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Singular Perturbations for Implicit port-Hamiltonian systems

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Abstract— **In this work, we present the standard Singular Perturbations technique applied to Implicit port-Hamiltonian systems. The investigation produces a structure-preserving reduced-order model if certain additional passivity conditions are satisfied. Moreover, such an investigation provides a different insight into the standard Singular Perturbations approach relating the negligible time constant parameters ε to energy parameters. We analyze the deviation between the complete system model and the reduced one via a Lyapunov-based approach. We then conclude the paper by applying the proposed reduced order model to a DC-motor example to show the effectiveness of the development.**

I. Introduction

The port-Hamiltonian approach to modeling and controlling complex physical systems constitutes a well-established framework that originated with the seminal work by van der Schaft and Maschke [1], [2]. For a comprehensive overview of this field, including control techniques, one can refer to [3], [4], [5].

Port-Hamiltonian systems (PHS) have the particular feature of describing all the main physical properties of the system under consideration, such as energy dissipation, passivity, and power conservation laws. Moreover, the formalism is very suitable for the interconnection of physical systems, preserving the passivity, stability, and structure in a larger port-Hamiltonian system, [6]. When lumped-parameters systems are interconnected or when even a standard spatial discretization of a distributed-parameter model is taken into account, that state space dimension of the system rapidly grows. Therefore, Model Order Reduction (MOR) techniques play a crucial role in the analysis and the control design of this kind of systems, [7].

The central goal of Model Order Reduction is to derive a more compact or simpler model from the original high-order model. As articulated in [7, Sec.1.3], the objectives of model reduction can be succinctly summarized as follows: gaining a deeper comprehension of the system, possibly obtaining a macroscopic perspective of its dynamics or input-output relationships; reducing computational demands for simulation and control design; and achieving simpler-to-implement control laws. Over the years, a plethora of techniques have been developed, and they can be classified based on the domain in which the original and approximated models are compared, i.e., the frequency or time domains. In particular, time-domain techniques involve the state-space realization of the system dynamics and amongst them, we find the Singular Perturbations [8], time scale separation [9], and more recent developments like those in [10]. All these approaches are based on a state truncation, that is part of the system's evolution is neglected, thus leaving only the part related to the slow evolution. The interested reader can find up-to-date surveys on these and other model order reduction techniques in [11], [12], as well as more classic references such as [13], [14], [15], and [16].

In the context of port-Hamiltonian systems, the application of these reduction techniques necessitates the preservation of additional desired properties in the reduced model. Alongside retaining stability and passivity, it is imperative to maintain the system's underlying structure.

To this end, all significant contributions to Model Order Reduction for port-Hamiltonian systems must ensure structural preservation. Among all, the first results on this line, rely on the balanced truncation technique [17] (see also [18] and [19] for more recent results), while relying on Krylov subspaces approach [20], Moment matching [21], [22], [23], on interpolation methods [24]. More recent advancements are based on symplectic Moder Order Reduction [25] and on tangential interpolation for descriptor system [26].

A distinctive and relevant approach, particularly tailored for port-Hamiltonian systems, is the Effort- and Flow-constraint reduction method [27], developed and expanded upon in [28] and [29]. Notably, the same authors claimed that the standard Singular Perturbations technique, as described in [30], is not suitable for a structure-preserving order reduction applied to PHS, see [27, Remark 6.2], [28, Sec. 3] and [29].

In light of evolving methodologies and novel perspectives in the field, this paper introduces a fresh analysis of the Singular Perturbations technique for model order reduction applied to port-Hamiltonian systems. The proposed investigation, indeed, yields a structure-preserving reduced order model for PHS, and it allows us to obtain a different (more physical) point of view on the Singular Perturbations technique and provides more insights into the negligible system time constant, relating them to the system energy parameters instead.

This paper is structured as follows. In Section II, we provide an overview of the framework within which we operate, introducing port-Hamiltonian systems and their realization on the Lagrange subspace in Section II-A. In Section II-B, we delineate the structure-preserving Model Order Reduction problem specific to port-Hamiltonian systems. In Section II-C, we provide a concise review of the standard Singular

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Perturbations technique. In Section III, we deliver the main result of this paper, demonstrating the application of the Singular Perturbations technique to Implicit port-Hamiltonian systems and establishing the structure-preserving nature of the reduced order model under specific passivity conditions. In Section IV, we conduct a Lyapunov-based analysis of the reduced order model's performance in terms of state and output deviation from the complete system model. We conclude by applying the method to a DC-motor system in Section V and summarizing our findings in Section VI.

II. Model order reduction for port-Hamiltonian systems

In this section, we begin by revisiting the definition of port-Hamiltonian systems employed in this paper. This definition builds upon recent developments where the implicit characterization of energy through reciprocal relations [31], and it leads to an implicit definition of port-Hamiltonian systems, also called descriptor[1] port-Hamiltonian systems [32], [33]. Subsequently, we will provide a general definition of singularly perturbed systems and outline how it applies to the descriptor port-Hamiltonian systems considered in this paper.

A. Port-Hamiltonian systems

1) Explicit port-Hamiltonian systems generated by a Hamiltonian function: Let us first recall the definition of passive port-Hamiltonian systems generated by an energy function $\mathcal{H} = \frac{1}{2}x^\top Q x$, where $Q = Q^\top \geq 0$, with dynamics

$$\begin{aligned} \dot{x} &= (J - R)\, Qx + (G - V)u \\ y &= (G + V)^\top Qx + (N + U)u, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $u \in \mathbb{R}^m$ and the matrices of opportune dimensions, where J and N are skew-symmetric, while $R \geq 0$ and U are symmetric matrices. Moreover, we have the passivity condition

$$W = \begin{bmatrix} R & V \\ V^\top & U \end{bmatrix} \geq 0 \tag{2}$$

as in [34], [28], [35], [3][ch.2], so that the energy balance equation gives us

$$\dot{\mathcal{H}}(x) = y^\top u - \begin{bmatrix} x^\top Q & u^\top \end{bmatrix} \begin{bmatrix} R & V \\ V^\top & U \end{bmatrix} \begin{bmatrix} Qx \\ u \end{bmatrix} \leq y^\top u$$

thus, providing passivity with storage function $\mathcal{H}$, since $\mathcal{H} \geq 0$ as it is a quadratic form.

2) Implicit port-Hamiltonian systems on Lagrange subspace: A more general formulation of port-Hamiltonian systems has been suggested where instead of defining a Hamiltonian function, one considers relations between the state variable x and the effort variable e. Maxwell's reciprocity conditions on these relations correspond to the pair of state and effort variables belonging to the Lagrangian subspace [31] which is defined as follows

$$\mathcal{L} = \left\{(x, e) \in \mathcal{X} \times \mathcal{X}^\star | S^\top x + P^\top e = 0\right\}$$

where the matrices $P, S \in \mathbb{R}^{n \times n}$, satisfy

$$P^\top S = S^\top P \text{ and } \operatorname{rank}\,[P \;\; S] = n.$$

In this paper, we shall use the equivalent definition, called *image representation*

$$\mathcal{L} = \left\{(x, e) \in \mathcal{X} \times \mathcal{X}^\star | \exists z \in \mathbb{R}^n \text{ s.t. } x = Pz,\ e = Sz\right\}.$$

In this context, a general port-Hamiltonian system with a resistive element can be described by the following system dynamics:

$$\begin{aligned} P\dot{z} &= (J - R)Sz + (G - V)u \\ y &= (G + V)^\top Sz + (N + U)u \end{aligned} \tag{3}$$

with the passivity property, related to the Hamiltonian function $\mathcal{H} = \frac{1}{2}z^\top SPz$, given by

$$W = \begin{bmatrix} R & V \\ V^\top & U \end{bmatrix} \geq 0.$$

Remark. *In the formalism in Sec.II-A.1, the Lagrange subspace is described by the graph of Q, see [31].*

For the purposes of this paper, we assume, without loss of generality[2], P and S to be block diagonal in the *slow-fast* coordinates and we rewrite the matrices accordingly, i.e.,

$$\begin{aligned} P &= \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} & M = J - R &= \begin{bmatrix} M_1 & M_{12} \\ M_{21} & M_2 \end{bmatrix} \\ S &= \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} & G - V &= \begin{bmatrix} G_1 - V_1 \\ G_2 - V_2 \end{bmatrix}. \end{aligned}$$

With this matrix realization, the system can be split into slow and fast dynamics:

$$\begin{aligned} \begin{pmatrix} P_1 \dot{z}_1 \\ P_2 \dot{z}_2 \end{pmatrix} &= \begin{bmatrix} M_1 S_1 & M_{12} S_2 \\ M_{21} S_1 & M_2 S_2 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{bmatrix} G_1 - V_1 \\ G_2 - V_2 \end{bmatrix} u \\ y &= \begin{bmatrix} G_1 + V_1 \\ G_2 + V_2 \end{bmatrix}^\top \begin{bmatrix} S_1 & \\ & S_2 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + (N + U)u, \end{aligned} \tag{4}$$

with $z_1 \in \mathbb{R}^{n_1}$ and $z_2 \in \mathbb{R}^{n_2}$ and their related matrices of suitable dimensions. Accordingly, we split the x state into $x_1 = P_1 z_1$ and $x_2 = P_2 z_2$. The choice of these mappings P_1 and P_2, (S_1 and S_2 respectively), has to be done 'wisely', according to the parameters that are going to be neglected. In other words, we have to construct these mappings, for the image representation of the Lagrange subspace $\mathcal{L}$, such that when $P_2 \to 0$ also $x_2 \to 0$, i.e. the latent variable z_2 is bounded when $P_2 \to 0$. Moreover, without loss of generality, we consider $M_2 S_2$ to be full rank.

Remark. *A particular class of port-Hamiltonian systems on Lagrange Subspace is that of DAE (or Constrained) port-Hamiltonian systems, as described in [32], [36], [31],*

[1] We call this type of system *descriptor* because of the structure of the dynamics, and DAE pH systems, in our nomenclature, are a particular (singular) case.

[2] Because Q is a symmetric matrix it can always be diagonalized, and thus P and S can always be taken as diagonal matrices.

[37], [38], [39]. In particular, the latter can be obtained by considering identically $P_2 = 0$ in (4)*, thus obtaining an additional algebraic relationship among the states along with their dynamical evolution.*

B. Model Order Reduction Problem

For PHS, the symmetry and positive semi-definiteness properties of the involved matrices play a crucial role in defining the passivity and stability of the involved system. Thus, any Model Order Reduction technique applied to PHS must be structure-preserving, i.e., it has to preserve such symmetry and definiteness properties of the system matrices in the reduced order model. In other words, when we consider the full order model (3), the Model Order Reduction problem is to find a reduced order model of the form

$$\begin{aligned} P'\dot{z}' &= (J' - R')\, S'z' + (G' - V')u \\ y &= (G' + V')^\top S'z' + (N' + U')\, u \end{aligned} \tag{5}$$

with $R' \geq 0$, $J' = -J'^\top$ and

$$W' = \begin{bmatrix} R' & V' \\ V'^\top & U' \end{bmatrix} \geq 0$$

with related Hamiltonian $\mathcal{H}' = \frac{1}{2} z'^\top P'^\top S' z'$.

C. Singular Perturbations

The standard Singular Perturbations setup, in the context of linear dynamical systems, can be expressed as follows:

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + A_{12} x_2 + B_1 u \\ \varepsilon \dot{x}_2 &= A_{21} x_1 + A_2 x_2 + B_2 u \\ y &= C_1 x_1 + C_2 x_2 + D u \end{aligned} \tag{6}$$

where ε is usually a very small time constant that allows us to characterize the time scale separation between the two dynamics, thus identifying the slow and fast system evolutions.

Assuming A_2 is Hurwitz, the standard procedure, leading to the approximated dynamics, involves imposing $\varepsilon = 0$ so to obtain the slow-driven behavior of x_2, i.e., $\bar{x}_2 = -A_2^{-1}(A_{21}x_1 + B_2 u)$ as a solution of the equivalent algebraic constraint

$$\varepsilon \dot{\bar{x}}_2 = A_{21} x_1 + A_2 \bar{x}_2 + B_2 u \underset{\varepsilon=0}{=} 0$$

To analyze the attractiveness property of the slow evolution $\bar{x}_2$, we introduce the deviation coordinate $\tilde{x}_2 = x_2 - \bar{x}_2$, whose dynamics reads as

$$\begin{aligned} \varepsilon \dot{\tilde{x}}_2 &= \varepsilon \dot{x}_2 - \varepsilon \dot{\bar{x}}_2 \\ &= A_{21} x_1 + A_2 (\tilde{x}_2 + \bar{x}_2) + B_2 u - \varepsilon \dot{\bar{x}}_2 \\ &= A_2 \tilde{x}_2 - \varepsilon \dot{\bar{x}}_2 \end{aligned} \tag{7}$$

that is, the deviation dynamics has a stable filter realization. Hence, because A_2 is full rank, setting $\varepsilon = 0$, we additionally have $\tilde{x}_2 = 0$. This means that we can approximate the system onto the slow manifolds by setting $x_2 = \bar{x}_2$ (or considering $\tilde{x}_2 = 0$) in the original dynamical equations. As a result, the reduced order system for the Singular Perturbations is given by

$$\begin{aligned} \dot{x}_1 &= \left(A_1 - A_{12} A_2^{-1} A_{21}\right) x_1 + \left(B_1 - A_{12} A_2^{-1} B_2\right) u \\ y &= \left(C_1 - C_2 A_2^{-1} A_{21}\right) x_1 + \left(D - C_2 A_2^{-1} B_2\right) u. \end{aligned}$$

Remark. *In this setting, system* (6) *is similar to the implicit port-Hamiltonian System* (4) *where equivalently $P_1 = I$ and $P_2 = \varepsilon I$.*

III. Main Result

We analyze how the Singular Perturbations approach applies to port-Hamiltonian systems on Lagrange subspace, i.e., PHS in the form (4), and what type of passivity properties we obtain from the system approximation to the slow dynamics. In particular, we aim to describe the dynamics (4), with state (z_1, z_2), via a particular change of coordinates that introduces a deviation state on the fast dynamics bringing to the state representation $(z_1, \tilde{z}_2)$. The introduction of the deviation state $\tilde{z}_2$ will be a crucial step in obtaining and characterizing the reduced order dynamics, with states $(\bar{z}_1, \bar{z}_2)$, obtained by projecting the whole dynamics on the slow manifold. To clarify the approach, we need to establish an equivalence between the first two sets of coordinates (thus involving a different dynamics). Afterward, we will establish an approximation relationship between the last two sets of coordinates. The schematic representation of this process is outlined below

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \underset{\substack{\text{Diff.}\\ \text{dyn.}}}{\sim} \begin{pmatrix} \bar{z}_1 \\ \tilde{z}_2 \end{pmatrix} \underset{P_2 = 0}{\approx} \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2(\bar{z}_1) \end{pmatrix}$$

We start presenting our main result with the following assumption.

Assumption III.1 (Time-scale separation)**.** *P_2 is the matrix of energy parameters associated with the fast eigenvalues (or the small time constants of the system) of the system dynamics $A = P^{-1}MS$. Moreover, the input u and its time derivative $\dot{u}$ are essentially bounded for all nonnegative t.*

We then introduce the matrices involved in the reduced order model, which are defined as follows:

$$\begin{aligned} J_1' &= J_1 - \frac{1}{2}\left(M_{12} M_2^{-1} M_{21} - M_{21}^\top M_2^{-\top} M_{12}^\top\right) \\ R_1' &= R_1 + \frac{1}{2}\left(M_{12} M_2^{-1} M_{21} + M_{21}^\top M_2^{-\top} M_{12}^\top\right) \\ G_1' &= G_1 - \frac{1}{2}\left(M_{21}^\top M_2^{-\top} + M_{12} M_2^{-1}\right)(G_2 - V_2) \\ V_1' &= V_1 - \frac{1}{2}\left(M_{21}^\top M_2^{-\top} - M_{12} M_2^{-1}\right)(G_2 - V_2) \\ N' &= N - \frac{1}{2}\Big((G_2 + V_2)^\top M_2^{-1} (G_2 - V_2) - \\ &\qquad (G_2 - V_2)^\top M_2^{-\top} (G_2 + V_2)\Big) \\ U' &= U - \frac{1}{2}\Big((G_2 + V_2)^\top M_2^{-1} (G_2 - V_2) + \\ &\qquad (G_2 - V_2)^\top M_2^{-\top} (G_2 + V_2)\Big). \end{aligned} \tag{8}$$

These matrices are central to our analysis, and they are employed to define the reduced-order model. With these definitions in place, we can now present the main result in the following theorem.

Theorem III.1. *Consider system* (4)*, with a time scale separation, i.e., Assumption III.1 holds, and assume P_1 full rank. Then, if matrices* (8) *satisfy $R_1' \geq 0$ and*

$$W' = \begin{bmatrix} R_1' & V_1' \\ V_1'^{\top} & U' \end{bmatrix} \geq 0$$

then, system (4) *can be reduced to the following PHS*

$$\begin{aligned} P_1 \dot{\bar{z}}_1 &= (J_1' - R_1')\, S_1 \bar{z}_1 + (G_1' - V_1')u \\ \bar{y} &= (G_1' + V_1')^{\top} S_1 \bar{z}_1 + (N' + U')\, u \end{aligned} \tag{9}$$

with Hamiltonian function

$$\mathcal{H} = \frac{1}{2} \bar{z}_1^{\top} P_1^{\top} S_1 \bar{z}_1.$$

Proof: Following the Singular Perturbations approach, we impose [3] $P_2 = 0$, to get the approximation

$$\bar{z}_2 := -S_2^{-1} M_2^{-1} \left(M_{21} S_1 z_1 + (G_2 - V_2)\, u \right). \tag{10}$$

By assumption, P_1 is full rank, and thus we can explicitly write

$$\begin{aligned} \dot{\bar{z}}_2 = -S_2^{-1} M_2^{-1} &\left[M_{21} S_1 P_1^{-1} \left(M_1 S_1 z_1 + M_{12} S_2 z_2 + \right.\right. \\ &\left.\left. (G_1 - V_1)\, u \right) \right] - S_2^{-1} M_2^{-1} (G_2 - V_2)\, \dot{u}. \end{aligned}$$

For the sake of readability, in the dynamics equation of (3), we temporarily consider $(G_i - V_i) = B_i$, $i = 1, 2$. Then, we define the deviation coordinates $\tilde{z}_2 = z_2 - \bar{z}_2$, and the corresponding dynamics is

$$\begin{aligned} P_2 \dot{\tilde{z}}_2 &= P_2 \dot{z}_2 - P_2 \dot{\bar{z}}_2 \\ &= M_{21} S_1 z_1 + M_2 S_2 z_2 + B_2 u + P_2 S_2^{-1} M_2^{-1} \cdot \\ &\quad \left[M_{21} S_1 P_1^{-1} \left(M_1 S_1 z_1 + M_{12} S_2 z_2 + B_1 u \right) + B_2 \dot{u} \right] \\ &= \Big(M_2 S_2 + \underline{P_2 S_2^{-1} M_2^{-1} M_{21} S_1 P_1^{-1} M_{12} S_2} \Big)\, \tilde{z}_2 + \\ &\quad \underline{P_2 S_2^{-1} M_2^{-1} M_{21} S_1 P_1^{-1}} \cdot \\ &\quad \left(M_1 S_1 - M_{12} M_2^{-1} M_{21} S_1 \right) z_1 + \\ &\quad \underline{P_2 S_2^{-1} M_2^{-1} M_{21} S_1 P_1^{-1}} \left(B_1 - M_{12} M_2^{-1} B_2 \right) u + \\ &\quad \underline{P_2 S_2^{-1} M_2^{-1} B_2 \dot{u}}. \end{aligned}$$

Considering $P_2 = 0$, for bounded signals u and $\dot{u}$, the evolution of the deviation dynamics is that of a stable autonomous linear system, i.e.,

$$P_2 \dot{\tilde{z}}_2 = M_2 S_2 \tilde{z}_2$$

which has $\tilde{z}_2 = 0$ as a stable equilibrium point. Thus, we obtain the dynamics of the resulting system by substituting in (4), the slow evolution of z_2, i.e., $z_2 = \bar{z}_2$, that is we consider the deviation coordinate $\tilde{z}_2 = 0$, we obtain

$$\begin{aligned} P_1 \dot{z}_1 &= \left(M_1 - M_{12} M_2^{-1} M_{21} \right) S_1 z_1 \\ &\quad + \left[(G_1 - V_1) - M_{12} M_2^{-1} (G_2 - V_2) \right] u \\ y &= \left[S_1^{\top} (G_1 + V_1) - M_{21}^{\top} M_2^{-\top} (G_2 + V_2) \right]^{\top} z_1 \\ &\quad + \Big[N + U - (G_2 + V_2)^{\top} M_2^{-1} (G_2 - V_2) \Big]\, u. \end{aligned}$$

[3]Note that, it is conceptually wrong to fix $\dot{z}_2 = 0$, because, even if on the slow manifold, the evolution of z_2 may still be present and have a non-null time derivative.

We can easily see that the system thus obtained has a port-Hamiltonian structure by considering

$$\begin{aligned} J_1' - R_1' &= M_1 - M_{12} M_2^{-1} M_{21} \\ J_1' &= J_1 - \mathrm{skew}\{ M_{12} M_2^{-1} M_{21} \} \\ R_1' &= R_1 - \mathrm{sym}\{ M_{12} M_2^{-1} M_{21} \} \\ G_1' - V_1' &= G_1 - V_1 - M_{12} M_2^{-1} (G_2 - V_2) \\ G_1' + V_1' &= G_1 + V_1 - M_{21}^{\top} M_2^{-\top} (G_2 - V_2) \\ N' + U' &= N + U - (G_2 + V_2)^{\top} M_2^{-1} (G_2 - V_2) \end{aligned}$$

from which we obtain matrices (8). $\square$

It's important to note that the matrix P_1 can be considered full rank without any loss of generality. This assumption holds because if P_1 were not full rank, it would imply that the related singular dynamics in the z_1 state should have been considered as part of the algebraic constraint associated with the fast evolution of the system.

Remark. *Fixing $P_2 = 0$, we obtain a flow-constraint [29], i.e., $\dot{x}_2 = P_2 \dot{z}_2 = 0$, but in the $\tilde{z}$ coordinates, this induces an effort-constraint [29] since $M_2 S_2$ is full rank, i.e.,*

$$P_2 \dot{\tilde{z}}_2 = M_2 S_2 \tilde{z}_2 + P_2 \dot{\bar{z}}_2 \underset{P_2 = 0}{\Longrightarrow} 0 = M_2 S_2 \tilde{z}_2$$

which implies that $\tilde{z}_2 = 0$, because $M_2 S_2$ is full rank, and thus the associated effort variable is zero $S_2 \tilde{z}_2 = 0$.

In general, the slow behavior of the system is affected by the fast or transient evolution of the dynamics, for non-zero initial conditions. This implies that the transient dynamics (such as $\tilde{z}_2$ in this paper) has an impact on the equivalent initial condition of the reduced order model, as investigated in [10]. With the approach presented here, the initial condition of z_2 is completely ignored and z_2 is thought to be initialized directly on the slow manifold, $z_2(0) = \bar{z}_2(0)$. This assumption simplifies the analysis but may not fully capture the initial transient behavior.

IV. Performance Evaluation of the Reduced Order Models

In the complete model where $P_2 \neq 0$, the system dynamics does not involve any algebraic constraint among the system's states. This implies that the reduced order model obtained in Th.III.1, i.e., (9), will show some state and output deviation with respect to the evolution of the complete system. To characterize this deviation, we consider the error state $\tilde{z}_1 = z_1 - \bar{z}_1$, where $\bar{z}_1$ is the state trajectory of the reduced order model (9), with $\bar{z}_1(0) = z_1(0)$. The output signal is given by the output error $\tilde{y} = y - \bar{y}$, and $\tilde{z}_2$ plays the role of the input signal. Thus, we have the following involved dynamics:

$$\begin{aligned} P_1 \dot{\tilde{z}}_1 &= \left(M_1 - M_{12} M_2^{-1} M_{21} \right) S_1 \tilde{z}_1 + M_{12} S_2 \tilde{z}_2 \\ &= (J_1' - R_1') S_1 \tilde{z}_1 + (\tilde{G}_1 - \tilde{V}_1) \tilde{z}_2 \\ \tilde{y} &= \left[(G_1 + V_1) - M_{21}^{\top} M_2^{-\top} S_2^{-\top} (G_2 + V_2) \right]^{\top} S_1 \tilde{z}_1 \\ &\quad + (G_2 + V_2)^{\top} S_2 \tilde{z}_2 \\ &= (\tilde{G}_1 + \tilde{V}_1)^{\top} S_1 \tilde{z}_1 + (\tilde{N}_1 + \tilde{U}_1) \tilde{z}_2 \end{aligned}$$

with matrices $\tilde{G}_1$, $\tilde{V}_1$, $\tilde{N}_1$, and $\tilde{U}_1$ defined accordingly. The obtained deviation dynamics has a port-Hamiltonian structure and it is stable by assumptions on R_1' in Th.III.1, but that are no a priory passivity guarantees from input $\tilde{z}_2$ and the output $\tilde{y}$.

However, because $\tilde{z}_1(0) = 0$ we can characterize the L_2 norm of the output deviation $\|\tilde{y}\|$ as a function of the L_2 norm of $\tilde{z}_2$. This can be achieved by exploiting the H_∞ norm of the deviation transfer function

$$\tilde{G}_1(s)=(\tilde{G}_1+\tilde{V}_1)^\top S_1(P_1 s-M_1' S_1)^{-1}(\tilde{G}_1-\tilde{V}_1)+\tilde{N}_1+\tilde{U}_1.$$

With this we can write $\|\tilde{y}\| \leq \|\tilde{G}_1(s)\|_\infty \|\tilde{z}_2\|$, where[4]

$$\|\tilde{G}_1(s)\|_\infty = \max_\omega \sigma\left(\tilde{G}_1(j\omega)\right).$$

To fully characterize the asymptotic evolution of the output deviation (and of the slow evolution of the state) it is important to describe the asymptotic behavior of $\|\tilde{z}_2\|$. This can be achieved through a Lyapunov analysis, as detailed in the following Theorem.

Theorem IV.1. *Under Assumption III.1, denote* $M = ess\sup\|\dot{\tilde{z}}_2\|$. *Then, for any positive* $\epsilon < \sqrt{2\sigma_{\min}(R_2)}$ *there exist a* δ*, such that* $2R_2 - \delta^2 P_2 S_2^{-1} \geq \epsilon^2$ *and the limit behaviour of* $\|\tilde{z}_2\|$ *is given by*

$$\lim_{t\to\infty} \|\tilde{z}_2\| \leq \frac{\sqrt{2}}{\delta\epsilon}\mu(S_2^\top P_2)\sqrt{\frac{\sigma_{\max}(S_2^\top P_2)}{\sigma_{\max}(S_2^\top S_2)}}M.$$

Proof: The proof runs on the analysis of the Lyapunov function $V = \frac{1}{2}\tilde{z}_2^\top S_2^\top P_2 \tilde{z}_2$. In particular, by taking its time derivative we have

$$\begin{aligned}
\dot{V} &= \tilde{z}_2^\top S_2^\top M_2 S_2 \tilde{z}_2 - \tilde{z}_2^\top S_2^\top P_2 \dot{\tilde{z}}_2\\
&\leq -z_2^\top S_2^\top R_2 S_2 \tilde{z}_2 + \frac{1}{2}\delta^2 \tilde{z}_2^\top S_2^\top P_2 \tilde{z}_2 + \frac{1}{2\delta^2}\dot{\tilde{z}}_2^\top S_2^\top P_2 \dot{\tilde{z}}_2\\
&\leq \frac{1}{2} z_2^\top S_2^\top (\delta^2 P_2 S_2^{-1} - 2R_2) S_2 \tilde{z}_2 + \frac{1}{2\delta^2}\dot{\tilde{z}}_2^\top S_2^\top P_2 \dot{\tilde{z}}_2\\
&\leq -\frac{\epsilon^2}{2}\frac{\sigma_{\min}(S_2^\top S_2)}{\sigma_{\max}(S_2^\top P_2)}V + \frac{1}{2\delta^2}\sigma_{\max}(S_2^\top P_2)M^2.
\end{aligned}$$

Then, by the comparison lemma, we have

$$\begin{aligned}
&V = \frac{1}{2}\tilde{z}_2^\top S_2^\top P_2 \tilde{z}_2 \leq\\
&\exp(-\alpha t)V(\tilde{z}_2(0)) + \frac{1}{\delta^2\epsilon^2}\frac{\sigma^2_{\max}(S_2^\top P_2)}{\sigma_{\min}(S_2^\top S_2)}M^2(1-\exp(-\alpha t))
\end{aligned}$$

with

$$\alpha = \frac{\epsilon^2}{2}\frac{\sigma_{\min}(S_2^\top S_2)}{\sigma_{\max}(S_2^\top P_2)}.$$

We can thus explicitly write $\|\tilde{z}_2\|^2 \leq V/\sigma_{\min}(S_2^\top P_2)$ and taking the limit for $t \to \infty$ gives

$$\lim_{t\to\infty} \|\tilde{z}_2\|^2 \leq \frac{2}{\delta^2\epsilon^2}\frac{\sigma_{\max}(S_2^\top P_2)}{\sigma_{\min}(S_2^\top P_2)}\frac{\sigma_{\max}(S_2^\top P_2)}{\sigma_{\min}(S_2^\top S_2)}M^2,$$

taking the square root will then prove the theorem. □

[4]By $\sigma(\Lambda)$ we mean the set of singular values of the matrix Λ.

V. Example: DC-motor

We consider the example of a DC-motor, with shaft angular momentum p_ω and magnetic flux variable ϕ with an associated Hamiltonian function

$$\mathcal{H} = \frac{1}{2}\begin{pmatrix} p_\omega & \phi \end{pmatrix}\begin{bmatrix} \frac{1}{J} & 0\\ 0 & \frac{1}{L}\end{bmatrix}\begin{pmatrix} p_\omega\\ \phi\end{pmatrix}$$

with associated dynamics

$$\begin{aligned}
\begin{pmatrix}\dot{p}_\omega\\ \dot{\phi}\end{pmatrix} &= \begin{bmatrix} -\beta & k_\tau\\ -k_e & -R\end{bmatrix}\begin{pmatrix}\dfrac{\partial\mathcal{H}}{\partial p_\omega}\\ \dfrac{\partial\mathcal{H}}{\partial\phi}\end{pmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{pmatrix}-\tau_\ell\\ \mathcal{V}\end{pmatrix}\\
y = \begin{pmatrix}\omega\\ \mathcal{I}\end{pmatrix} &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{pmatrix}\dfrac{\partial\mathcal{H}}{\partial p_\omega}\\ \dfrac{\partial\mathcal{H}}{\partial\phi}\end{pmatrix}
\end{aligned}$$

where ω and τ_ℓ are the shaft angular velocity and the load torque, k_τ and k_e are the torque and back e.m.f. constants, while $R, \mathcal{I}$, and $\mathcal{V}$ are the resistance, current, and applied voltage of the electrical circuit, respectively.

Defining $z_1 = p_\omega$ and $z_2 = \mathcal{I} = \partial_\phi\mathcal{H} = \phi/L$ we re-write the energy in the latent variable coordinates via the *total* Legendre transform $\mathcal{H}^\star(p_\omega, \mathcal{I}) = p_\omega^2/(2J) + L\mathcal{I}^2/2$ and the dynamics in the z coordinates reads as

$$\begin{aligned}
\begin{pmatrix}\dot{p}_\omega\\ L\dot{\mathcal{I}}\end{pmatrix} &= \begin{bmatrix} -\beta & k_\tau\\ -k_e & -R\end{bmatrix}\begin{pmatrix}\dfrac{1}{J}p_\omega\\ \mathcal{I}\end{pmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{pmatrix}-\tau_\ell\\ \mathcal{V}\end{pmatrix}\\
y = \begin{pmatrix}\omega\\ \mathcal{I}\end{pmatrix} &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{J} & 0\\ 0 & 1\end{bmatrix}\begin{pmatrix}p_\omega\\ \mathcal{I}\end{pmatrix}
\end{aligned}$$

where the matrices P and S here read as $P = \mathrm{diag}(1, L)$ and $S = \mathrm{diag}(J^{-1}, 1)$. Following the Singular Perturbations procedure, we impose $L = 0$, thus obtaining $\mathcal{I} = (\mathcal{V} - k_e(\partial\mathcal{H}^\star)/(\partial p_\omega))/R$ and hence the reduced order dynamics reads as

$$\begin{aligned}
\dot{p}_\omega &= -\left(\beta + \frac{k_\tau k_e}{R}\right)\frac{\partial\mathcal{H}^\star}{\partial p_\omega} + \begin{bmatrix}1 & \dfrac{k_\tau}{R}\end{bmatrix}\begin{pmatrix}-\tau_\ell\\ \mathcal{V}\end{pmatrix}\\
y = \begin{pmatrix}\omega\\ \mathcal{I}\end{pmatrix} &= \begin{bmatrix}1\\ -\dfrac{k_e}{R}\end{bmatrix}\frac{\partial\mathcal{H}^\star}{\partial p_\omega} + \begin{bmatrix}0 & 0\\ 0 & \dfrac{1}{R}\end{bmatrix}\begin{pmatrix}-\tau_\ell\\ \mathcal{V}\end{pmatrix}.
\end{aligned}$$

One can easily appreciate that the system is in the port-Hamiltonian form (3) and but does not preserve passivity for all possible system parameters. Indeed, we have

$$W' = T\begin{bmatrix}\beta + \dfrac{k_\tau k_e}{R} & -\dfrac{k_\tau + k_e}{2R} & 0\\ -\dfrac{k_\tau + k_e}{2R} & \dfrac{1}{R} & 0\\ 0 & 0 & 0\end{bmatrix}T^\top, T = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$

whose first principal minor is always positive and the second one is nonnegative if and only if $2R\beta - (k_\tau^2 + k_e^2) \geq 0$, that may not be true in general (e.g., take β or R sufficiently small). Thus, applying the standard singular perturbation approach does not always lead to a structure-preserving, thus passive, reduced-order model.

VI. Conclusions

We apply the standard Singular Perturbations method to Implicit port-Hamiltonian systems. Our analysis provides a structure-preserving port-Hamiltonian reduced order model, under certain additional passivity assumptions. The resulting reduced order model is moreover defined on the Lagrange subspace of the slow dynamics.

Exploiting the port-Hamiltonian formalism, we deepen the understanding of the Singular Perturbation technique, offering new and more physically insightful connections between the negligible time constant parameterization and an 'energetic' parameterization.

We complete the analysis by investigating the asymptotic deviation between the original system and the obtained reduced-order model. We then conclude the work with an analytical example, showing the effects of the developed analysis.

It is worth highlighting that in this paper we have not dealt with possible singularities in the structural matrices (such as neglecting resistive terms) and thus in the Dirac structure. This case is currently under consideration and may be the subject of future work on the topic.

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