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A Game-Theoretical Control Framework for Transactive Energy Trading in Energy Communities

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Abstract— **Under the umbrella of non-cooperative game theory, we formulate a transactive energy framework to model and control energy communities comprised of heterogeneous agents including (yet not limited to) prosumers, energy storage systems, and energy retailers. The underlying control task is defined as a generalized Nash equilibrium problem (GNEP), which must be solved in a distributed fashion. To solve the GNEP, we formulate a Gauss-Seidel-type alternating direction method of multipliers algorithm, which is guaranteed to converge under strongly monotone pseudo-gradient mappings. As such, we provide sufficient conditions on the private cost and energy pricing functions of the community members, so that the strong monotonicity of the overall pseudo-gradient is ensured. Finally, the proposed framework and the effectiveness of the solution method are illustrated through a numerical simulation.**

I. Introduction

As the advent of distributed energy resource technologies has increased the independence of grid actors from central power providers [1], the problem of efficiently controlling the overall operational apparatus has not yet thoroughly been solved [2]. Renewable energy sources, such as photovoltaic, eolic, and hydropower, have opened up the path to dynamic energy communities, whose members (nodes) are not merely passive loads, but active agents capable of steering the dynamics of the energy market, e.g., prosumers and energy storage systems (ESSs) [3]. Given that market dynamics often stimulate selfish behaviours, the community members can be modelled as non-cooperative agents characterized by private objectives and operational constraints, which are tightly coupled to the decisions of the remaining actors. Therefore, non-cooperative game theory [4] is a powerful tool to model, design, and analyze energy communities, and the concept of generalized Nash equilibrium (GNE) [5] yields a reasonable solution for the underlying multi-agent decision-making task. Furthermore, the modern architecture of energy communities, whose members are often geographically distant, calls for non-centralized algorithmic approaches, able to take over the limitations of centralized frameworks, such as low scalability and privacy weaknesses.

The problem of distributed GNE computation in multi-agent systems has recently received significant attention [6], [7], [8], [9], and its role in energy communities has been studied both from the classical and evolutionary game theoretical perspectives [10], [11], [12]. In fact, in our previous work [13], we formulate the control task of the energy community as a GNE problem (GNEP), where the agents' decisions correspond to energy transactions between prosumers, ESSs, and a single energy retailer (ER). Nonetheless, the modelling in [13] only considers inter-agent couplings through the constraints and not through the private cost functions of the community members, and the allowed energy transactions are ruled by a particular bipartite graph topology.

Motivated by these previous works, in this paper, we extend the framework in [13] to a more general energy market setup, where the community members have private cost functions coupled with each other through the monetary price at which each member sells its energy. Besides, these energy prices are allowed to be dependent on aggregate demands, thus coupling the decisions of multiple community members. Moreover, the inter-agent energy transactions and communication are ruled by an arbitrarily connected and undirected graph, and we take in the (multiple) energy retailers as active participants in the energy community, characterizing them with their own objectives to pursue selfishly. To solve the underlying GNEP in a distributed fashion, we leverage the results in [9] to formulate a Gauss-Seidel-type alternating direction method of multipliers (ADMM) distributed GNE computation algorithm. Note that, although Gauss-Seidel-type methods tend to converge faster than Jacobi-type ones [9], [8], the former requires ordered sequential computations which would not suitably scale for large energy communities. To overcome such a drawback, we reformulate the underlying GNEP in an equivalent form, and we solve it following a two-block iterative process, i.e., energy transactions computation

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and energy prices update, where each block can be executed in parallel over the energy community regardless of the total number of members. ADMM-based decompositions have found popularity in the game-theoretical context, as they provide a solid baseline for developing decentralized equilibrium seeking schemas [14]. Finally, as the main technical contribution, we provide sufficient conditions on the private cost and energy pricing functions to ensure the strong monotonicity of the overall pseudo-gradient mapping, which is in turn a sufficient condition to guarantee the convergence of the considered ADMM-type algorithm. As such, the provided sufficient conditions are useful in the management of energy communities, as the cost and pricing functions can be designed to ensure convergence to a GNE.

Notations: Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$ denote the sets of real, non-negative real, and positive real numbers, respectively. Let $\mathbb{Z}_{\geq a}$ be the set of integers not less than $a \in \mathbb{Z}_{\geq 1}$, and we let $\mathbb{B} = \{0,1\}$. Given a set $\mathcal{S} = \{1,2,\dots,N\}$, $\mathrm{col}(\cdot)_{i\in\mathcal{S}}$ and $\mathrm{diag}(\cdot)_{i\in\mathcal{S}}$ denote the column stack and the (block) diagonal stack operations ordered by the set $\mathcal{S}$, respectively, where the natural ordering of the elements in $\mathcal{S}$ is preserved in the concatenations. On the other hand, $\phi_{\hat{\mathcal{S}}}\colon \hat{\mathcal{S}} \to \{1,2,\dots,|\hat{\mathcal{S}}|\}$ yields the relative natural ordering of the elements of $\hat{\mathcal{S}}$. Namely, if $\mathcal{S} = \{1,2,3,4,5\}$ and $\hat{\mathcal{S}} = \{2,4,5\}$, then $\phi_{\hat{\mathcal{S}}}(2) = 1$, $\phi_{\hat{\mathcal{S}}}(4) = 2$, and $\phi_{\hat{\mathcal{S}}}(5) = 3$. Given a symmetric matrix $\mathbf{S}$, we let $\lambda_{\max}(\mathbf{S})$ denote the maximum eigenvalue of $\mathbf{S}$, and $\mathbf{S} \succeq 0$ denotes that $\mathbf{S}$ is positive semi-definite. Throughout the paper, $\|\cdot\|$ and $\|\cdot\|_\infty$ denote the Euclidean and infinity norms, respectively, and $|\cdot|$ yields the cardinality when applied to a set. Operators ∇ and D yield the gradient and Jacobian matrix when applied to differentiable scalar-valued and vector-valued functions, respectively (we view gradients as column vectors by default). Besides, a subindex is included to ∇ and D when specifying partial differentiation. Given a scalar-valued function $f\colon \mathcal{D} \to \mathbb{R}$ and some $\theta \in \mathbb{R}_{>0}$, we say that $f(\cdot)$ is θ-strongly convex if it holds that $g(\mathbf{z}) = f(\mathbf{z}) - (\theta/2)\mathbf{z}^\top\mathbf{z}$ is convex for every $\mathbf{z} \in \mathcal{D}$. Given a vector-valued function $\mathbf{h}\colon \mathcal{D} \to \mathbb{R}^m$ with domain $\mathcal{D} \subseteq \mathbb{R}^m$, we say that $\mathbf{h}(\cdot)$ is L-Lipschitz continuous if there exists some $L \in \mathbb{R}_{>0}$ such that $\|\mathbf{h}(\mathbf{z}) - \mathbf{h}(\tilde{\mathbf{z}})\| \leq L\|\mathbf{z} - \tilde{\mathbf{z}}\|$, for all $\mathbf{z}, \tilde{\mathbf{z}} \in \mathcal{D}$; we say that $\mathbf{h}(\cdot)$ is monotone if $(\mathbf{h}(\mathbf{z}) - \mathbf{h}(\tilde{\mathbf{z}}))^\top(\mathbf{z} - \tilde{\mathbf{z}}) \geq 0$, for all $\mathbf{z}, \tilde{\mathbf{z}} \in \mathcal{D}$; and we say that $\mathbf{h}(\cdot)$ is μ-strongly monotone if there exists a $\mu \in \mathbb{R}_{>0}$ such that $(\mathbf{h}(\mathbf{z}) - \mathbf{h}(\tilde{\mathbf{z}}))^\top(\mathbf{z} - \tilde{\mathbf{z}}) \geq \mu\|\mathbf{z} - \tilde{\mathbf{z}}\|^2$, for all $\mathbf{z}, \tilde{\mathbf{z}} \in \mathcal{D}$. If $\mathbf{h}(\cdot)$ is continuously differentiable, then a sufficient and necessary condition for μ-strong monotonicity is that $\mathrm{D}\,\mathbf{h}(\mathbf{z}) + \mathrm{D}\,\mathbf{h}(\mathbf{z})^\top - 2\mu\mathbf{I}_m \succeq 0$, for all $\mathbf{z} \in \mathcal{D}$. Throughout the paper, $\mathbf{I}_n$ is the $n \times n$ identity matrix, $\mathbf{1}_n$ ($\mathbf{0}_n$) is the column vector with n ones (zeros), $\mathbf{0}_{n\times m}$ is the $n \times m$ matrix of zeros, and $\otimes$ denotes the Kronecker product. Finally, $\mathcal{U}[a,b]$ is the uniform random distribution over $[a,b] \subset \mathbb{R}$.

II. Problem Statement

Consider an energy community with $N \in \mathbb{Z}_{\geq 2}$ agents indexed by the set $\mathcal{A} = \{1,2,\dots,N\}$. The interaction and communication among agents is characterized by the connected[1] and undirected graph $\mathcal{G} = (\mathcal{A}, \mathcal{E})$, where the agents correspond to the nodes, and $\mathcal{E} \subset \mathcal{A} \times \mathcal{A}$ is the set of edges (by convention we assume that there are no self-loops, i.e., $(i,i) \notin \mathcal{E}$, for all $i \in \mathcal{A}$). If $(i,j) \in \mathcal{E}$, then we say that agents $i, j \in \mathcal{A}$ are neighbors and thus can interact and communicate with each other, and since $\mathcal{G}$ is undirected, $(i,j) \in \mathcal{E} \Leftrightarrow (j,i) \in \mathcal{E}$. Hence, we denote $\mathcal{A}_i = \{j \in \mathcal{A}\colon (i,j) \in \mathcal{E}\}$ as the set of neighbors of agent i, and we let $N_i = |\mathcal{A}_i|$ ($N_i \geq 1$ due to the connectivity of $\mathcal{G}$).

In the considered energy community, every agent $i \in \mathcal{A}$ is allowed to trade energy with its neighbors over $T \in \mathbb{Z}_{\geq 1}$ ordered time slots $t_1 < t_2 < \cdots < t_T$, where t_k represents a generic time slot and $k \in \mathcal{T} = \{1,2,\dots,T\}$. As such, for all $(i,j,t) \in \mathcal{A} \times \mathcal{A}_i \times \mathcal{T}$, let $\hat{x}_{ijt} \in \mathbb{R}_{\geq 0}$ denote the energy that agent i buys from agent j at time t, let $\check{x}_{ijt} \in \mathbb{R}_{\geq 0}$ denote the energy that agent i sells to agent j at time t, and let $p_{ijt} \in \mathbb{R}_{\geq 0}$ denote the monetary price at which agent i sells its energy to agent j at time t. Clearly, for an energy trade to be attainable, it is necessary that $\hat{x}_{ijt} = \check{x}_{jit}$ and $\check{x}_{ijt} = \hat{x}_{jit}$, i.e., agents must agree on their energy exchanges. Let z be a placeholder notation for either $\hat{x}$, $\check{x}$, or p, and define the vectorization given by

$$\begin{array}{ll} \mathbf{z}_{ij} = \mathrm{col}\,(z_{ijt})_{t\in\mathcal{T}} & \mathbf{z} = \mathrm{col}\,(\mathbf{z}_i)_{i\in\mathcal{A}} \\ \mathbf{z}_i = \mathrm{col}\,(\mathbf{z}_{ij})_{j\in\mathcal{A}_i} & \mathbf{z}_{-i} = \mathrm{col}\,(\mathbf{z}_{ji})_{j\in\mathcal{A}_i} \end{array} \tag{1}$$

where $n_i = TN_i$ and $n = \sum_{i\in\mathcal{A}} n_i$. That is, the vectors $\hat{\mathbf{x}}_{ij}$, $\check{\mathbf{x}}_{ij}$, $\mathbf{p}_{ij}$, $\hat{\mathbf{x}}_i$, $\check{\mathbf{x}}_i$, $\mathbf{p}_i$, $\hat{\mathbf{x}}$, $\check{\mathbf{x}}$, $\mathbf{p}$, $\hat{\mathbf{x}}_{-i}$, $\check{\mathbf{x}}_{-i}$, and $\mathbf{p}_{-i}$, are all constructed following the ordering in (1). Namely, $\hat{\mathbf{x}}_i$ ($\check{\mathbf{x}}_i$) yields the energy that agent i buys from (sells to) its neighbors, $\hat{\mathbf{x}}_{-i}$ ($\check{\mathbf{x}}_{-i}$) yields the energy that the neighbors of agent i buy from (sell to) agent i, and $\mathbf{p}_{-i}$ yields the energy selling prices that the neighbors of agent i offer to agent i. Under the considered framework, every agent $i \in \mathcal{A}$ is thus responsible for computing its own decision $(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \mathbf{p}_i)$ subject to the constraints given by

$$(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i) \in \mathcal{X}_i \tag{2a}$$
$$\hat{\mathbf{x}}_i = \check{\mathbf{x}}_{-i} \tag{2b}$$
$$\check{\mathbf{x}}_i = \hat{\mathbf{x}}_{-i} \tag{2c}$$
$$\mathbf{p}_i \in \mathbb{R}^{n_i}_{\geq 0}. \tag{2d}$$

Here, $\mathcal{X}_i \subseteq \mathbb{R}^{2n_i}_{\geq 0}$ defines the local energy-related constraint set of agent i, and the constraints in (2b)-(2c) impose an agreement between neighbouring agents regarding their energy trades. Therefore, the decision of agent i is feasible only if $(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \mathbf{p}_i) \in \Omega_i(\hat{\mathbf{x}}_{-i}, \check{\mathbf{x}}_{-i}) \times \mathbb{R}^{n_i}_{\geq 0}$, with

$$\Omega_i(\hat{\mathbf{x}}_{-i}, \check{\mathbf{x}}_{-i}) = \left\{ (\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i) \in \mathcal{X}_i\colon \begin{array}{l} \hat{\mathbf{x}}_i = \check{\mathbf{x}}_{-i} \\ \check{\mathbf{x}}_i = \hat{\mathbf{x}}_{-i} \end{array} \right\}.$$

Consequently, the set of feasible collective decisions for the entire energy community is given by $\Omega \times \mathbb{R}^n_{\geq 0}$, where $\Omega = \big\{(\hat{\mathbf{x}}, \check{\mathbf{x}}) \in \prod_{i\in\mathcal{A}} \mathcal{X}_i\colon \hat{\mathbf{x}} = \mathbf{B}\check{\mathbf{x}}\big\}$. Here, $\mathbf{B} = \mathrm{col}\,(\mathbf{B}_i)_{i\in\mathcal{A}} \in \mathbb{B}^{n\times n}$, and $\mathbf{B}_i \in \mathbb{B}^{n_i \times n}$ is the (unique) matrix that satisfies $\mathbf{z}_{-i} = \mathbf{B}_i\mathbf{z}$, for any placeholder $z \in \{\hat{x}, \check{x}, p\}$ and all $i \in \mathcal{A}$.

[1] If the graph were disconnected, then each connected component could be treated as a separate energy community.

More precisely, $\mathbf{B}_i$ has the block structure given by $\mathbf{B}_i = \begin{bmatrix}\mathbf{B}_{i1} & \mathbf{B}_{i2} & \cdots & \mathbf{B}_{iN}\end{bmatrix}$, where

$$\mathbf{B}_{ij} = \begin{cases} \mathbf{0}_{n_i \times n_j}, & \text{if } j \notin \mathcal{A}_i \\ \mathbf{W}_{ij} \otimes \mathbf{I}_T, & \text{if } j \in \mathcal{A}_i, \end{cases}$$

and $\mathbf{W}_{ij} \in \mathbb{B}^{N_i \times N_j}$ is the matrix that has a 1 at the $(\text{row} = \phi_{\mathcal{A}_i}(j), \text{column} = \phi_{\mathcal{A}_j}(i))$ position and a 0 elsewhere, i.e., $\mathbf{W}_{ij} = \mathbf{e}_{\phi_{\mathcal{A}_i}(j)} \mathbf{e}_{\phi_{\mathcal{A}_j}(i)}^\top$, where $\mathbf{e}_k$ is k-th column of $\mathbf{I}_n$.

Based on the considered framework, every agent $i \in \mathcal{A}$ computes its own decision $(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \mathbf{p}_i)$ to simultaneously solve the optimization problems (OPs) given by

$$\min_{\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i} f_i\left(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \mathbf{p}_i, \mathbf{p}_{-i}\right) \text{ s.t. } \left(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i\right) \in \Omega_i\left(\hat{\mathbf{x}}_{-i}, \check{\mathbf{x}}_{-i}\right) \quad (3a)$$

$$\min_{\mathbf{p}_i} \frac{\rho_i}{2} \left\|\mathbf{p}_i - \mathbf{g}_i\left(\hat{\mathbf{x}}_{-i}\right)\right\|^2 \text{ s.t. } \mathbf{p}_i \in \mathbb{R}^{n_i}. \quad (3b)$$

Here, $f_i \colon \mathbb{R}_{\geq 0}^{4n_i} \to \mathbb{R}$ is the local cost function of agent i, $\rho_i \in \mathbb{R}_{>0}$ is a weighting parameter, and $\mathbf{g}_i \colon \mathbb{R}_{\geq 0}^{n_i} \to \mathbb{R}_{\geq 0}^{n_i}$ is the (non-negative) local pricing function of agent i. Note that the unique solution of the OP in (3b) is $\mathbf{p}_i = \mathbf{g}_i\left(\hat{\mathbf{x}}_{-i}\right)$ and the constraint $\mathbf{p}_i \in \mathbb{R}_{\geq 0}^{n_i}$ is enforced by the co-domain of $\mathbf{g}_i(\cdot)$ (we define such a computation as an OP for convenience). Hence, solving the OP in (3a) yields the energy transactions that agent i should execute to minimize its operational costs $f_i(\cdot,\cdot,\cdot,\cdot)$, whilst solving the OP in (3b) yields the energy-selling prices of agent i. Besides, notice that for a single agent i the OP in (3a) is coupled to the OP in (3b) through $\mathbf{p}_i$, while for multiple agents the OPs in (3) are coupled to each other through $\hat{\mathbf{x}}_{-i}$, $\check{\mathbf{x}}_{-i}$, and $\mathbf{p}_{-i}$. As such, simultaneously solving the OPs in (3) for every agent $i \in \mathcal{A}$ is equivalent to solving the GNEP stated in Definition 1.

Definition 1: The GNEP for the energy community is to compute a collective decision $(\hat{\mathbf{x}}^*, \check{\mathbf{x}}^*, \mathbf{p}^*) \in \Omega \times \mathbb{R}_{\geq 0}^n$ such that

$$f_i\left(\hat{\mathbf{x}}_i^*, \check{\mathbf{x}}_i^*, \mathbf{p}_i^*, \mathbf{p}_{-i}^*\right) \leq f_i\left(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \mathbf{p}_i^*, \mathbf{p}_{-i}^*\right)$$
$$\left\|\mathbf{p}_i^* - \mathbf{g}_i\left(\hat{\mathbf{x}}_{-i}^*\right)\right\|^2 \leq \left\|\mathbf{p}_i - \mathbf{g}_i\left(\hat{\mathbf{x}}_{-i}^*\right)\right\|^2,$$

for all $\left(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i\right) \in \Omega_i\left(\hat{\mathbf{x}}_{-i}^*, \check{\mathbf{x}}_{-i}^*\right)$, all $\mathbf{p}_i \in \mathbb{R}_{\geq 0}^{n_i}$, and all $i \in \mathcal{A}$. Such a collective decision $(\hat{\mathbf{x}}^*, \check{\mathbf{x}}^*, \mathbf{p}^*)$ is termed as a GNE for the energy community.

Namely, the GNEP is the task of computing a GNE, which is a collective (feasible) decision $(\hat{\mathbf{x}}^*, \check{\mathbf{x}}^*, \mathbf{p}^*)$ where no agent can further decrease its local costs by unilaterally deviating from the GNE. In that sense, a GNE is a self-enforceable agreement among the energy community. In this paper, we focus on the so-called variational GNE (vGNE), which is a particular type of GNE that can be linked to the solution of an underlying variational inequality [15].

Definition 2: A collective decision $(\hat{\mathbf{x}}^*, \check{\mathbf{x}}^*, \mathbf{p}^*)$ is a vGNE if $(\hat{\mathbf{x}}^*, \check{\mathbf{x}}^*, \mathbf{p}^*) \in \text{SOL}\left(\Omega \times \mathbb{R}_{\geq 0}^n, \mathbf{q}(\cdot,\cdot,\cdot)\right)$, where $\text{SOL}\left(\Omega \times \mathbb{R}_{\geq 0}^n, \mathbf{q}(\cdot,\cdot,\cdot)\right)$ denotes the set of solutions of the variational inequality $\text{VI}\left(\Omega \times \mathbb{R}_{\geq 0}^n, \mathbf{q}(\cdot,\cdot,\cdot)\right)$ defined as: find $(\hat{\mathbf{x}}^*, \check{\mathbf{x}}^*, \mathbf{p}^*) \in \Omega \times \mathbb{R}_{\geq 0}^n$ such that

$$\begin{bmatrix} \hat{\mathbf{x}} - \hat{\mathbf{x}}^* \\ \check{\mathbf{x}} - \check{\mathbf{x}}^* \\ \mathbf{p} - \mathbf{p}^* \end{bmatrix}^\top \mathbf{q}\left(\hat{\mathbf{x}}^*, \check{\mathbf{x}}^*, \mathbf{p}^*\right) \geq 0, \quad \forall \left(\hat{\mathbf{x}}, \check{\mathbf{x}}, \mathbf{p}\right) \in \Omega \times \mathbb{R}_{\geq 0}^n.$$

Standing Assumption 1: For all $i \in \mathcal{A}$, the functions $f_i(\cdot,\cdot,\cdot,\cdot)$ and $\mathbf{g}_i(\cdot)$ are continuously differentiable, $f_i(\cdot,\cdot,\mathbf{p}_i,\mathbf{p}_{-i})$ is (jointly) convex for every fixed $(\mathbf{p}_i,\mathbf{p}_{-i})$, and $\nabla_{\hat{\mathbf{x}}_i} f_i\left(\hat{\mathbf{x}}_i,\check{\mathbf{x}}_i,\cdot,\cdot\right)$ and $\nabla_{\check{\mathbf{x}}_i} f_i\left(\hat{\mathbf{x}}_i,\check{\mathbf{x}}_i,\cdot,\cdot\right)$ are $\hat{L}_i$-Lipschitz continuous and $\check{L}_i$-Lipschitz continuous for every fixed $(\hat{\mathbf{x}}_i,\check{\mathbf{x}}_i)$, respectively. Moreover, the pseudo-gradient

$$\mathbf{q}\left(\hat{\mathbf{x}},\check{\mathbf{x}},\mathbf{p}\right) = \begin{bmatrix} \text{col}\left(\nabla_{\hat{\mathbf{x}}_i} f_i\left(\hat{\mathbf{x}}_i,\check{\mathbf{x}}_i,\mathbf{p}_i,\mathbf{p}_{-i}\right)\right)_{i\in\mathcal{A}} \\ \text{col}\left(\nabla_{\check{\mathbf{x}}_i} f_i\left(\hat{\mathbf{x}}_i,\check{\mathbf{x}}_i,\mathbf{p}_i,\mathbf{p}_{-i}\right)\right)_{i\in\mathcal{A}} \\ \text{col}\left(\rho_i\left(\mathbf{p}_i - \mathbf{g}_i(\hat{\mathbf{x}}_{-i})\right)\right)_{i\in\mathcal{A}} \end{bmatrix} \in \mathbb{R}^{3n}$$

is μ-strongly monotone. Finally, Ω is a closed convex set with a non-empty relative interior.

Under Standing Assumption 1, it follows that there exists a unique vGNE for the considered energy community [15, Theorem 2.3.3]. Besides, from [5, Theorem 3.9] it holds that every vGNE is also a GNE (yet the converse is not true in general). Consequently, computing a vGNE is sufficient to solve the GNEP of Definition 1.

Furthermore, in Proposition 1 we provide sufficient conditions on the functions $f_i(\cdot,\cdot,\cdot,\cdot)$ and $\mathbf{g}_i(\cdot)$ to guarantee the μ-strong monotonicity of the pseudo-gradient $\mathbf{q}(\cdot,\cdot,\cdot)$.

Proposition 1: Suppose that every agent $i \in \mathcal{A}$ has functions $f_i(\cdot,\cdot,\cdot,\cdot)$ and $\mathbf{g}_i(\cdot)$ of the form

$$\begin{aligned} f_i\left(\hat{\mathbf{x}}_i,\check{\mathbf{x}}_i,\mathbf{p}_i,\mathbf{p}_{-i}\right) &= \psi_i\left(\hat{\mathbf{x}}_i,\check{\mathbf{x}}_i\right) + \mathbf{p}_{-i}^\top \hat{\mathbf{x}}_i - \mathbf{p}_i^\top \check{\mathbf{x}}_i \\ \mathbf{g}_i\left(\hat{\mathbf{x}}_{-i}\right) &= \mathbf{Q}_i \hat{\mathbf{x}}_{-i} + \mathbf{r}_i, \end{aligned} \quad (4)$$

where $\psi_i \colon \mathbb{R}_{\geq 0}^{2n_i} \to \mathbb{R}$ is twice continuously differentiable and θ_i-strongly convex in all its arguments, $\mathbf{Q}_i \in \mathbb{R}_{\geq 0}^{n_i \times n_i}$, and $\mathbf{r}_i \in \mathbb{R}_{\geq 0}^{n_i}$. Moreover, denote $\underline{\theta} = \min_{i\in\mathcal{A}} \theta_i$ and $\bar{\lambda} = \max_{i\in\mathcal{A}} \lambda_{\max}\left(\mathbf{Q}_i^\top \mathbf{Q}_i\right)$, and let $\rho_i = \rho \in \mathbb{R}_{>0}$, for all $i \in \mathcal{A}$. If there exists a $\mu \in (0, \rho)$ such that

$$\underline{\theta} - \mu \geq \frac{\max\left\{2,\, \rho^2 \bar{\lambda}\right\}}{\rho - \mu}, \quad (5)$$

then the pseudo-gradient $\mathbf{q}(\cdot,\cdot,\cdot)$ is μ-strongly monotone[2].

We remark that (4) encompasses objectives and pricing schemas as in [11], where the former collects the local operational costs, while the latter includes aggregative pricing functions as illustrated in Section IV. Besides, (5) can be verified by solving a quadratic inequality.

We now proceed to describe the three types of agents we consider in the energy community: ERs, prosumers, and ESSs. Nonetheless, we highlight that other types of agents might also fit the considered framework, e.g., plug-in electric vehicles, and energy hubs. The number of agents of each type and the topology of the graph $\mathcal{G}$ are arbitrary as long as the non-emptiness of the feasible set $\Omega \times \mathbb{R}_{\geq 0}^n$ is guaranteed according to Standing Assumption 1.

Energy Retailers

ERs constitute the traditional power providers in centralized distribution networks, being mainly characterized by thermal-based production plants, and thus able to serve

[2]For the sake of brevity, the proof of Proposition 1 is available at `https://fastupload.io/8ZPeUai6kdMyliE/file`

medium-to-large districts. Their presence is still relevant in the modern energy community since they can provide almost-steady power throughput, and hence compensate for the intermittent generation by renewable-based sources. Therefore, ERs are active grid agents, collected in the set $\mathcal{R} \subset \mathcal{A}$, so that each ER $i \in \mathcal{R}$ is characterized by an outward energy availability $\check{a}_{it} \in \mathbb{R}_{\geq 0}$, i.e., the maximum aggregate energy that can be sold to the agents in $\mathcal{A}_i$ at time $t \in \mathcal{T}$, and a maximum inward energy availability $\hat{a}_{it} \in \mathbb{R}_{\geq 0}$, i.e., the maximum aggregate energy that can be absorbed from neighbouring agents in $\mathcal{A}_i$ at time $t \in \mathcal{T}$. As such, each ER's local constraints set is given by

$$\mathcal{X}_i = \left\{ (\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i) \in \mathbb{R}^{2n_i}_{\geq 0} : \begin{array}{l} \mathbf{I}_2 \otimes \mathbf{1}^\top_{N_i} \otimes \mathbf{I}_T \begin{bmatrix} \check{\mathbf{x}}_i \\ \hat{\mathbf{x}}_i \end{bmatrix} \leq \begin{bmatrix} \check{\mathbf{a}}_i \\ \hat{\mathbf{a}}_i \end{bmatrix} \\ \hat{x}_{ijt} \leq \overline{\hat{x}}_{ijt},\ \check{x}_{ijt} \leq \overline{\check{x}}_{ijt} \\ \forall j \in \mathcal{A}_i, t \in \mathcal{T} \end{array} \right\}, \tag{6}$$

for all $i \in \mathcal{R}$, where $\overline{\hat{x}}_{ijt}$ and $\overline{\check{x}}_{ijt}$ represent local energy-transmission limits for the i-th ESS, $\hat{\mathbf{a}}_i := \mathrm{col}(\hat{a}_{it})_{t \in \mathcal{T}}$ and $\check{\mathbf{a}}_i := \mathrm{col}(\check{a}_{it})_{t \in \mathcal{T}}$. Namely, the constraints set in (6) ensure that the aggregate energy inflow and outflow of an ER do not exceed its respective availability.

Prosumers

The backbone of the modern energy community is composed of prosumers, which are grid agents equipped with their own means of generation in addition to their local energy demand. Often, those private sources are renewable, e.g., eolic, photovoltaic, or hydroelectric, with a gross power production capable of fulfilling domestic energy requirements for industrial appliances. Hence, let us indicate the prosumer set with $\mathcal{P} \subset \mathcal{A}$, with each $i \in \mathcal{P}$ being characterized by the difference between its local energy generation and demand $\delta_{it} \in [\underline{\delta}_i, \overline{\delta}_i] \subset \mathbb{R}$. Namely, for $\delta_{it} = 0$, the i-th prosumer is capable of exact self-sustenance, while for $\delta_{it} > 0$ and $\delta_{it} < 0$, the i-th prosumer has an energy *deficit* and *surplus*, respectively. Therefore, $\underline{\delta}_i$ and $\overline{\delta}_i$ indicate the maximum demand and generation that prosumer i can attain, respectively. For every prosumer, its interactions with its neighbours aim at ensuring its overall energy balance. Therefore, the i-th prosumer's local constraints set is defined as

$$\mathcal{X}_i = \left\{ (\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i) \in \mathbb{R}^{2n_i}_{\geq 0} : \begin{array}{l} \mathbf{1}^\top_{N_i} \otimes \mathbf{I}_T (\check{\mathbf{x}}_i - \hat{\mathbf{x}}_i) = \boldsymbol{\delta}_i \\ \hat{x}_{ijt} \leq \overline{\hat{x}}_{ijt},\ \check{x}_{ijt} \leq \overline{\check{x}}_{ijt} \\ \forall j \in \mathcal{A}_i, t \in \mathcal{T} \end{array} \right\},$$

with $\boldsymbol{\delta}_i := \mathrm{col}(\delta_{it})_{t \in \mathcal{T}}$, and $\overline{\hat{x}}_{ijt}, \overline{\check{x}}_{ijt} \in \mathbb{R}_{>0}$, for all $i \in \mathcal{P}$. Here, $\overline{\hat{x}}_{ijt}$ and $\overline{\check{x}}_{ijt}$ represent local energy-transmission limits for prosumer i.

Energy Storage Systems

To take advantage of peak-generation periods, it is convenient to equip the grid with ESSs. The task of ESSs is to provide energy reserves during low-generation time windows, e.g., at night when referring to photovoltaic systems. Thus, let us define $\mathcal{S} \subset \mathcal{A}$ as the set of ESSs, with each $i \in \mathcal{S}$ characterized by a first-order dynamics of the form[3]

$$s_{i(t+1)} = \alpha_i s_{it} + \sum_{j \in \mathcal{A}_i} \left(\hat{\eta}_i \hat{x}_{ijt} - \frac{1}{\check{\eta}_i} \check{x}_{ijt} \right), \tag{7}$$

where $s_{it} \in \mathbb{R}_{\geq 0}$ is the total stored energy in the i-th ESS at time t, and $\alpha_i, \hat{\eta}_i, \check{\eta}_i \in (0, 1)$ are the leakage coefficient, the charging efficiency, and the discharging efficiency, respectively. Equivalently, given an initial stored energy $s_{i0} \in \mathbb{R}_{\geq 0}$, and defining $\hat{x}_{ij0} = \check{x}_{ij0} = 0$, it follows that

$$s_{it}(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i) = \alpha_i^t s_{i0} + \sum_{\tau=1}^{t} \alpha_i^{t-\tau} \sum_{j \in \mathcal{A}_i} \left(\hat{\eta}_i \hat{x}_{ij\tau} - \frac{1}{\check{\eta}_i} \check{x}_{ij\tau} \right).$$

In fact, for all times $t \in \mathcal{T}$, it is required that $0 \leq s_{it}(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i) \leq \overline{s}_i$, where $\overline{s}_i \in \mathbb{R}_{>0}$ is the maximum storage capacity of the i-th ESS. Therefore, the local constraints set of the i-th ESS is

$$\mathcal{X}_i = \left\{ (\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i) \in \mathbb{R}^{2n_i}_{\geq 0} : \begin{array}{l} 0 \leq s_{it}(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i) \leq \overline{s}_i \\ \hat{x}_{ijt} \leq \overline{\hat{x}}_{ijt},\ \check{x}_{ijt} \leq \overline{\check{x}}_{ijt} \\ \forall j \in \mathcal{A}_i, t \in \mathcal{T} \end{array} \right\},$$

III. The Proposed Approach

To state our proposed approach to solve the GNEP in Definition 1, we reformulate the OPs in (3) in an equivalent yet more convenient form. For all $(i, j, t) \in \mathcal{A} \times \mathcal{A}_i \times \mathcal{T}$, let $y_{ijt} \in \mathbb{R}$ be an auxiliary variable to be computed by agent i, and define $\mathbf{y}_{ij}$, $\mathbf{y}_i$, $\mathbf{y}$, and $\mathbf{y}_{-i}$ according to (1). By introducing constraint $\check{x}_{ijt} = y_{ijt}$, constraints (2b)-(2c) can be equivalently stated as the four constraints: $\hat{\mathbf{x}}_i = \mathbf{y}_{-i}$, $\check{\mathbf{x}}_{-i} = \mathbf{y}_{-i}$, $\check{\mathbf{x}}_i = \mathbf{y}_i$, and $\hat{\mathbf{x}}_{-i} = \mathbf{y}_i$. As such, the decision of each agent $i \in \mathcal{A}$ now regards the tuple $(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \mathbf{y}_i, \mathbf{p}_i) \in \tilde{\Omega}_i(\mathbf{y}_{-i}) \times \Phi_i(\check{\mathbf{x}}_i) \times \mathbb{R}^{n_i}_{\geq 0}$, with

$$\tilde{\Omega}_i(\mathbf{y}_{-i}) = \{(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i) \in \mathcal{X}_i : \hat{\mathbf{x}}_i = \mathbf{y}_{-i}\}$$
$$\Phi_i(\check{\mathbf{x}}_i) = \{\mathbf{y}_i \in \mathbb{R}^{n_i} : \mathbf{y}_i = \check{\mathbf{x}}_i\}.$$

On the other hand, the feasible set regarding such augmented decisions for the entire energy community is given by $\Psi \times \mathbb{R}^n_{\geq 0}$, where

$$\Psi = \left\{ (\hat{\mathbf{x}}, \check{\mathbf{x}}, \mathbf{y}) \in \prod_{i \in \mathcal{A}} \mathcal{X}_i \times \mathbb{R}^n : \begin{array}{l} \hat{\mathbf{x}} = \mathbf{B}\mathbf{y} \\ \check{\mathbf{x}} = \mathbf{y} \end{array} \right\},$$

and by Standing Assumption 1 it holds that Ψ is a closed convex set with non-empty relative interior.

For every $i \in \mathcal{A}$, the OPs in (3) can then be equivalently redefined as

$$\min_{\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i} f_i(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \mathbf{p}_i, \mathbf{p}_{-i}) \ \text{ s.t. } \ (\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i) \in \tilde{\Omega}_i(\mathbf{y}_{-i}) \tag{8a}$$

$$\min_{\mathbf{y}_i, \mathbf{p}_i} \frac{\rho_i}{2} \left\| \mathbf{p}_i - \mathbf{g}_i(\hat{\mathbf{x}}_{-i}) \right\|^2 \ \text{ s.t. } \ (\mathbf{y}_i, \mathbf{p}_i) \in \Phi_i(\check{\mathbf{x}}_i) \times \mathbb{R}^{n_i}. \tag{8b}$$

Here, recall that the constraint $\mathbf{p}_i \in \mathbb{R}^{n_i}_{\geq 0}$ is enforced by the co-domain of $\mathbf{g}_i(\cdot)$. Note that in contrast to the OP in

[3]Note that higher-order dynamics for s_{it} might be used, as long as the resulting $\mathcal{X}_i$ is closed and convex.

(3a), for a given agent i, the OP in (8a) is decoupled from the decisions $\hat{\mathbf{x}}_{-i}$ and $\check{\mathbf{x}}_{-i}$ of other agents, i.e., the inter-agent coupling in (8a) is only due to variables $\mathbf{p}_{-i}$ and $\mathbf{y}_{-i}$. Therefore, simultaneously solving (8a) for all i and under a fixed pair $(\mathbf{y}', \mathbf{p}')$ is equivalent to solving the OP given by

$$\min_{\hat{\mathbf{x}},\check{\mathbf{x}}} \sum_{i\in\mathcal{A}} f_i\left(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \mathbf{p}'_i, \mathbf{p}'_{-i}\right) \text{ s.t. } (\hat{\mathbf{x}}, \check{\mathbf{x}}) \in \prod_{i\in\mathcal{A}} \tilde{\Omega}_i\left(\mathbf{y}'_{-i}\right), \tag{9}$$

which is separable over $\mathcal{A}$. Similarly, the inter-agent coupling in (8b) is only obtained through variable $\hat{\mathbf{x}}_{-i}$. Thus, simultaneously solving (8b) for all i and under a fixed pair $(\hat{\mathbf{x}}', \check{\mathbf{x}}')$ is equivalent to solving the OP given by

$$\min_{\mathbf{y},\mathbf{p}} \sum_{i\in\mathcal{A}} \frac{\rho_i}{2} \left\|\mathbf{p}_i - \mathbf{g}_i\left(\hat{\mathbf{x}}'_{-i}\right)\right\|^2 \text{ s.t. } (\mathbf{y}, \mathbf{p}) \in \prod_{i\in\mathcal{A}} \Phi_i\left(\check{\mathbf{x}}'_i\right) \times \mathbb{R}^{n_i}, \tag{10}$$

which is separable over $\mathcal{A}$ as well. Based on these observations, we remark that Gauss-Seidel ADMM-type GNEP solving methods [9] can be applied to the OPs in (8) following a three-block iterative scheme rather than iterating over the total number of agents. Consequently, in this paper, we adapt [9, Algorithm 4.1] to our framework. Note that [9, Algorithm 4.1] enjoys the simple structure of the celebrated ADMM algorithm, and as a Gauss-Seidel-type method it tends to converge faster than its Jacobi-type counterpart [8].

For every agent $i \in \mathcal{A}$, let $\hat{\mathbf{u}}_i \in \mathbb{R}^{n_i}$ and $\check{\mathbf{u}}_i \in \mathbb{R}^{n_i}$ be the Lagrange multipliers associated to the coupling constraints $\hat{\mathbf{x}}_i = \mathbf{y}_{-i}$ and $\check{\mathbf{x}}_i = \mathbf{y}_i$, respectively. Besides, define $\hat{\mathbf{u}}, \check{\mathbf{u}} \in \mathbb{R}^n$ using the ordering in (1). Let $k \in \mathbb{Z}_{\geq 0}$ denote the iteration index, and let $\hat{\mathbf{x}}^k$, $\check{\mathbf{x}}^k$, $\mathbf{y}^k$, $\mathbf{p}^k$, $\hat{\mathbf{u}}^k$, and $\check{\mathbf{u}}^k$, denote the values of the corresponding optimization variables at iteration k. Applying [9, Algorithm 4.1] to (9)-(10) yields the (sequential) updates given by

$$\left(\hat{\mathbf{x}}^{k+1}, \check{\mathbf{x}}^{k+1}\right) = \underset{(\hat{\mathbf{x}},\check{\mathbf{x}})\in\mathcal{X}}{\arg\min} \left\{ \sum_{i\in\mathcal{A}} f_i\left(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \mathbf{p}_i^k, \mathbf{p}_{-i}^k\right) + \begin{bmatrix}\hat{\mathbf{u}}^k \\ \check{\mathbf{u}}^k\end{bmatrix}^\top \begin{bmatrix}\hat{\mathbf{x}} \\ \check{\mathbf{x}}\end{bmatrix} + \frac{\gamma_1}{2} \left\| \begin{bmatrix}\hat{\mathbf{x}} - \hat{\mathbf{x}}^k \\ \check{\mathbf{x}} - \check{\mathbf{x}}^k\end{bmatrix} \right\|^2 + \frac{\beta}{2} \left\| \begin{bmatrix}\hat{\mathbf{x}} - \mathbf{B}\mathbf{y}^k \\ \check{\mathbf{x}} - \mathbf{y}^k\end{bmatrix} \right\|^2 \right\} \tag{11a}$$

$$\left(\mathbf{y}^{k+1}, \mathbf{p}^{k+1}\right) = \underset{(\mathbf{y},\mathbf{p})\in\mathbb{R}^{2n}}{\arg\min} \left\{ \sum_{i\in\mathcal{A}} \frac{\rho_i}{2} \left\|\mathbf{p}_i - \mathbf{g}_i\left(\hat{\mathbf{x}}_{-i}^{k+1}\right)\right\|^2 - \begin{bmatrix}\hat{\mathbf{u}}^k \\ \check{\mathbf{u}}^k\end{bmatrix}^\top \begin{bmatrix}\mathbf{B}\mathbf{y} \\ \mathbf{y}\end{bmatrix} + \frac{\gamma_2}{2} \left\| \begin{bmatrix}\mathbf{y} - \mathbf{y}^k \\ \mathbf{p} - \mathbf{p}^k\end{bmatrix} \right\|^2 + \frac{\beta}{2} \left\| \begin{bmatrix}\hat{\mathbf{x}}^{k+1} - \mathbf{B}\mathbf{y} \\ \check{\mathbf{x}}^{k+1} - \mathbf{y}\end{bmatrix} \right\|^2 \right\} \tag{11b}$$

$$\begin{bmatrix}\hat{\mathbf{u}}^{k+1} \\ \check{\mathbf{u}}^{k+1}\end{bmatrix} = \begin{bmatrix}\hat{\mathbf{u}}^k \\ \check{\mathbf{u}}^k\end{bmatrix} + \beta \begin{bmatrix}\hat{\mathbf{x}}^{k+1} - \mathbf{B}\mathbf{y}^{k+1} \\ \check{\mathbf{x}}^{k+1} - \mathbf{y}^{k+1}\end{bmatrix}, \tag{11c}$$

where $\mathcal{X} = \prod_{i\in\mathcal{A}} \mathcal{X}_i$, and $\gamma_1, \gamma_2, \beta \in \mathbb{R}_{>0}$ are constant parameters of the algorithm. Now, note that (11b) has the closed-form solution given by

$$\mathbf{y}^{k+1} = \frac{1}{\gamma_2 + 2\beta}\left(\beta\mathbf{B}\hat{\mathbf{x}}^{k+1} + \beta\check{\mathbf{x}}^{k+1} + \gamma_2\mathbf{y}^k + \mathbf{B}\hat{\mathbf{u}}^k + \check{\mathbf{u}}^k\right)$$

$$\mathbf{p}^{k+1} = \left(\mathbf{P} + \gamma_2\mathbf{I}_n\right)^{-1}\left(\operatorname{col}\left(\rho_i\mathbf{g}_i\left(\hat{\mathbf{x}}_{-i}^{k+1}\right)\right)_{i\in\mathcal{A}} + \gamma_2\mathbf{p}^k\right),$$

Algorithm 1: ADMM Distributed GNE computation

1 Set parameters $\gamma_1, \gamma_2, \beta \in \mathbb{R}_{>0}$.

2 Initialize $\hat{\mathbf{x}}_i^0, \check{\mathbf{x}}_i^0, \mathbf{y}_i^0, \mathbf{p}_i^0, \hat{\mathbf{u}}_i^0, \check{\mathbf{u}}_i^0 \in \mathbb{R}_{\geq 0}^{n_i}$, $\forall i \in \mathcal{A}$.

3 Every agent $i \in \mathcal{A}$ receives $\mathbf{y}_{-i}^k$, $\mathbf{p}_{-i}^k$, and computes:

$$\left(\hat{\mathbf{x}}_i^{k+1}, \check{\mathbf{x}}_i^{k+1}\right) = \underset{(\hat{\mathbf{x}}_i,\check{\mathbf{x}}_i)\in\mathcal{X}_i}{\arg\min} \left\{h_i\left(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \mathbf{v}_i^k\right)\right\}$$

4 Every agent $i \in \mathcal{A}$ receives $\hat{\mathbf{x}}_{-i}^{k+1}$, $\check{\mathbf{x}}_{-i}^{k+1}$, $\hat{\mathbf{u}}_{-i}^k$, $\check{\mathbf{u}}_{-i}^k$, and computes:

$$\mathbf{y}_i^{k+1} = \frac{\beta\hat{\mathbf{x}}_{-i}^{k+1} + \beta\check{\mathbf{x}}_i^{k+1} + \gamma_2\mathbf{y}_i^k + \hat{\mathbf{u}}_{-i}^k + \check{\mathbf{u}}_i^k}{\gamma_2 + 2\beta}$$

$$\mathbf{p}_i^{k+1} = \frac{\rho_i\mathbf{g}_i\left(\hat{\mathbf{x}}_{-i}^{k+1}\right) + \gamma_2\mathbf{p}_i^k}{\rho_i + \gamma_2}$$

$$\hat{\mathbf{u}}_i^{k+1} = \hat{\mathbf{u}}_i^k + \beta\left(\hat{\mathbf{x}}_i^{k+1} - \boldsymbol{\omega}_i\right)$$

$$\check{\mathbf{u}}_i^{k+1} = \check{\mathbf{u}}_i^k + \beta\left(\check{\mathbf{x}}_i^{k+1} - \mathbf{y}_i^{k+1}\right),$$

with $\boldsymbol{\omega}_i = \frac{\beta\hat{\mathbf{x}}_i^{k+1} + \beta\check{\mathbf{x}}_{-i}^{k+1} + \gamma_2\mathbf{y}_{-i}^k + \hat{\mathbf{u}}_i^k + \check{\mathbf{u}}_{-i}^k}{\gamma_2 + 2\beta}$.

5 If the termination criterion is met, then stop. Otherwise, update $k \leftarrow k+1$ and go back to Step 3.

with $\mathbf{P} = \operatorname{diag}\left(\rho_i\mathbf{I}_{n_i}\right)_{i\in\mathcal{A}} \in \mathbb{R}_{\geq 0}^{n\times n}$. Thus, using the facts that $\mathbf{B}\mathbf{z} = \operatorname{col}\left(\mathbf{z}_{-i}\right)_{i\in\mathcal{A}}$, $\mathbf{B} = \bar{\mathbf{B}}^\top$, and $\mathbf{B}^\top\mathbf{B} = \mathbf{I}_n$, the updates in (11) yield our proposed Algorithm 1, where we have defined $\mathbf{v}_i^k := \left(\hat{\mathbf{x}}_i^k, \check{\mathbf{x}}_i^k, \mathbf{y}_i^k, \mathbf{y}_{-i}^k, \mathbf{p}_i^k, \mathbf{p}_{-i}^k, \hat{\mathbf{u}}_i^k, \check{\mathbf{u}}_i^k\right)$ and

$$h_i\left(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \mathbf{v}_i^k\right) = f_i\left(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \mathbf{p}_i^k, \mathbf{p}_{-i}^k\right) + \begin{bmatrix}\hat{\mathbf{u}}_i^k \\ \check{\mathbf{u}}_i^k\end{bmatrix}^\top \begin{bmatrix}\hat{\mathbf{x}}_i \\ \check{\mathbf{x}}_i\end{bmatrix} + \frac{\gamma_1}{2} \left\| \begin{bmatrix}\hat{\mathbf{x}}_i - \hat{\mathbf{x}}_i^k \\ \check{\mathbf{x}}_i - \check{\mathbf{x}}_i^k\end{bmatrix} \right\|^2 + \frac{\beta}{2} \left\| \begin{bmatrix}\hat{\mathbf{x}}_i - \mathbf{y}_{-i}^k \\ \check{\mathbf{x}}_i - \mathbf{y}_i^k\end{bmatrix} \right\|^2.$$

The effectiveness of Algorithm 1 is certified by Corollary 1, which provides sufficient conditions to guarantee its asymptotic convergence to a GNE of the energy community.

Corollary 1: If $\gamma_1, \gamma_2, \beta \in \mathbb{R}_{>0}$ and γ_2 satifies that $\gamma_2 > (1/\mu)\left(4\beta^2 + \sum_{i\in\mathcal{A}}\left(\hat{L}_i^2 + \check{L}_i^2\right)\right)$, then the iterations of Algorithm 1 converge strongly to the unique vGNE of the GNEP of Definition 1. That is, as $k \to \infty$, it holds that $\left(\hat{\mathbf{x}}^k, \check{\mathbf{x}}^k, \mathbf{p}^k\right) \to \left(\hat{\mathbf{x}}^*, \check{\mathbf{x}}^*, \mathbf{p}^*\right)$, where $\left(\hat{\mathbf{x}}^*, \check{\mathbf{x}}^*, \mathbf{p}^*\right)$ is a GNE[4].

IV. An Illustrative Numerical Simulation

In this section, we illustrate the proposed framework through a numerical simulation over a 24 hours period, i.e., $T = 24$. As such, consider an energy community comprised of 2 ERs, 9 prosumers, and 3 ESSs, i.e., $N = 14$. Without loss of generality, we set

$$f_i\left(\hat{\mathbf{x}}_i, \check{\mathbf{x}}_i, \mathbf{p}_i, \mathbf{p}_{-i}\right) = \chi_i\left\|\hat{\mathbf{x}}_i + \check{\mathbf{x}}_i\right\|^2 + \mathbf{p}_{-i}^\top\hat{\mathbf{x}}_i - \mathbf{p}_i^\top\check{\mathbf{x}}_i$$

$$\mathbf{g}_i\left(\hat{\mathbf{x}}_{-i}\right) = \frac{\nu_i}{N_i}\mathbf{1}_{N_i} \otimes \operatorname{col}\left(\sum_{j\in\mathcal{A}_i} \hat{x}_{jit}\right)_{t\in\mathcal{T}},$$

[4]For the sake of brevity, the proof of Corollary 1 is available at `https://fastupload.io/8ZPeUai6kdMyliE/file`

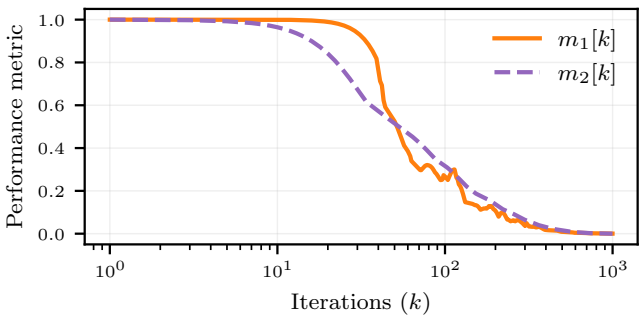

Fig. 1. Evolution of the selected performance metrics over 10^3 iterations of Algorithm 1. Without loss of generality, the initial condition is taken as $\hat{\mathbf{x}}_i^0, \check{\mathbf{x}}_i^0, \mathbf{y}_i^0, \mathbf{p}_i^0, \hat{\mathbf{u}}_i^0, \check{\mathbf{u}}_i^0 = \mathbf{0}_{n_i}$, for all $i \in \mathcal{A}$.

for all $i \in \mathcal{A}$. Here, $\chi_i, \nu_i \in \mathbb{R}_{>0}$ are constant parameters. Note that $f_i(\cdot,\cdot,\cdot,\cdot)$ regards quadratic energy-transmission-related costs, weighted by χ_i, while the considered pricing function $\mathbf{g}_i(\cdot)$ corresponds to a linear map on the averaged aggregate energy request of the neighbors of agent i, weighted by ν_i. In fact, such a pricing function can be rewritten as $\mathbf{g}_i(\hat{\mathbf{x}}_{-i}) = \mathbf{Q}_i \hat{\mathbf{x}}_{-i}$, with $\mathbf{Q}_i = (\nu_i/N_i)\left(\mathbf{1}_{N_i}\mathbf{1}_{N_i}^\top\right) \otimes \mathbf{I}_T$. Thus, $\mathbf{Q}_i^\top \mathbf{Q}_i = \nu_i \mathbf{Q}_i$, and $\lambda_{\max}\left(\mathbf{Q}_i^\top \mathbf{Q}_i\right) = \nu_i^2$. Hence, the considered functions fit the setup of Proposition 1 with $\underline{\theta} = 2\min_{i\in\mathcal{A}} \chi_i$ and $\overline{\lambda} = \max_{i\in\mathcal{A}} \nu_i^2$. For simplicity, for our numerical experiments we set $\rho_i = 1$, and we randomly sample $\chi_i \sim \mathcal{U}[1.5, 2]$ and $\nu_i \sim \mathcal{U}[0.5, 1]$, for all $i \in \mathcal{A}$. Therefore, (5) simplifies to $\underline{\theta} \geq \mu + (2/(1-\mu))$. Since $\underline{\theta} \geq 3$, it follows that (5) is satisfied with $\mu = 0.26$. Finally, note that for the considered functions, it follows that $\hat{L}_i = \check{L}_i = 1$.

Regarding the graph $\mathcal{G}$, we consider a random undirected topology plus a star graph with an ER as the central node. For the agents' parameters, we sample $\overline{\hat{x}}_{ijt}, \overline{\check{x}}_{ijt} \sim \mathcal{U}[4, 6]$ kWh, $\hat{a}_{it}, \check{a}_{it} \sim \mathcal{U}[50, 80]$ kWh, $\delta_{it} = G_{it} - D_{it}$ where $G \sim \mathcal{N}[T/2, \sigma_i]$ kWh represents the energy generation (with $\sigma_i \sim \mathcal{U}[0, 2]$), and $D \sim \mathcal{U}[0, 2]$ kWh represents the demand, $\overline{s}_i \sim \mathcal{U}[20, 30]$ kWh, $s_{ij0} \sim \mathcal{U}[0, \overline{s}_i/N_i]$ kWh, and $\alpha_i, \hat{\eta}_i, \check{\eta}_i \sim \mathcal{U}[0.95, 0.98]$, for all the corresponding $i \in \mathcal{A}$, $j \in \mathcal{A}_i$, and $t \in \mathcal{T}$. Nonetheless, we numerically check that Ω is non-empty so that Standing Assumption 1 holds. Moreover, although the simulation data is synthetic, G_{it} follows the typical bell-shaped curve of photovoltaic generation so that the obtained results follow plausible trends.

Finally, regarding the parameters of Algorithm 1, we let $\gamma_1 = \beta = 0.5$, and we set $\gamma_2 = 112$. Thus, the sufficient condition of Corollary 1 is satisfied and the convergence of Algorithm 1 to the unique GNE of the energy community is guaranteed. In fact, Fig. 1 depicts the evolution of the performance metrics $m_1[k] = \left\|\hat{\mathbf{x}}^k - \mathbf{B}\check{\mathbf{x}}^k\right\|_\infty / \left\|\hat{\mathbf{x}}^1 - \mathbf{B}\check{\mathbf{x}}^1\right\|_\infty$, $m_2[k] = \|\mathbf{c}^k - \mathbf{c}^*\|_\infty / \|\mathbf{c}^1 - \mathbf{c}^*\|_\infty$, where $\mathbf{c} = \mathrm{col}(\hat{\mathbf{x}}, \check{\mathbf{x}}, \mathbf{p})$ and $\mathbf{c}^* = \mathrm{col}(\hat{\mathbf{x}}^*, \check{\mathbf{x}}^*, \mathbf{p}^*)$ is the unique GNE of the energy community. Namely, metric $m_1[k]$ measures the satisfaction of the agreement constraints in (2b)-(2c), for all $i \in \mathcal{A}$, while $m_2[k]$ yields the infinity-norm distance to the GNE (both metrics are normalized over the first iteration's results). That is, if $m_1[k] = 0$, then the computed energy transactions at iteration k are attainable over the energy community and if $m_2[k] = 0$, then the computed solution at iteration k is a GNE for the energy community. As shown, in Fig. 1, the selected metrics indeed converge asymptotically to 0, verifying the effectiveness of Algorithm 1.

V. Concluding Remarks

In this paper, we have formulated an energy transactive framework to model energy communities comprised of (but not limited to) prosumers, energy storage systems, and energy retailers. The underlying control task is formulated as a so-called generalized Nash equilibrium problem, and a distributed Gauss-Seidel-type alternating direction method of multipliers algorithm is devised to solve it. Furthermore, sufficient conditions on the local cost and energy pricing functions are provided, so that the convergence of the algorithm is guaranteed. Future work should seek to extend the framework to even more general energy communities, with additional optimization variables and subject to uncertainties.

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