$ the control input, $e(t) \in \mathbb{R}$ the regulation error, and $\omega(t) = [r(t),\ d(t)^\top]^\top \in \mathbb{R}^\nu$ the exogenous signal representing the disturbances $d(t) \in \mathbb{R}^{\nu-1}$ and/or reference signal $r(t) \in \mathbb{R}$. In this setup, $e(t) = y(t) - r(t)$ with $y(t) = Cx(t) + Du(t) + F_d d(t) \in \mathbb{R}$ the system output and $F_d \in \mathbb{R}^{1\times(\nu-1)}$.

Usually, exogenous signals are modeled as the solutions of systems in implicit form [20]

$$\dot{\omega}(t) = S\omega(t), \qquad \omega(t_0) = \omega_0. \tag{2}$$

where $S \in \mathbb{R}^{\nu\times\nu}$. However, this model cannot generate signals that are not differentiable for all times. To address this class of signals, we express ω as the solution of a generator in explicit form [26], [27, Section 5.1]

$$\omega(t) = \Lambda(t,t_0)\,\omega(t_0), \qquad \omega(t_0) = \omega_0, \tag{3}$$

with $\Lambda(t,t_0) \in \mathbb{R}^{\nu\times\nu}$ such that $\Lambda(t_0,t_0) = I$ and $\Lambda(t_2,t_0) = \Lambda(t_2,t_1)\Lambda(t_1,t_0)$ for any $t_0 \le t_1 \le t_2$. If ω is differentiable for all times and there exists an S such that (2) is valid, then $\Lambda(t,t_0) = e^{S(t-t_0)}$. However, the generator (3) can represent a wider range of signals when compared with (2). For instance, it can describe signals generated by a time-varying system of the form

$$\dot{\omega}(t) = S(t)\omega(t), \qquad \omega(t_0) = \omega_0, \tag{4}$$

in which case $\Lambda(t,t_0)$ is the transition matrix associated to (4) [28, Section 3]. Moreover, generator (3) can also express signals provided by possibly time-varying hybrid systems of the form

$$\dot{\omega}(t,k) = S(t,k)\omega(t,k), \quad \omega(t,k+1) = J(t,k)\omega(t,k), \tag{5}$$

where $S(t,k) \in \mathbb{R}^{\nu\times\nu}$ and $J(t,k) \in \mathbb{R}^{\nu\times\nu}$. This hybrid system flows and jumps according to some hybrid time domain to be specified. In addition, note that generator (3) is inherently a more direct representation of exogenous signals when compared with other modeling frameworks. For example, a square wave, indicated by the symbol $\sqcap$, can be expressed by (3) by setting $\Lambda(t,t_0) = \sqcap(t)$, directly, without the need of specifying whether this signal is generated by, *e.g.* a nonlinear system $\sqcap(t) = \mathrm{sign}(\sin(t-t_0))$, or by the hybrid system (5) for an opportune selection of constant matrices S and J.

The main target of our study is the rejection of disturbances and/or tracking of references which are non-smooth exogenous signals. We stress that, differently from the literature of output regulation of hybrid systems, we do not assume periodicity. With respect to non-smoothness, we focus on piecewise-continuous signals. This is not a restrictive requirement as it does not exclude any signal of practical interest, *e.g.* (possibly time-varying) square waves, triangular waves. Note that the generator (3) requires some additional properties to bring it closer to the standard output regulation setting. For instance, one would like uniqueness of $\omega(t)$ for all times. We now discuss a series of assumptions that make the signal ω well-behaved. To guarantee that the solution generated by (3) is unique, we require that $\Lambda(t,t_0)$ is non-singular for all times. Also this assumption is not restrictive. For instance, any signals representable by (2) can be expressed by (3) with $\Lambda(t,t_0) = e^{S(t-t_0)}$, which is invertible for all times. Similarly, invertibility is a standard property of the transition matrix of (4). More generally, given a signal of interest, it is always possible to construct $\Lambda(t,t_0)$ invertible, possibly by inflating its dimension. For instance, consider a square wave $\sqcap(t) = \mathrm{sign}(\sin(t))$ with $t_0 = 0$ as an example. Then $\Lambda(t,0) = \sqcap(t)$ is not invertible for all times. However, one can easily construct a non-singular T-periodic $\Lambda(t,0)$ by specific phase shifts of the form

$$\Lambda(t,0) = \begin{bmatrix} \sqcap\left(\frac{2\pi}{T}t + \frac{\pi}{2}\right) & -\sqcap\left(\frac{2\pi}{T}t\right) \\ \sqcap\left(\frac{2\pi}{T}t\right) & \sqcap\left(\frac{2\pi}{T}t + \frac{\pi}{2}\right) \end{bmatrix}, \tag{6}$$

in which case the determinant of $\Lambda(t,0)$ is constantly one. If more than one signal is considered, then an invertible $\Lambda(t,0)$ can be constructed as a block diagonal matrix with each block in the form (6).

The aforementioned properties of the exogenous signal generator are formalized in the following assumption.

Assumption 1: The matrix-valued function Λ is piecewise continuous and non-singular for all times.

Another assumption related to the boundedness of the exogenous signal is added.

Assumption 2: The matrix-valued functions Λ and Λ^{-1} are bounded for all times.

This assumption is a generalization of the standard hypothesis that the exogenous system $\dot{\omega} = s(\omega)$ in the traditional nonlinear output regulation problem is Poisson stable [29, Section 8.1]. In that case, all the eigenvalues of the matrix $[\frac{\partial s}{\partial \omega}]_{\omega=0}$ lie on the imaginary axis. Then Assumption 2 holds trivially.

We are now ready to formulate the full-information linear output regulation problem driven by a non-periodic explicit signal generator.

Problem 1 (Full-information Output Regulation Problem): Consider system (1) interconnected with the exogenous system (3) under Assumptions 1 and 2. The output regulation problem consists in designing a regulator u such that the following two conditions are satisfied.

(S) The closed-loop system obtained by interconnecting system (1), the exogenous system (3), and the regulator with $\omega(t) \equiv 0$ is asymptotically stable.

(R) The closed-loop system obtained by interconnecting system (1), the exogenous system (3), and the regulator satisfies

$$\lim_{t\to\infty} e(t) = 0, \tag{7}$$

uniformly for any $x(t_0) \in \mathbb{R}^n$, $\omega(t_0) \in \mathbb{R}^\nu$,

To ensure that the problem can be solved, another standard assumption on system (1) is introduced.

Assumption 3: The pair (A, B) is stabilizable.

III. Steady-state of the Closed-loop System

In this section, we propose the form of the dynamic state feedback regulator and we characterize the steady-state of the closed-loop system obtained by interconnecting system (1), the exogenous system (3), and the regulator.

To solve Problem 1, we look for a dynamic state feedback controller of the form

$$u(t) = Kx(t) + \Gamma(t)\omega(t), \tag{8}$$

where $K \in \mathbb{R}^{1\times n}$ and $\Gamma(t) \in \mathbb{R}^{1\times\nu}$. Note that differently from the standard linear theory of smooth regulation, Γ is a time-varying matrix. The reason for this choice will be clear later. Moreover, to guarantee the existence of the steady-state response $x_{ss}(t)$[3], we require that Γ is bounded and piecewise continuous. With this setup, if condition (S) is satisfied and Assumptions 1 and 2 hold, the steady-state solution $x_{ss}(t)$ exists for all times.

By interconnecting system (1), generator (3) and controller (8), we obtain the closed-loop system

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_c(t)\omega(t), \\ e(t) &= C_c x(t) + D_c(t)\omega(t), \\ \omega(t) &= \Lambda(t,t_0)\,\omega_0, \end{aligned} \tag{9}$$

where $A_c = A + BK$, $B_c(t) = E + B\Gamma(t)$, $C_c = C + DK$, and $D_c(t) = F + D\Gamma(t)$. We now characterize the steady-state response of the closed-loop system (9) under the assumption that the condition (S) is satisfied by some choice of gain K. For brevity, when $t_0 = 0$ we use the notation $\Lambda(t) = \Lambda(t,0)$.

Lemma 1: Consider the closed-loop system (9). Suppose Assumptions 1, 2, and 3 hold and that requirement (S) is satisfied by a properly selected K. For any bounded piecewise continuous Γ, the steady-state response of x is given by

$$x_{ss}(t) = \Pi_\infty(t)\omega(t), \tag{10}$$

where $\Pi_\infty(t) \in \mathbb{R}^{n\times\nu}$ is the matrix-valued function

$$\Pi_\infty(t) = \left(\int_{-\infty}^{t} e^{A_c(t-\tau)} B_c(\tau)\Lambda(\tau)d\tau\right)\Lambda(t)^{-1}. \tag{11}$$

The proof of Lemma 1 is omitted for reasons of space. From now on, we drop the subscript in Π_∞. Thus, in the following by Π, we mean Π_∞.

[3]We use the definition of steady state given in [30].

IV. Necessary and Sufficient Conditions for Regulation

This section focuses on providing the necessary and sufficient conditions for the solution of Problem 1. Note that the feedforward term D in system (1) can be either zero or non-zero, which is a factor related to the existence of a solution. This will be discussed in the next section.

We present the solution to the full-information problem in the next theorem.

Theorem 1: Consider Problem 1. Suppose Assumptions 1, 2, and 3 hold. Then there exists a matrix K and a bounded piecewise continuous matrix Γ such that the regulator (8) solves the full-information output regulation problem if and only if there exist bounded piecewise continuous matrices Π and Δ that solve the regulator equations

$$\begin{aligned} \Pi(t) &= \left(\int_{-\infty}^{t} e^{A(t-\tau)}(E + B\Delta(\tau))\Lambda(\tau)d\tau\right)\Lambda(t)^{-1}, \\ 0 &= \lim_{t\to+\infty} C\Pi(t) + D\Delta(t) + F. \end{aligned} \tag{12}$$

The proof of Theorem 1 is omitted for reasons of space. By Theorem 1, if bounded and piecewise continuous matrices Π and Δ solving the regulator equations (12) can be found, then the control law

$$u(t) = Kx(t) + (\Delta(t) - K\Pi(t))\omega(t) \tag{13}$$

solves Problem 1. Note that K is any matrix such that the closed-loop system is asymptotically stable, *i.e.* $\sigma(A+BK) \subset \mathbb{C}_{<0}$.

It is not straightforward to check the solvability conditions and compute solutions of the regulator equations (12). To remedy this problem, we introduce the following corollary that transforms (12) into differential-algebraic equations.

Corollary 1: Suppose Assumptions 1, 2, and 3 hold. There exist bounded piecewise continuous matrices Π and Δ that solve the regulator equations (12) if and only if there exist steady-state solutions $\Psi^*(t)$ and $\Delta^*(t)$ solving

$$\begin{aligned} \dot{\Psi}^*(t) &= A\Psi^*(t) + (B\Delta^*(t) + E)\Lambda(t), \\ 0 &= C\Psi^*(t) + (D\Delta^*(t) + F)\Lambda(t), \end{aligned} \tag{14}$$

for all $t \geq 0$, where Δ^* is piecewise continuous.

The proof of Corollary 1 is omitted for reasons of space.

V. Solvability of the Regulator Equations

We have presented the solution to the output regulation problem. Finding this solution relies on solving the regulator equations (12). We now study under which conditions the regulator equations are solvable for any matrices E and F.

Given the generality of Λ considered so far, the presence of the feedforward term D plays a pivotal role in the tracking/rejection ability of the linear system (1). To be more specific, if $D = 0$, the posed regulation problem cannot be solved (without feedforward term and with a bounded regulator[4]) for exogenous signals that are discontinuous (*e.g.*

[4]Note that if u is allowed to be an impulsive control then the problem with discontinuous signals may be solved without feedforward term. This is left as a future research direction.

possibly time-varying square waves). Note that this makes sense: a discontinuous signal cannot be tracked by Cx, which is a continuous signal, without the help of a feedforward term Du that can cancel the jump. Therefore, we study the solvability of the case $D=0$ restricting the class of inputs (to *e.g.* possibly time-varying triangular waves). We introduce the following assumption that helps us to guarantee the existence of a solution to the regulation problem when $D=0$.

Assumption 4: The matrix-valued function $\Lambda(t,t_0)$ is piecewise differentiable[5].

Assumption 4 assumes that ω has a finite number of non-differentiable points in any finite interval and is semi-differentiable at each non-differentiable point, *i.e.* both left and right derivatives exist, although different. However, considering the class of inputs that we target (*e.g.* triangular waves), the assumption does not practically restrict the applicability of the result.

Now we propose the solvability condition of the regulator equation (12) by considering $D=0$ and $D\neq 0$ separately. More specifically, under Assumptions 1, 2, and 3, there exist bounded piecewise continuous matrices Π and Δ solving the regulator equations (12) for any E and F if the following solvability requirement holds.

(SR) When $D=0$, if Assumption 4 holds and system (1) is minimum-phase with a unitary relative degree. When $D\neq 0$, if system (1) is minimum-phase.

The formal derivation of this argument is omitted for reasons of space.

Remark 1: Assumption 2 can be relaxed and the obtained results still hold in the case in which Λ is exponentially bounded, *i.e.* there exist positive numbers k, α, such that $\|\Lambda(t,t_0)\|\leq ke^{\alpha(t-t_0)}$ for all t. In this case, while Λ is unbounded, the solutions Π and Δ to the regulator equations (12) are still bounded. In this case we cannot talk of "steady states", but the property $\lim_{t\to+\infty}(x(t)-\Pi(t)\omega(t))=0$ still holds. This will be shown by a numerical example in the next section. A formal formulation of the problem in this more general case will be retained as future work.

VI. Numerical Simulations

In this section, we illustrate the theory proposed in this paper by providing two numerical examples: a system with $D=0$ and bounded Λ, and a system with $D\neq 0$ and unbounded Λ. We have randomly generated the matrices

[5] By definition, this means that for any $t_b>t_a\geq 0$, there exists a finite subdivision $t_a=t_0<t_1<\cdots<t_{n-1}<t_n=t_b$ of $[t_a,\,t_b]$ such that $\Lambda(t)$ is continuously differentiable in each subinterval $[t_{i-1},\,t_i]$ for any $i=1,2,\cdots,n$. Note that the derivative at t_{i-1} is understood as the right derivative and the derivative at t_i is understood as the left derivative [31, Definition 3.1].

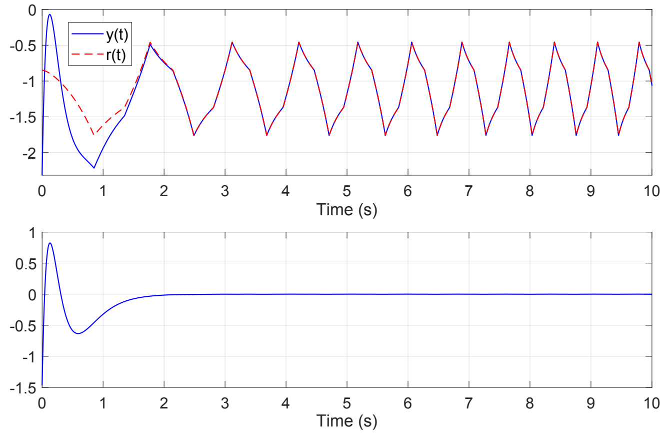

Fig. 1. Top graph: time history of the output response of the system $y(t)=Cx(t)+F_d d(t)$ (solid blue) and of the reference signal $r(t)$ (red dashed). Bottom graph: time history of the regulation error $e(t)$.

in system (1) in MATLAB as

$$A=\begin{bmatrix}0.8407 & 3.1776 & -0.7474 & -0.7711\\ -3.9223 & -2.3927 & -1.8728 & -4.0577\\ 4.0631 & 0.9436 & -3.3852 & 0.9852\\ 3.7965 & -4.7749 & -3.2123 & -0.2908\end{bmatrix},$$
$$B=\begin{bmatrix}1.9595 & 1.9989 & 1.3853 & -4.6640\end{bmatrix}^\top,$$
$$C=\begin{bmatrix}-4.3119 & -1.8040 & 0.3086 & 1.5445\end{bmatrix}, \quad (15)$$
$$E=\begin{bmatrix}0 & -0.9238 & 0.3133\\ 0 & 3.1998 & -1.7485\\ 0 & 2.1836 & -3.9437\\ 0 & 4.6865 & 1.1096\end{bmatrix},$$
$$F=\begin{bmatrix}-1 & 2.7880 & -0.7655\end{bmatrix}.$$

The value of D is different in each example.

A. System with Zero Feedforward Matrix and Bounded Exogenous Signals

In this example, we have set the feedforward matrix $D=0$. Note that the system with matrices (15) is minimum phase with unitary relative degree. Hence, the regulation problem is solvable. In accordance with Assumption 4, we consider an exogenous system composed of non-periodic triangular and sinusoidal waves such as

$$\Lambda(t)=$$
$$\begin{bmatrix}\nabla\left(\frac{2\pi}{T}t^{\frac{3}{2}}+\frac{\pi}{2}\right)+1 & \frac{1}{2}\nabla\left(\frac{2\pi}{T}t^{\frac{3}{2}}\right)+\frac{1}{2} & \sin\left(\frac{2\pi}{T}t^{\frac{3}{2}}\right)\\ -\sin\left(\frac{2\pi}{T}t^{\frac{3}{2}}\right) & \nabla\left(\frac{2\pi}{T}t^{\frac{3}{2}}+\frac{\pi}{2}\right)+1 & \frac{1}{2}\nabla\left(\frac{2\pi}{T}t^{\frac{3}{2}}\right)+\frac{1}{2}\\ \frac{1}{2}\nabla\left(\frac{2\pi}{T}t^{\frac{3}{2}}\right)+\frac{1}{2} & \sin\left(\frac{2\pi}{T}t^{\frac{3}{2}}\right) & \nabla\left(\frac{2\pi}{T}t^{\frac{3}{2}}+\frac{\pi}{2}\right)+1\end{bmatrix},$$

where ∇ is the triangular wave defined as $\nabla(t)=\frac{4}{T}\int_0^t\sqcap(\tau)d\tau-1=\frac{4}{T}\int_0^t\mathrm{sign}(\sin(\tau))d\tau-1$ and $T=\pi$. Note that since Λ is not periodic, it cannot be tracked with the available hybrid output regulation theory. The initial conditions have been randomly selected to be $x(0)=\begin{bmatrix}0.4898 & -0.3756 & 0.2057 & 0.4224\end{bmatrix}^\top$ and $\omega(0)=\begin{bmatrix}-0.8481 & -0.5202 & 0.1964\end{bmatrix}^\top$. We first determine the value of the matrix K. We have selected

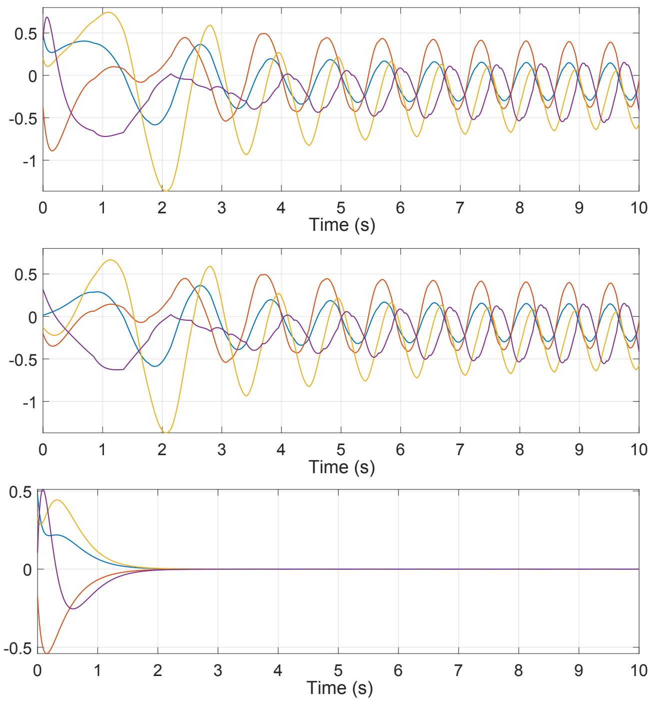

Fig. 2. Top graph: time history of the state $x(t)$. Middle graph: time history of the steady state $x_{ss}(t) = \Pi(t)\omega(t)$. Bottom graph: time history of the difference $x(t) - x_{ss}(t)$.

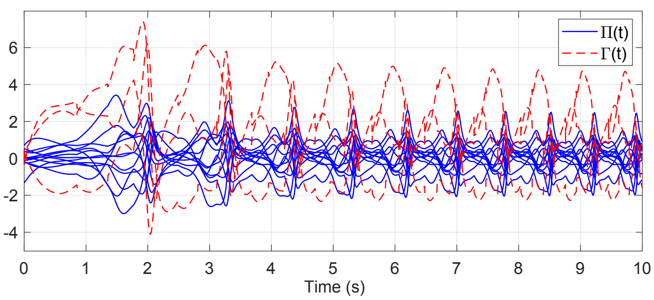

Fig. 3. Time history of $\Pi(t)$ (solid blue) and $\Gamma(t)$ (red dashed).

the closed-loop poles as $-5+2i$, $-5-2i$, -3, -10, which correspond to the stabilizing feedback matrix $K = \begin{bmatrix} -7.2020 & -3.0845 & 2.2956 & 0.1446 \end{bmatrix}$.

The computation of the matrices $\Pi(t)$ and $\Gamma(t)$ has been implemented by following Corollary 1, *i.e.* solving (14). Fig. 1 shows the time history of the system output $y(t)$ and of the reference signal $r(t)$ (top graph) and displays the time history of the regulation error $e(t)$ (bottom graph). The figure shows that the output of the system is asymptotically tracking the given reference signal while rejecting the disturbances. To verify the steady-state analysis presented in Section III, Fig. 2 depicts the time history of the state $x(t)$ (top graph) and of the computed steady state $\Pi(t)\omega(t)$ (middle graph), and displays their difference $x(t) - \Pi(t)\omega(t)$ (bottom graph). Fig. 3 shows that Π and Γ are bounded. The result is in accordance with Lemma 1 as $\lim_{t\to+\infty}(x(t) - \Pi(t)\omega(t)) = 0$.

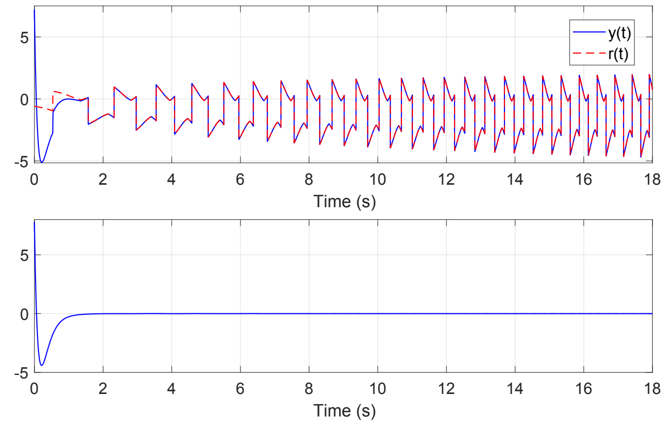

Fig. 4. Top graph: time history of the output response of the system $y(t) = Cx(t) + Du(t) + F_d d(t)$ (solid blue) and of the reference signal $r(t)$ (red dashed). Bottom graph: time history of the regulation error $e(t)$.

B. System with Non-Zero Feedforward Matrix and Unbounded Exogenous Signals

In this example, we have randomly set the feedforward matrix $D = -4.4605$, which guarantees the satisfaction of (SR). Consequently, we can consider a more complex exogenous system that contains logarithmically magnified non-periodic square, triangular, and sinusoidal waves namely

$$\Lambda(t) = \log(t+2)\times \begin{bmatrix} \sqcap\left(\frac{2\pi}{T}t^{\frac{3}{2}} + \frac{3\pi}{4}\right) & \frac{1}{2}\nabla\left(\frac{2\pi}{T}t^{\frac{3}{2}}\right)+\frac{1}{2} & \cos\left(\frac{2\pi}{T}t^{\frac{3}{2}}\right)-1 \\ -\cos\left(\frac{2\pi}{T}t^{\frac{3}{2}}\right)+1 & \sqcap\left(\frac{2\pi}{T}t^{\frac{3}{2}} + \frac{3\pi}{4}\right) & -\sin\left(\frac{2\pi}{T}t^{\frac{3}{2}}\right) \\ 0 & \sqcap\left(\frac{2\pi}{T}t^{\frac{3}{2}} + \frac{\pi}{2}\right)-1 & \sqcap\left(\frac{2\pi}{T}t^{\frac{3}{2}} + \frac{3\pi}{4}\right) \end{bmatrix},$$

where $\sqcap(t) = \text{sign}(\sin(t))$ and $T = \pi$. The initial conditions and matrix K have been selected as before. Fig. 4 shows the time history of the system output $y(t)$ and of the reference signal $r(t)$ (top graph) and displays the time history of the regulation error $e(t)$ (bottom graph). Thus, the results show that the output of the system is asymptotically tracking the reference also in this case. Fig. 5, similarly to Fig. 2, compares the time histories of $x(t)$ and $\Pi(t)\omega(t)$. Fig. 6 shows the trajectories of $\Pi(t)$ and $\Gamma(t)$. In particular, when Λ is not bounded, $\Pi\omega$ diverges with Λ, but Π and Γ are still bounded.

VII. Conclusion

The full-information output regulation problem for a linear system interconnected with a generator in explicit form that produces (possibly) non-periodic non-smooth exogenous signals has been addressed. In this paper, we have proposed a dynamic state feedback regulator and characterized the steady-state response of the closed-loop system obtained by interconnecting the linear system, the exogenous system, and the state-feedback regulator. We have then presented the solution of the problem in the form of regulator equations. Depending on the existence of the feedforward path, we have discussed the solvability of the two different cases separately. Finally, we have illustrated our results by means of two examples. Future work will focus on solving the error

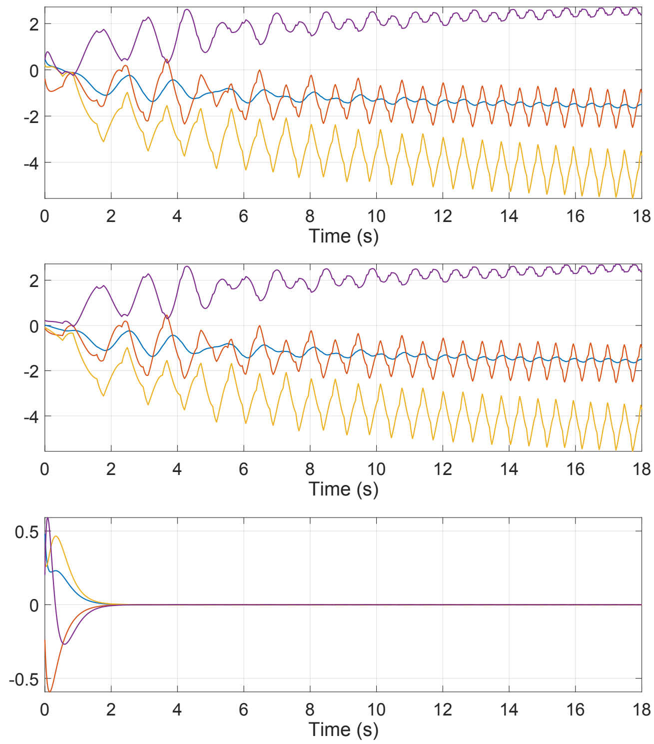

Fig. 5. Top graph: time history of the state $x(t)$. Middle graph: time history of the steady state $x_{ss}(t) = \Pi(t)\omega(t)$. Bottom graph: time history of the difference $x(t) - x_{ss}(t)$.

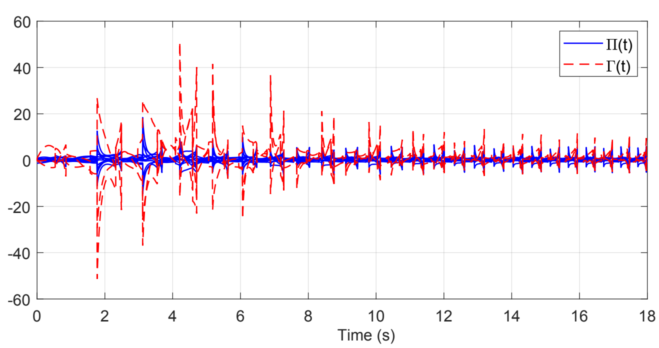

Fig. 6. Time history of $\Pi(t)$ (solid blue) and $\Gamma(t)$ (red dashed).

feedback problem, producing an internal model principle, and generalizing the results to the case with unbounded Λ and the case of impulsive regulators.

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