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Event-Triggered Distributed State Estimation based on Asymptotic Kalman-Bucy Filter*

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Abstract— **Distributed state estimation is a relevant research topic due to its application opportunities in different fields, such as multi-robot cooperation and control of large-scale networked systems. In addition, event-triggering mechanisms have been studied in recent years to reduce communication between network nodes without significantly compromising the desired behavior. In this work, we contribute a distributed algorithm to estimate the state of a stochastic system under event-triggered communication. The proposal uses consensus on the state estimates, and it takes advantage of the asymptotic form of the well-known Kalman-Bucy filter so that only state information needs to be transmitted during the online execution. We provide guarantees of boundedness of the error covariance for the state estimates under event-triggered communication. Moreover, we show that the centralized optimal solution can be recovered when the event threshold is decreased, which is an improvement with respect to existing event-triggered estimators in the stochastic context. Finally, we show via simulation that the proposal effectively reduces communication without sacrificing the quality of the estimates, and it improves performance with respect to previous approaches.**

I. Introduction

Distributed state estimation is a relevant research topic due to its application opportunities in different fields, such as multi-robot cooperation and control of large-scale networked systems [1]. This problem consists of producing state estimates of a dynamical system from the measurement information captured by a network of sensors, in which each sensor node has access to its local measurement and communication with neighboring nodes. The goal is for all network nodes to estimate the plant's full state adequately.

The distributed implementation brings several advantages, such as redundancy that reduces the risk of single-point failure and the uncertainty of the state estimates [2]. Moreover, the combination of information from different sensors allows the reconstruction of the full state of the plant from each node, even if the system is not observable using only the local measurements. On the other hand, the main disadvantage of distributed estimation is the high communication load required between the network nodes. Constant transmissions of information may be detrimental to systems with resource constraints. Examples of such systems are battery-powered devices, transmission networks that are shared by several nodes, or systems with limited bandwidth [3].

Event-triggered strategies have been widely studied in recent years as a possible solution for the problem of constrained communication, achieving significant interest in the context of networked systems and wireless sensor networks (see, for example, [3], [4] and the references therein). They aim to reduce communication through a decision mechanism that monitors the behavior of the setup and chooses the necessary transmission instants accordingly, so that satisfactory performance is maintained.

In the context of distributed state estimation, several works have proposed event-triggered state estimators for different classes of systems. In terms of the event-triggering mechanism, a variety of schemes also exist: monitoring the measured signal and transmitting once its value differs from the last transmitted one [5], [6], [7], [8]; applying the triggering condition to the local state estimate [9], [10]; or evaluating the innovation of the measurements, i.e. the difference between the measured signal and the predicted measurement based on the estimates [11], [12], [13]. However, most of these works feature discrete-time systems, and the works that consider continuous-time systems [14], [15] do not typically include disturbances such as stochastic noise. To the best of our knowledge, the only work that considers the case of continuous-time systems affected by stochastic noise in this context is [16], in which a dynamic consensus algorithm is exploited to estimate the average measurement of the sensor network. This approach is based on [2] but includes event-triggered communication on the consensus step. However, while the boundedness of the consensus error is shown under mild assumptions, this bound is given regarding the average measurement, not on the resulting state estimates. Moreover, in the stochastic context, guarantees of recovering the optimal solution of the centralized filter as the frequency of events increases are not achieved by the previously mentioned works, both considering discrete or continuous-time filters.

This work proposes a different approach to event-triggered distributed state estimation for stochastic systems. The proposed algorithm performs consensus on the state estimates so that stronger guarantees of performance can be provided in comparison to the proposal from [16]. In particular, we prove the boundedness of the error covariance of the estimates under event-triggered communication and stochastic noise, and we show that the optimal centralized solution can be

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recovered as the event threshold is decreased. This optimality aspect is a novel contribution for event-triggered distributed estimators in the stochastic context, regardless of continuous or discrete-time formulations. Furthermore, we validate the proposal via simulation examples, which show a performance improvement compared to the approach from [16].

A. Notation

Let $\mathbf{I}_n$ denote the $n \times n$ identity matrix and let $\mathbb{1} = [1, \dots, 1]^\top$. The operator $\| \bullet \|$ denotes the Euclidean norm, and $\otimes$ is the Kronecker product. Let $\mathsf{cov}\{\bullet, *\} = \mathbb{E}\{(\bullet - \mathbb{E}\{*\})(\bullet - \mathbb{E}\{*\})^\top\}$ denote the covariance operator, with $\mathsf{cov}\{\bullet\} = \mathsf{cov}\{\bullet, \bullet\}$. The notation $\mathsf{diag}(\bullet_i)$ represents the diagonal composition of the matrices indexed by i.

II. Problem Statement

Consider a linear time-invariant system, with dynamics described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{w}(t), \quad t \geq 0 \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times n_\mathbf{w}}$ and $\mathbf{w}(t) \in \mathbb{R}^{n_\mathbf{w}}$ represents an unknown input, accounting for non-modelled dynamics or disturbances. As usual in Kalman filtering literature, let $\mathbf{w}(t)$ be an $n_\mathbf{w}$-dimensional Wiener process with $\mathsf{cov}\{\mathbf{w}(s), \mathbf{w}(r)\} = \mathbf{W} \min(s, r)$ [17, Page 63] and interpret (1) as a Stochastic Differential Equation (SDE). Then, $\mathbf{x}(t)$ is normally distributed with known mean $\mathbf{x}_0$ and covariance matrix $\mathbf{P}_0$ for the initial condition $\mathbf{x}(0)$.

The system described by (1) is observed by a network of N sensors. We consider that the topology of the sensor network can be described by an undirected graph $\mathcal{G}$, with the set of sensor nodes being $\mathcal{V} = \{1, \dots, N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denoting the edge set that describes communication links between sensor nodes. Let $\mathbf{A}_\mathcal{G} = [a_{ij}] \in \{0, 1\}^{N \times N}$ be the adjacency matrix of the graph, which has elements $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise, and $\mathbf{Q}_\mathcal{G}$ be the Laplacian matrix. We define $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ as the set of neighbors of node i.

Each sensor in the network has access to a local measurement of the form

$$\mathbf{y}_i(t) = \mathbf{C}_i\mathbf{x}(t) + \mathbf{v}_i(t), \forall t \geq 0$$

with $\mathbf{C}_i \in \mathbb{R}^{n_\mathbf{y} \times n}$ and $\mathbf{v}_i(t)$ being Gaussian white noise with covariance $\mathbf{R}_i(t)$.

The goal for the sensor network is to jointly estimate the state of the system (1) in a distributed fashion, so that each node uses its local measurement as well as communication with its neighboring nodes. Moreover, in order to save resources in the communication process, each node features an event-triggering mechanism that decides when to broadcast information to the neighbors.

Assumption 1: The pair $(\mathbf{A}, \mathbf{C})$, where $\mathbf{C}$ is defined as $\mathbf{C} = [\mathbf{C}_1^\top, \dots, \mathbf{C}_N^\top]^\top$, is observable.

Remark 1: Note that Assumption 1 requires the system to be collectively observable using the combined information from all sensor nodes, but it does not require local observability.

III. Optimal Centralized Solution

The optimal centralized solution for the problem of continuous-time state estimation in the stochastic context is known to be the Kalman-Bucy filter [18]. This filter provides estimates $\hat{\mathbf{x}}(t)$ for the state $\mathbf{x}(t)$, as well as an error covariance matrix $\mathbf{P}(t) = \mathbb{E}\{(\mathbf{x}(t) - \hat{\mathbf{x}}(t))(\mathbf{x}(t) - \hat{\mathbf{x}}(t))^\top\}$ for the estimation. The centralized implementation, considering the measurements from all the sensors in the network, is given by

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{K}(t)\left(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)\right) \\ \mathbf{K}(t) &= \mathbf{P}(t)\mathbf{C}^\top\mathbf{R}^{-1} \\ \dot{\mathbf{P}}(t) &= \mathbf{A}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^\top + \mathbf{B}\mathbf{W}\mathbf{B}^\top - \mathbf{K}(t)\mathbf{R}\mathbf{K}(t)^\top \end{aligned} \tag{2}$$

with $\mathbf{C}$ defined in Assumption 1, $\mathbf{R} = \mathsf{diag}(\mathbf{R}_i)$ and $\mathbf{y}(t) = [\mathbf{y}_1(t)^\top, \dots, \mathbf{y}_N(t)^\top]^\top$. Moreover, for linear-time invariant systems, the error covariance matrix $\mathbf{P}(t)$ converges to an asymptotic solution $\mathbf{P}_\infty$, given by the following Riccati equation

$$0 = \mathbf{A}\mathbf{P}_\infty + \mathbf{P}_\infty\mathbf{A}^\top + \mathbf{B}\mathbf{W}\mathbf{B}^\top - \mathbf{P}_\infty\mathbf{C}^\top\mathbf{R}^{-1}\mathbf{C}\mathbf{P}_\infty \tag{3}$$

which provides the asymptotic form of the filter with constant gain $\mathbf{K}_\infty = \mathbf{P}_\infty\mathbf{C}\mathbf{R}^{-1}$.

IV. Distributed Asymptotic Filter under Event-Triggered Communication

In order to achieve distributed state estimation with reduced communication, we propose the use of an asymptotic filter featuring a consensus step on the state estimates, in which information is exchanged between nodes in an event-triggered fashion.

A. Event-Triggering Mechanism

Due to the consensus step requiring communication of the state estimates between nodes, we propose the use of the following event-triggering mechanism to decide the sequence of communication events $\{\tau_k^i\}_{k=0}^\infty$ at some node $i \in \mathcal{V}$:

$$\tau_{k+1}^i = \inf\{t - \tau_k^i > \underline{\tau} \ \mid \ \|\hat{\mathbf{x}}_i(t) - \hat{\mathbf{x}}_i(\tau_k^i)\| \geq \delta_i\} \tag{4}$$

where δ_i is a user-defined event threshold and $\underline{\tau} > 0$ is included as time regularization, in order to guarantee a minimum inter-event time. Including time regularization is a common strategy to ensure Zeno-freeness in continuous-time systems [19]. In practice, this parameter can be set as close to zero as necessary, e.g. to the sampling time used to approximate the continuous-time behaviour.

The interpretation of the event-triggering mechanism (4) is straightforward: the current state estimate $\hat{\mathbf{x}}(t)$ of node i is transmitted to its neighbors when it differs from the value $\hat{\mathbf{x}}(\tau_k^i)$ transmitted at the last event by more than the threshold δ_i. This mechanism ensures that the error in the knowledge that a node $j \in \mathcal{N}_i$ has of the estimate at node i is bounded by the event threshold.

B. Distributed Filter

For the distributed filter, we adopt a similar approach as in [20], where a distributed version of the asymptotic Kalman-Bucy filter is proposed. This filter is shown to recover the performance of the asymptotic centralized solution.

Here, we adapt the filter from [20] to include event-triggered communication of the state estimates, as opposed to considering continuous communication between the nodes. By doing so, we obtain the following filter:

$$\begin{aligned}\mathbf{K}_i =& N\mathbf{P}_\infty \mathbf{C}_i^\top \mathbf{R}_i^{-1} \\ \dot{\hat{\mathbf{x}}}_i(t) =& \mathbf{A}\hat{\mathbf{x}}_i(t) + \mathbf{K}_i\left(\mathbf{y}_i(t) - \mathbf{C}_i\hat{\mathbf{x}}_i(t)\right) \\ &+ \kappa \mathbf{P}_\infty \sum_{j\in\mathcal{N}_i}\left(\hat{\mathbf{x}}_j(\tau_t^j) - \hat{\mathbf{x}}_i(t)\right)\end{aligned} \tag{5}$$

where κ is the consensus gain and we define $\tau_t^j = \max\{\tau_k^j \leq t\}$ as the last event triggered at node j prior to time t, recalling that each node has a different triggering sequence.

Note that if $\delta_i = 0, \forall i \in \mathcal{V}$, the filter with full communication is recovered. Moreover, note that using the asymptotic form of the filter requires the pre-computation of the asymptotic covariance matrix $\mathbf{P}_\infty$. As in [20], this can be done by computing the average inverse covariance matrix of the network $\mathbf{C}^\top\mathbf{R}^{-1}\mathbf{C} = \sum_{i=1}^N \mathbf{C}_i^\top \mathbf{R}_i^{-1}\mathbf{C}_i$ through the *Push-Sum* algorithm, which is a diffusion protocol for the computation of values over graphs. Then, once the nodes reach agreement on the value of $\mathbf{C}^\top\mathbf{R}^{-1}\mathbf{C}$, each node can solve the Riccati equation (3) to obtain $\mathbf{P}_\infty$. For brevity, we refer the reader to [20] for additional details on how $\mathbf{P}_\infty$ may be computed. The number of nodes N can also be computed similarly, if it is not a known parameter for the nodes.

From the point of view of saving communication resources, using the asymptotic form of the Kalman-Bucy filter is beneficial, since it only requires the communication of $\hat{\mathbf{x}}_i(t)$ at events, rather than having to transmit both $\hat{\mathbf{x}}_i(t)$ and its corresponding error covariance matrix $\mathbf{P}_i(t)$.

Remark 2: Note that the gains $\mathbf{K}_i, \mathbf{P}_\infty$ in (5) are relevant in regards to the optimal centralized solution. As $\delta_i \to 0, \kappa \to \infty$, (5) recovers the performance of the centralized Kalman-Bucy filter (2), where $\mathbf{P}_\infty$ is the asymptotic covariance of the state estimates produced by the filter. Finally, note that high gain arguments as for κ are common in other popular observers, particularly under unknown inputs similar to the case considered in this work.

C. Stability Analysis

In this Section, we provide formal guarantees of stability of the proposed filter. In particular, we show that the true error covariance of the state estimates is bounded under event-triggered communication, i.e. the estimates are bounded in mean square error. Moreover, we show that the performance of the optimal centralized solution can be recovered as the event threshold $\delta_i \to 0$.

Proposition 1: There exists κ_0 such that $\forall\kappa > \kappa_0$ the error covariance of the state estimates produced by (5) is bounded. In particular, for $\delta_i \to 0, \kappa \to \infty$, the true error covariance of the estimates tends to the asymptotic covariance of the optimal centralized solution, $\mathbf{P}_\infty$, as $t \to \infty$.

Proof: Define the estimation error of a node as $\mathbf{e}_i(t) = \mathbf{x}(t) - \hat{\mathbf{x}}_i(t)$ and let $\mathbf{e}(t) = [\mathbf{e}_1(t)^\top, \ldots, \mathbf{e}_N(t)^\top]^\top$. Additionally, define the true error covariance of the estimates as $\mathbf{E}(t) = \mathsf{cov}\{\mathbf{e}(t)\}$.

Note that the event-triggering mechanism (4) introduces an error in the consensus term given by $\mathbf{d}_i(t) = \hat{\mathbf{x}}_i(t) - \hat{\mathbf{x}}_i(\tau_t^i)$, with $\|\mathbf{d}_i(t)\| \leq \delta_i$.

Considering the system dynamics (1) and the filter (5), we can write the error dynamics of the estimate in a node as

$$\begin{aligned}\dot{\mathbf{e}}_i(t) =& \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}_i(t) \\ =& \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{w}(t) - \mathbf{A}\hat{\mathbf{x}}_i(t) - \mathbf{K}_i\left(\mathbf{y}_i(t) - \mathbf{C}_i\hat{\mathbf{x}}_i(t)\right) \\ &- \kappa\mathbf{P}_\infty \sum_{j\in\mathcal{N}_i}\left(\hat{\mathbf{x}}_j(\tau_t^j) - \hat{\mathbf{x}}_i(t)\right) \\ =& \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{w}(t) - \mathbf{A}\hat{\mathbf{x}}_i(t) - \mathbf{K}_i\mathbf{C}_i\mathbf{x}(t) - \mathbf{K}_i\mathbf{v}_i(t) \\ &+ \mathbf{K}_i\mathbf{C}_i\hat{\mathbf{x}}_i(t) - \kappa\mathbf{P}_\infty\sum_{j\in\mathcal{N}_i}\left(\hat{\mathbf{x}}_j(t) - \mathbf{d}_j(t) - \hat{\mathbf{x}}_i(t)\right) \\ =& \left(\mathbf{A} - \mathbf{K}_i\mathbf{C}_i\right)\mathbf{e}_i(t) + \mathbf{n}_i(t) \\ &- \kappa\mathbf{P}_\infty\sum_{j\in\mathcal{N}_i}\left(\hat{\mathbf{x}}_j(t) - \hat{\mathbf{x}}_i(t)\right) + \kappa\mathbf{P}_\infty\sum_{j\in\mathcal{N}_i}\mathbf{d}_j(t) \\ =& \left(\mathbf{A} - \mathbf{K}_i\mathbf{C}_i\right)\mathbf{e}_i(t) + \mathbf{n}_i(t) \\ &- \kappa\mathbf{P}_\infty\sum_{j\in\mathcal{N}_i}\left(\mathbf{e}_i(t) - \mathbf{e}_j(t)\right) + \kappa\mathbf{P}_\infty\sum_{j\in\mathcal{N}_i}\mathbf{d}_j(t)\end{aligned}$$

where we have defined $\mathbf{n}_i(t) = \mathbf{B}\mathbf{w}(t) - \mathbf{K}_i\mathbf{v}_i(t)$. For the aggregate error of the nodes, $\mathbf{e}(t)$, we can write the dynamics as follows, taking into account the Laplacian $\mathbf{Q}_\mathcal{G}$ and adjacency $\mathbf{A}_\mathcal{G}$ matrices of the graph:

$$\begin{aligned}\dot{\mathbf{e}}(t) =& \mathsf{diag}\left(\mathbf{A} - \mathbf{K}_i\mathbf{C}_i\right)\mathbf{e}(t) + \mathbf{n}(t) \\ &- \kappa\left(\mathbf{Q}_\mathcal{G}\otimes\mathbf{P}_\infty\right)\mathbf{e}(t) + \kappa\left(\mathbf{A}_\mathcal{G}\otimes\mathbf{P}_\infty\right)\mathbf{d}(t) \\ =& \mathbf{A}(\kappa)^*\mathbf{e}(t) + \mathbf{n}(t) + \kappa\left(\mathbf{A}_\mathcal{G}\otimes\mathbf{P}_\infty\right)\mathbf{d}(t)\end{aligned} \tag{6}$$

with $\mathsf{diag}\left(\mathbf{A} - \mathbf{K}_i\mathbf{C}_i\right)$ being the diagonal composition of the matrices $\left(\mathbf{A} - \mathbf{K}_i\mathbf{C}_i\right)$ for all nodes $i \in \mathcal{V}$, $\mathbf{A}(\kappa)^* = \mathsf{diag}\left(\mathbf{A} - \mathbf{K}_i\mathbf{C}_i\right) - \kappa\left(\mathbf{Q}_\mathcal{G}\otimes\mathbf{P}_\infty\right)$, $\mathbf{n}(t) = [\mathbf{n}_1(t)^\top, \ldots, \mathbf{n}_N(t)^\top]^\top$ and $\mathbf{d}(t) = [\mathbf{d}_1(t)^\top, \ldots, \mathbf{d}_N(t)^\top]^\top$.

Note that (6) resembles the error dynamics obtained in [20], with the exception of the error term $\kappa\left(\mathbf{A}_\mathcal{G}\otimes\mathbf{P}_\infty\right)\mathbf{d}(t)$ induced by the event-triggered communication. Recall that if $\delta_i = 0$, this term vanishes and the error system from [20] is recovered.

According to [20, Theorem 1], there exists a $\kappa_0 > 0$ such that $\forall\kappa > \kappa_0$ the matrix $\mathbf{A}(\kappa)^*$ in (6) is Hurwitz. Moreover, note that the covariance of the noise disturbance $\mathbf{n}(t)$ is constant. This can be observed since $\mathsf{cov}\{\mathbf{n}_i(t)\} = \mathbf{B}\mathbf{W}\mathbf{B}^\top + \mathbf{K}_i\mathbf{R}_i\mathbf{K}_i^\top$ and

$$\begin{aligned}&\mathsf{cov}\{\mathbf{n}_i(t), \mathbf{n}_j(t)\} \\ &= \mathsf{cov}\{\mathbf{B}\mathbf{w}(t), \mathbf{B}\mathbf{w}(t)\} + \mathsf{cov}\{\mathbf{B}\mathbf{w}(t), \mathbf{K}_j\mathbf{v}_j(t)\} \\ &+ \mathsf{cov}\{\mathbf{K}_i\mathbf{v}_i(t), \mathbf{B}\mathbf{w}(t)\} + \mathsf{cov}\{\mathbf{K}_i\mathbf{v}_i(t), \mathbf{K}_j\mathbf{v}_j(t)\} \\ &= \mathsf{cov}\{\mathbf{B}\mathbf{w}(t), \mathbf{B}\mathbf{w}(t)\} = \mathbf{B}\mathbf{W}\mathbf{B}^\top\end{aligned}$$

Hence, $\mathsf{cov}\{\mathbf{n}(t)\} = \mathbb{1}\mathbb{1}^\top \otimes \mathbf{B}\mathbf{W}\mathbf{B}^\top + \mathsf{diag}(\mathbf{K}_i\mathbf{R}_i\mathbf{K}_i^\top)$.

For the event-triggered error term, we have the following covariance:

$$\begin{aligned}&\mathsf{cov}\{\kappa\left(\mathbf{A}_{\mathcal{G}}\otimes\mathbf{P}_{\infty}\right)\mathbf{d}(t)\} = \\ &= \kappa^2\left(\mathbf{A}_{\mathcal{G}}\otimes\mathbf{P}_{\infty}\right)\mathsf{cov}\{\mathbf{d}(t)\}\left(\mathbf{A}_{\mathcal{G}}\otimes\mathbf{P}_{\infty}\right)^{\top}\end{aligned}$$

Recalling that $\|\mathbf{d}_i(t)\| \le \delta_i$, this implies

$$\|\mathbf{d}(t)\| \le \bar{\delta}\sqrt{N}$$

with $\bar{\delta} := \max_i \delta_i$, then

$$\|\mathbf{d}(t) - \mathbb{E}\{\mathbf{d}(t)\}\| \le 2\bar{\delta}\sqrt{N}$$

Therefore the elements of the vector

$$\mathbf{q}(t) = [q_1(t), \ldots, q_{nN}(t)]^{\top} := \mathbf{d}(t) - \mathbb{E}\{\mathbf{d}(t)\}$$

satisfy $|q_{\mu}(t)| \le 2\bar{\delta}\sqrt{N}$ for all $\mu \in \{1, \ldots, nN\}$.

As a result,

$$|m_{\mu\nu}(\bar{\delta})| \le 4\bar{\delta}^2 N$$

with $m_{\mu\nu}(\bar{\delta}) := \mathbb{E}\{q_{\mu}(t)q_{\nu}(t)\}$ and $m_{\mu\nu}(0) = 0$. Henceforth,

$$\mathsf{cov}\{\mathbf{d}(t)\} = \mathbb{E}\{\mathbf{q}(t)\mathbf{q}(t)^{\top}\} \equiv [m_{\mu\nu}(\bar{\delta})]$$

implying $\mathsf{cov}\{\mathbf{d}(t)\}$ is a bounded function of $\bar{\delta}$ and $\mathsf{cov}\{\mathbf{d}(t)\} \equiv \mathbf{0}$ when $\bar{\delta} = 0$.

Then, the proof of boundedness for the error covariance follows from $\mathbf{A}(\kappa)^*$ being Hurwitz, and the covariances for the terms $\mathbf{n}(t)$ and the event-triggered error being bounded.

Finally, recalling that when $\delta_i \to 0$ the algorithm from [20] is recovered, [20, Theorem 3] ensures that the true covariance of the estimates produced at each node of the network tends to the asymptotic solution of the centralized Kalman-Bucy filter (2), i.e. $\mathsf{cov}\{\mathbf{e}_i(t)\} \to \mathbf{P}_{\infty}$, as $\kappa \to \infty$, $t \to \infty$.

■

V. Discussion

In this Section, we discuss the main differences between our proposal and the approach taken in [16], which is, to the best of our knowledge, the only other work that considers a similar setup for continuous-time stochastic systems.

In particular, the algorithm proposed in [16] is as follows. First, define the average measurement of the network in information form $\bar{\mathbf{z}}(t)$ and its inverse covariance matrix $\bar{\mathbf{Z}}$:

$$\bar{\mathbf{z}}(t) := \frac{1}{N}\sum_{i=1}^{N}\mathbf{C}_i^{\top}\mathbf{R}_i^{-1}\mathbf{y}_i(t)$$

$$\bar{\mathbf{Z}} := \frac{1}{N}\sum_{i=1}^{N}\mathbf{C}_i^{\top}\mathbf{R}_i^{-1}\mathbf{C}_i$$

Each node computes its estimates $\hat{\mathbf{z}}_i(t)$ of $\bar{\mathbf{z}}(t)$ via the following event-triggered consensus algorithm:

$$\begin{aligned}\dot{\mathbf{p}}_i(t) &= -\gamma\mathbf{p}_i(t) + \kappa\sum_{j\in\mathcal{N}_i}\left(\hat{\mathbf{z}}_i(t) - \hat{\mathbf{z}}_j(\tau_t^j)\right)\\ \hat{\mathbf{z}}_i(t) &= \mathbf{C}_i^{\top}\mathbf{R}_i^{-1}\mathbf{y}_i(t) - \mathbf{p}_i(t)\end{aligned} \tag{7}$$

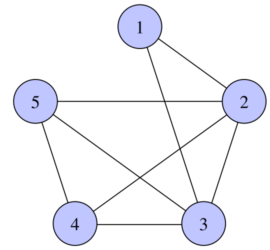

Fig. 1. Graph representing the sensor network used in the experiments.

with the event-triggering mechanism being

$$\tau_{k+1}^i = \inf\{t - \tau_k^i > \underline{\tau} \;\;|\;\; \|\hat{\mathbf{z}}_i(t) - \hat{\mathbf{z}}_i(\tau_k^i)\| \ge \delta_i\}$$

The inverse covariance matrix is updated at events by requesting $\hat{\mathbf{Z}}_j(\tau_{k-}^i)$ from the neighbors $j \in \mathcal{N}_i$ prior to the event τ_k^i, averaging the values as

$$\hat{\mathbf{Z}}_i(\tau_k^i) = \frac{\hat{\mathbf{Z}}_i(\tau_{k-}^i) + \sum_{j\in\mathcal{N}_i}\hat{\mathbf{Z}}_j(\tau_{k-}^i)}{J_i + 1}$$

where $J_i := \sum_{j=1}^{N} a_{ij}$, and broadcasting $\hat{\mathbf{Z}}_i(\tau_k^i)$ so that all neighbours also update their estimate to the same value. Finally, nodes input their estimates of the average measurement and covariance matrix to their local Kalman-Bucy filter:

$$\begin{aligned}\dot{\hat{\mathbf{x}}}_i(t) =& \mathbf{A}\hat{\mathbf{x}}_i(t) + N\mathbf{P}_i(t)\hat{\mathbf{z}}_i(t) - N\mathbf{P}_i(t)\hat{\mathbf{Z}}_i(t)\hat{\mathbf{x}}_i(t)\\ \dot{\mathbf{P}}_i(t) =& \mathbf{A}\mathbf{P}_i(t) + \mathbf{P}_i(t)\mathbf{A}^{\top} + \mathbf{B}\mathbf{W}\mathbf{B}^{\top} - N\mathbf{P}_i(t)\hat{\mathbf{Z}}_i(t)\mathbf{P}_i(t)\end{aligned}$$

Note that this algorithm performs consensus on the average measurement of the network, rather than the state estimates. Therefore, while boundedness of the consensus error on $\bar{\mathbf{z}}(t)$ is proven in [16, Proposition 1], providing guarantees on the error of the resulting state estimates is not straightforward. In contrast, our proposal (5) performs consensus directly on the state estimates, rather than the measurements, enabling the analysis of the error covariance of the state estimates and providing a direct relation between this covariance and the event threshold, rather than the less explicit bound on the consensus error given in [16]. Moreover, for the proof of convergence of the consensus algorithm, an assumption is needed on the bounds of noise of the signals, which we no longer need for the analysis presented here.

Finally, from the point of view of reducing communication, the proposal here only requires the communication of state estimates at events, rather than both the information-form measurement and its inverse covariance matrix.

VI. Simulation Examples

To illustrate the effectiveness of the proposal, consider the sensor network with $N = 5$ sensors, described by Figure 1, which observes the trajectory of a moving target on a plane. Let the state vector $\mathbf{x}(t) = [x(t), y(t), v_x(t), v_y(t)]^{\top}$, where the position of the object in Cartesian coordinates is given by $(x(t), y(t))$ and the corresponding velocities are

$(v_x(t), v_y(t))$. The system can be described by (1) with

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the covariance of the process noise $\mathbf{w}(t)$ is given by $\mathbf{W} = \mathbf{I}_{n_\mathbf{w}}$. The sensors obtain a measurement of the state of the plant in the form $\mathbf{y}_i(t) = \mathbf{C}_i\mathbf{x}(t) + \mathbf{v}_i(t)$, with

$$\mathbf{C}_1 = \mathbf{C}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$
$$\mathbf{C}_3 = \mathbf{C}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$
$$\mathbf{C}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

and noise covariances for $\mathbf{v}_i(t)$ given by $\mathbf{R}_1 = 0.01$, $\mathbf{R}_2 = 0.015$, $\mathbf{R}_5 = 0.02$ and

$$\mathbf{R}_3 = \mathbf{R}_4 = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}$$

Note that the plant is not locally observable from all nodes, only from 3 and 4, which will also perform better than if they used only their local data.

For the simulation tests, we have initialized the true system state to $\mathbf{x}_0 = [1, 1, 1, 1]^\top$. Since initial conditions are often unknown in practice, we have initialized the estimates to $\hat{\mathbf{x}}(0) = [0, 0, 0, 0]^\top$ and $\mathbf{P}(0) = \mathbf{I}_n$. We have used a simulation step of $h = 10^{-4}$, which also acts as the minimum inter-event time $\underline{\tau}$ in (4).

First, we compare the performance of our event-triggered proposal to the full communication algorithm. We have set the consensus gain $\kappa = 50$ in (5), and the event threshold $\delta_i = 0.1 \ \forall i \in \mathcal{V}$ in (4) for the event-triggered case. Recall that, for $\delta_i = 0$, (5) becomes the full communication case.

Figures 2 and 3 show the estimation results for the full communication and event-triggered case, respectively. While the result in terms of the state estimates appears similar, the event-triggered case uses, on average for a node, only 0.3% of the communication slots available for the node to send information, considering that in the full communication case communication occurs at every simulation step.

As usual in event-triggered state estimation, a trade-off usually exists between the frequency of communication and the resulting estimation error, i.e. increasing δ_i in the event-triggering mechanism reduces communication, at the cost of increasing the estimation error [21]. To evaluate the effect of different event thresholds in the communication-error trade-off, we have run simulations for a range of values of δ_i, and computed the communication rate $\mathcal{C}_s$ and mean estimation error $\mathcal{E}_s$ of a sensor node for each simulation as follows:

$$\mathcal{C}_s = \frac{\sum_{i=1}^{N} e_i \, h}{N \, T_f}$$
$$\mathcal{E}_s = \frac{1}{N \, T_f} \sum_{i=1}^{N} \int_0^{T_f} \|\hat{\mathbf{x}}_i(t) - \mathbf{x}(t)\| \mathrm{d}t$$

where e_i is the number of events triggered at node i, h is the simulation step and T_f is the total time for each simulation,

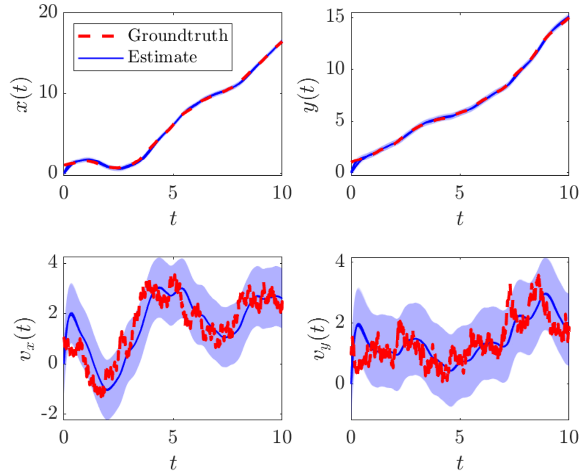

Fig. 2. State estimates of all nodes in the network, using full communication ($\delta_i = 0$, $\forall i \in \mathcal{V}$).

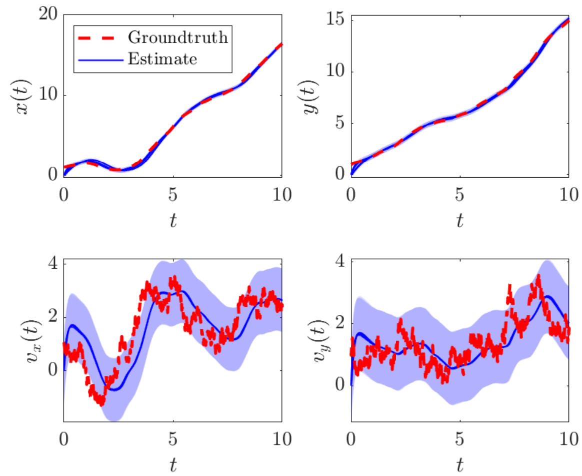

Fig. 3. State estimates of all nodes in the network, using event-triggered communication with $\delta_i = 0.1$, $\forall i \in \mathcal{V}$. Communication is reduced to 0.3% of the full communication case for each node, on average.

which we have set to $T_f = 10$. Note that the communication rate is normalized between 0 (no communication) and 1 (full communication).

To account for the presence of stochastic noise, we have run $S = 10$ simulations for each value of δ_i, and then averaged the results to obtain

$$\mathcal{C} = \frac{1}{S} \sum_{s=1}^{S} \mathcal{C}_s, \quad \mathcal{E} = \frac{1}{S} \sum_{s=1}^{S} \mathcal{E}_s$$

With these values, we have obtained the results shown in Figure 4. We have computed the trade-off curves for different values of the consensus gain κ in (5), namely $\kappa = 100$ and $\kappa = 1000$, in order to see how the interaction between the event threshold and consensus gain may affect the resulting estimation error. As seen in the Figure, communication can be very significantly reduced without impacting the

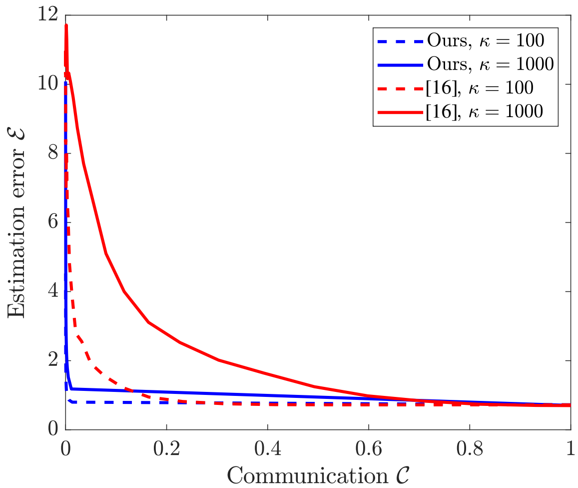

Fig. 4. Communication vs. estimation error trade-off, for different consensus gains κ. Our proposal greatly reduces communication without degrading performance with respect to the full communication case. Moreover, it is robust with respect to the choice of κ and it improves performance with respect to the proposal from [16].

estimation error. Additionally, the algorithm is robust with respect to the choice of κ, even for low communication rates, and does not need $\kappa \to \infty$ in practice to produce adequate results.

Moreover, we have computed the same communication-error curves for the method proposed in [16], again with similar consensus gains (we have fixed $\gamma = 5$ in (7)). The results are also represented in Figure 4. As shown in the Figure, this algorithm can also significantly reduce communication, but the error values for low communication rates increase more significantly than with our proposal here. This may be due to our proposal having the triggering condition monitoring the actual estimates, rather than the measured signals as in [16]. Additionally, the effect of the consensus gains is much more significant for [16], where a high value for κ can severely amplify the error induced by the event-triggering mechanism as communication is reduced. Thus, not only has our proposal an advantage from the point of view of formal analysis, but it also results in a smaller estimation error for similar communication rates in practice.

VII. Conclusions

In this work, we have presented a distributed state estimator with event-triggered communication based on the asymptotic form of the Kalman-Bucy filter. In opposition to prior literature, it features consensus on the state estimates rather than on the average measured signal of the network, which allows stronger guarantees of performance and improved results in practice. We have proven that the estimates are bounded in mean square error through formal analysis and that the centralized solution can be recovered by decreasing the event threshold. Additionally, we have validated the proposal through simulation experiments, which show that our proposal achieves a significant reduction in communication compared to its full communication counterpart, without sacrificing performance, as well as lower estimation errors for the same communication rates compared to prior work. Future research will address additional problems, such as communication delays or losses and directed graphs.

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