2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

Minimal water consumption for a crop fertirrigation model

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***Abstract*— In this work, we consider a simplified model of crop fertirrigation as a non-autonomous controlled system, with soil moisture, nitrogen content, and biomass as state variables and the delivered water flow rate as input. We study the problem of minimizing the total water quantity delivered during the agricultural season under the constraints that the crops are not suffering from water or nitrogen stress at any time. We establish sufficient conditions for the feasibility of the problem and depict several control strategies depending on the initial nitrogen content. In particular, we show that this problem can exhibit an infinity of singular trajectories of the same cost.**

I. INTRODUCTION

In many regions of the world, and especially in the arid ones, agricultural production faces water scarcity, leading to difficulties in satisfying food population needs. Crops irrigation with treated wastewater, instead of fresh water, is a solution for preserving water resources. This practice increasingly attracts decision-makers attention, but some essential technical and sanitary issues (such as time-varying fluctuations, pipes clogging, and undesirable nano-particles or pathogens) must be addressed.

The underlying idea is to preserve, at the water treatment step, the nutrients that are beneficial for crop growth, notably nitrogen, so that no additional nutrients, which may spread and contaminate soils, have to be brought. In this regard, irrigation with reused water amounts to fertirrigation (i.e., irrigation with nutrients supplied in water). In the present work, we propose to address the problem of optimizing crop production via fertirrigation. Although many works about optimization with classical irrigation can be found in the literature [17], [15], [14], [10], [2], and several commercial software are available on the market [4], [13], the optimization of fertirrigation has been comparatively much less considered, apart some recent works [11], [9], [7]. With classical irrigation that covers the crop's water needs during the whole agricultural season, the soil's nitrogen concentration can be diluted to the point that it no longer meets the crop demands if the initial nitrogen content is insufficient. This often causes farmers to supply nitrogen at the beginning of the season with the risk of spreading unnecessary quantities of nitrogen in soils. Bringing nitrogen with water appears to be a safe solution, provided that its concentration is appropriate and does not end up with too much nitrogen in the soil at the end of the season.

Here also, one faces a dilemma between nitrogen dilution and water supply, which we will find in the optimization problem. In the former works [11], [9], [7], numerical solutions have been proposed but there was not yet mathematical analysis of optimal control under constraints, as we do here. The outline of this work is as follows. The considered model of the crop is introduced in Section II. The constrained optimization problem is formulated in Section III, along with the analysis of conditions of the existence of solutions. The structure of the optimal solutions is investigated in Section IV. Some numeric illustrations and discussions are given in Section V.

II. THE FERTIRRIGATION MODEL

Following existing literature [16], [12] and former works [8], [2], [7], we consider a simplified crop model in greenhouse suitable for decision-making, where we explicitly consider a nitrogen compartment. The dynamics includes soil water balance, soil nitrogen balance, and biomass production as follows:

$$\dot{S}(t)=k_1\left(-\varphi(t)K_S(S(t))-(1-\varphi(t))K_R(S(t))+k_2u(t)\right) \tag{1}$$

$$\dot{N}(t) = -k_3\varphi(t)K_S(S(t))f\left(\frac{N(t)}{S(t)}\right) + k_4C_N^{in}u(t) \tag{2}$$

$$\dot{B}(t) = k_5\varphi(t)K_S(S(t))f\left(\frac{N(t)}{S(t)}\right) \tag{3}$$

where S denotes the soil humidity level (between 0 and 1), N and B (in $\mathbb{R}_+$) are the nitrogen content and the biomass per unit of soil surface, and k_i, $i = 1, \ldots, 5$ are positive constants. The time function φ is the crop radiation interception efficiency related to the canopy cover, which is supposed to be C^1 and increasing during the season period $[0, T]$ with values in $[0, 1]$, where times 0 and T correspond to seeding and harvesting dates. The terms $\varphi(t)K_S(S(t))$ and $(1-\varphi(t))\,K_R(S(t))$ represent the crop transpiration and soil evaporation at time t depending on the water content S. The functions K_S, K_R, and f are piecewise continuous functions whose classical forms can be found in the agronomic literature:

$$K_S(S) = \begin{cases} 0, & S \in [0, S_w] \\ \frac{S-S_w}{S^\star - S_w}, & S \in [S_w, S^\star] \\ 1, & S \in [S^\star, 1] \end{cases}$$

$$K_R(S) = \begin{cases} 0, & S \in [0, S_h] \\ \frac{S-S_h}{1-S_h}, & S \in [S_h, 1] \end{cases}$$

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where $0 < S_h < S_w < S^\star < 1$,

$$f(C_N) = \begin{cases} \frac{C_N}{\eta_C}, & C_N \in [0, \eta_C] \\ 1, & C_N > \eta_C > 0 \end{cases}$$

where numbers S_h, S_w, $S^\star$, η_C describe thresholds corresponding to change of modes in the growth. In particular, the crop suffers from *hydric stress* when the value of the function K_S is not maximal (that is, when $S < S^\star$), and from *nitrogen stress* when the value of the function f is not maximal (that is when $N < \eta_C S$). Parameters C_N^{in} (concentration of nitrogen in the delivered water) and $u_{\max}$ (maximum water flow) are the operating characteristics of the fertirrigation system. The control u is the flow rate of the delivered water. Given an initial value $S(0) = S_0$, a function u is *admissible* for S_0 if it is Lebesgue measurable on $[0, T]$ with values in $[0, u_{\max}]$ and the corresponding solution S remains in $[0, 1]$, which amounts to impose the constraint on the control

$$S = 1 \Rightarrow u \leq \frac{1}{k_2}.$$

From the agronomic viewpoint, this means that when the soil is saturated, i.e., $S = 1$, the irrigation has to be regulated to avoid water loss.

The authors have already considered this model to estimate in real-time simultaneously the model parameters and the nitrogen content of the soil (N), while measuring the humidity (S) and the produced biomass (B), for any persistently exciting control $u(\cdot)$ [5].

III. The optimization problem

From (3), one can see that the biomass production at time T is maximal if one has

$$K_S(S(t)) f\left(\frac{N(t)}{S(t)}\right) = 1, \quad t \in [0, T]$$

that is, when there is no water or nitrogen stress during the whole season. That amounts to having trajectories lying in the subset defined by the two constraints

$$E := \{(S, N) \in [0, 1] \times \mathbb{R}_+, | \; S \geq S^\star \; N \geq \eta_C S\}.$$

We shall look for irrigation strategies to minimize the total quantity of delivered water while ensuring maximal biomass production. This can be formulated via the following constrained optimization problem:

Problem $\mathscr{P}$**:** For $(S_0, N_0) \in E$ and $\mathcal{U}(S_0)$ the set of admissible controls u for S_0, we seek

$$\inf_{u \in \mathcal{U}(S_0)} \int_0^T u(\tau) d\tau,$$

subject to the state constraint

$$\left(S^u_{0,S_0}(t), N^u_{0,S_0,N_0}(t)\right) \in E, \quad t \in [0, T]$$

where $(S^u_{0,S_0}, N^u_{0,S_0,N_0})$ denote the solution of (1)-(2) with $S(0) = S_0$, $N(0) = N_0$.

We give here sufficient conditions for the problem $\mathscr{P}$ to be feasible. We have chosen the viability approach [1] which consists in showing the the domain E. Another (equivalent) approach based on projected dynamical systems [3] could have been used.

Definition 3.1: The domain E is *viable* if for any initial condition $(t_0, S_0, N_0) \in [0, T] \times E$, there exists an admissible control $u \in \mathcal{U}(S_0)$ such that the solution of (1)-(2) with $S(t_0) = S_0$, $N(t_0) = N_0$ verifies $(S(t), N(t)) \in E$ for any $t \in [t_0, T]$.

For convenience, we define the numbers

$$C_1 = \eta_C k_1 - k_3, \quad C_2 = k_4 C_N^{in} - \eta_C k_1 k_2.$$

Proposition 3.2: The domain E is viable in the sense of Definition 3.1 if the condition

$$C_N^{in} \geq \underline{C_N^{in}} := \frac{k_2}{k_4} \max\left\{\eta_C k_1 \left(1 - K_R(S^\star)\right), k_3\right\} \tag{4}$$

and $u_{\max} \geq \underline{u_{\max}}$ with

$$\underline{u_{\max}} := \begin{cases} \max\left\{\dfrac{1}{k_2}, \dfrac{-C_1}{C_2}\right\} & \text{if } C_1 < 0 \text{ and } C_2 > 0, \\ \dfrac{1}{k_2} & \text{otherwise.} \end{cases} \tag{5}$$

are fulfilled.

Proof: At $S = S^\star$, one has from equation (1)

$$\dot{S} = k_1 \left(-\varphi - (1 - \varphi) K_R(S^\star) + k_2 u\right).$$

A necessary condition to have $\dot{S}(t) \geq 0$ is that $u(t) \in [0, u_{\max}]$ satisfies

$$u(t) \geq \frac{\varphi(t) + (1 - \varphi(t)) K_R(S^\star)}{k_2}.$$

for any possible $t \in [0, T]$, which implies the condition

$$u_{\max} \geq \max_{t \in [0,T]} \frac{\varphi(t) + (1 - \varphi(t)) K_R(S^\star)}{k_2} = \frac{1}{k_2}. \tag{6}$$

When $N = \eta_C S$, one has from equations (1)-(2)

$$\begin{aligned} \dot{N}(t) - \eta_C \dot{S}(t) = k_1 \eta_C \left(\varphi(t) + (1 - \varphi(t)) K_R(S(t))\right) \\ - k_3 \varphi(t) + (k_4 C_N^{in} - \eta_C k_1 k_2) u(t). \end{aligned}$$

Then, the condition ensuring $\dot{N} - \eta_C \dot{S} \geq 0$ is that for any $(t, S) \in [0, T] \times [S^\star, 1]$, there exists u with $u(t) \in [0, u_{\max}]$ such that

$$\begin{aligned} (k_1 \eta_C - k_3) \varphi(t) + k_1 \eta_C (1 - \varphi(t)) K_R(S(t)) \\ + (k_4 C_N^{in} - \eta_C k_1 k_2) u(t) \geq 0 \end{aligned}$$

which is implied by

$$\begin{aligned} \min\left\{k_1 \eta_C - k_3, k_1 \eta_C K_R(S^\star)\right\} \\ + \left(k_4 C_N^{in} - \eta_C k_1 k_2\right) u(t) \geq 0, \end{aligned}$$

that is,

$$\min\left\{C_1, k_1 \eta_C K_R(S^\star)\right\} + C_2 u(t) \geq 0. \tag{7}$$

At $S = 1$ one should have also $\dot{S} \leq 0$, that is $u \leq \frac{1}{k_2}$, along with

$$\min\left\{C_1, k_1 \eta_C\right\} + C_2 u(t) \geq 0. \tag{8}$$

Let us distinguish cases depending on the signs of C_1, C_2.

i. If $C_1 < 0$ and $C_2 \leq 0$, clearly condition (7) cannot be fulfilled for a non-negative u.

ii. If $C_1 < 0$ and $C_2 > 0$, one has to have

$$k_2 C_1 + C_2 \geq 0 \tag{9}$$

for condition (8) to be fulfilled with $u \leq \frac{1}{k_2}$. Condition (7) implies that $u_{\max}$ satisfies

$$u_{\max} \geq \frac{-C_1}{C_2} > 0$$

and combining with condition (6)

$$u_{\max} \geq \max\left\{\frac{1}{k_2}, \frac{-C_1}{C_2}\right\}.$$

iii. If $C_1 \geq 0$ with $C_2 \geq 0$, conditions (7) and (8) are fulfilled for any $u \geq 0$. Therefore, only condition (6) needs to be satisfied.

iv. If $C_1 \geq 0$ with $C_2 < 0$, u has to satisfy

$$u \leq \frac{\min\left\{C_1, k_1\eta_C K_R(S^\star)\right\}}{-C_2}.$$

This last condition is compatible with condition (6) if

$$k_2 \min\left(C_1, k_1\eta_C K_R(S^\star)\right) + C_2 \geq 0. \tag{10}$$

So condition (8) with $u \leq \frac{1}{k_2}$ is necessarily satisfied. Finally, $u_{\max}$ has simply to satisfy condition (6).

Note that conditions for the set E to be viable in case ii with (9), or case iii, or case iv with (10), are satisfied when the single condition (10) is satisfied. Equivalently, this condition is fulfilled if and only if (4) is verified. Finally, the upper bound of $u_{\max}$ has to fulfill condition (5) depending on the case ii, iii or iv. ■

IV. Analysis of the optimal solutions

We assume the following conditions

$$C_N^{in} \geq \underline{C_N^{in}}, \quad u_{\max} \geq \underline{u_{\max}} \tag{11}$$

are fulfilled, so that the set E is viable, accordingly to Proposition 3.2, and Problem $\mathscr{P}$ is feasible for any (S_0, N_0) in E. For simplicity, we shall consider initial conditions with $S_0 = 1$ (and thus $N_0 \geq \eta_C$) only, which are often met in practice (assuming the soil humidity S to be maximal at the beginning of the agronomic season). Let us begin by highlighting the next property:

Lemma 4.1: If there exists an admissible control such that the solution remains in E and verifies $N(T) = \eta_C S^\star$, then it is optimal.

Proof: From equation (2), solutions in the set E satisfy

$$N(t) = N_0 - k_3\int_0^t \varphi(\tau)d\tau + k_4 C_N^{in}\int_0^t u(\tau)d\tau, \quad t \in [0,T].$$

Therefore, minimizing $\int_0^T u(\tau)d\tau$ amounts to minimizing $N(T)$ among all solutions in E. In particular, if there exists an admissible control such that the solution remains in E and verifies $N(T) = \eta_C S^\star$, which is the smallest value of N in the set E, then this control is necessarily optimal. ■

The optimal control problem $\mathscr{P}$ has been solved numerically using the software Bocop [18]. We have observed that the optimal solution always *saturates* the constraints (in the sense that along an optimal solution there exists a time t such that one has $S(t) = S^\star$ or $N(t) = \eta_C S(t)$), possibly at the final time only, provided that the time horizon T is sufficiently large. This has guided us to distinguish several optimal strategies depending on which constraint is saturated. Sketches of proofs of optimality of all these strategies are given in the report [6].

A. The S-strategy

Definition 4.2: We will call *S-strategy* the following control input:

$$u_S(t,S) := \begin{cases} 0, & S(t) > S^\star, \\ u_S^{\text{sing}}(t), & S(t) = S^\star \end{cases}$$

$$\text{where} \quad u_S^{\text{sing}}(t) := \frac{1}{k_2}(\varphi(t) + (1-\varphi(t))K_R(S^\star))$$

Note that under condition (5), this control takes values in $[0, u_{\max}]$ and is thus admissible. This strategy consists of no irrigation until the humidity level reaches the threshold $S^\star$ (if possible), and then maintaining the humidity S constant at the value $S^\star$ up to the final time. Let t_S be the hitting time of the set $\{S = S^\star\}$:

$$t_S := \sup\left\{t \in [0,T] \mid S_{0,1}^0(t) > S^\star\right\}, \quad S_S^\star := S_{0,1}^0(t_S).$$

where $(S_{0,1}^0, N_{0,1,N_0}^0)$ is the solution of (1)-(2) with the zero control and $S(0) = 1$, $N(0) = N_0$.

From this time instant, using the control u_S^{sing}, one obtains for $t \in [t_S, T]$

$$\dot N - \eta_C \dot S = \dot N = \left(C_1 + \frac{C_2}{k_2}\right)(\varphi + (1-\varphi)K_R(S^\star)) + k_3(1-\varphi)K_R(S^\star)$$

$$\text{where} \quad C_1 + \frac{C_2}{k_2} = \frac{k_4}{k_2}C_N^{in} - k_3$$

is non-negative under the condition $C_N^{in} \geq \underline{C_N^{in}}$. Therefore, one has the property

$$\dot N(t) - \eta_C \dot S(t) = \dot N(t) \geq 0, \quad t \in [t_S, T]$$

with the S-strategy, which is feasible if the nitrogen constraint $N \geq \eta_C S$ is not violated before time t_S. That amounts to have

$$N_0 \geq N_0^\flat := \max\left\{\eta_C, \eta_C S_S^\star + k_3\int_0^{t_S}\varphi(t)dt\right\}.$$

Remark 4.3: From equations (1)-(2), one gets

$$\dot N - \eta_C \dot S = C_1\varphi + (1-\varphi)\,k_1\eta_C K_R(S) + C_2 u.$$

For $S > S^\star$ and $u = 0$, one then obtains the property $N(t) - \eta_C S(t) \geq 0$ for any $t \in [0, t_S]$, when the condition

$$C_S := C_1\varphi(t_S) + k_1\eta_C K_R(S_S^\star)(1-\varphi(t_S)) \geq 0 \tag{12}$$

is fulfilled. This implies that the S-strategy necessarily satisfies both constraints, and it has been shown that it is optimal in this case [6].

In the following, we focus on cases when condition (12) is not fulfilled.

B. The NS-strategy

We generalize the former S-strategy when condition (12) is not fulfilled with $N_0 < N_0^\flat$ (i.e., when the trajectory with $u = 0$ hits the nitrogen constraint before reaching $S = S^\star$).

Definition 4.4: We will call *NS-strategy* the following control input:

$$u_{NS}(t,S,N) = \begin{cases} 0, & S > \dfrac{N}{\eta_C}, \\ \max\{0, u_N^{\text{sing}}(t,S)\}, & S = \dfrac{N}{\eta_C} > S^\star, \\ u_S^{\text{sing}}(t), & S = S^\star \end{cases}$$

with $u_N^{\text{sing}}(t,S) := \dfrac{C_1\varphi(t) + k_1\eta_C K_R(S(t))(1-\varphi(t))}{-C_2}$

and u_S^{sing} is as before.

Note that when (12) is not satisfied, one has $C_1 < 0$ and condition (11) gives $C_2 \geq -k_2 C_1 > 0$. Then for any $(t,S) \in [0,T] \times [0,1]$, one has $u_N^{\text{sing}}(t,S) \leq \frac{-C_1}{C_2}$, which is upper bounded by $u_{\max}$ with condition (5). The control u_{NS} is thus admissible.

This strategy consists of no irrigation until the nitrogen constraint is saturated (i.e., $N = \eta_C S$) and then staying on the boundary of this constraint until it possibly reaches the boundary of water constraint $S = S^\star$ (from which one stays on the boundary as with the S-strategy). The trajectory effectively reaches $S = S^\star$ when $N_0 \in [N_0^\dagger, N_0^\flat]$ where $N_0^\dagger$ is defined as

$$N_0^\dagger := \inf\left\{N_0 \in [\eta_C, N_0^\flat] \mid \exists t \in [0,T], N_{0,1,N_0}^{u_{NS}}(t) = \eta_C S^\star\right\}$$

(note that one has $N_{0,1,N_0}^0(t_S) = N_{0,1,N_0}^{u_{NS}}(t_S) = \eta_C S^\star$ for any $N_0 \leq N_0^\flat$, and thus the number $N_0^\dagger$ is well defined). It has been shown that this strategy is optimal in this case [6].

Remark 4.5: The solution with $N_0 = N_0^\dagger$ and control u_{NS} reaches the point $(S^\star, \eta_C S_S^\star)$ in E at the final time T.

C. The N-strategy

When $N_0^\dagger > \eta_C$, the NS-strategy generates a trajectory that does not hit the set $\{S = S^\star\}$, and we found numerical values $N_0 \in [\eta_C, N_0^\dagger)$ for which the NS-strategy is not optimal. Surprisingly, a completely different strategy, which also stays away from the water stress, is optimal.

Definition 4.6: We will call *N-strategy* the control input:

$$u_N(t,t_c) := \begin{cases} \bar{u}, & t < t_c \\ 0 & t \geq t_c \end{cases}$$

where $\bar{u} := \dfrac{1}{k_2}$ and $t_c \in [0,T]$ is a commutation time.

This control is admissible with condition (5). This strategy consists of maintaining the humidity level equal to 1, continuously irrigating from the beginning until time t_c, when one stops the irrigation up to the final time T. Let us define

$$N_0^\sharp := \eta_C S_S^\star + k_3 \int_0^T \varphi(t)dt - k_4 C_N^{in} \bar{u}\ \underline{t}_c$$

where $\quad \underline{t}_c = \inf\{t \in [0,T] \mid S_{T,S_S^\star}^0(t) < 1\}$

($S_{T,S_S^\star}^0(t)$ with $t < T$ denotes the solution in backward time from $S(T) = S^\star$). Note that t_c is well defined as the solution S with the control $u = 0$ is decreasing in forward time and $S_{0,1}^0(T) \leq S_{0,1}^0(t_S) = S_S^\star$. When $N_0^\sharp \geq \eta_C$, this implies that for any $N_0 \in [\eta_C, N_0^\sharp]$, there exists a unique $t_c^\star \in [\underline{t}_c, T]$ such that the solution with control $u_N(\cdot, t_c^\star)$ verifies exactly $N(T) = \eta_C S(T) \geq \eta_C S^\star$. The commutation time $t_c = t_c^\star$ is such that the nitrogen constraint is saturated only at the final time, and such a solution is optimal [6].

Remark 4.7: The solution for $N_0 = N_0^\sharp$ with the control u_N and commutation time $\underline{t}_c$ reaches the point $(S_S^\star, \eta_C S_S^\star)$ in E at the final time T.

D. The singular strategies

Note that when one has $N_0^\sharp < N_0^\dagger$, initial conditions with $N_0 \in (N_0^\sharp, N_0^\dagger)$ are not covered by the former cases. One has the following property.

Proposition 4.8: Assume $N_0^\sharp < N_0^\dagger$, where $N_0^\sharp$ and $N_0^\dagger$ are defined in subsections IV-C and IV-B. Let $u^\sharp$ and $u^\dagger$ be the optimal open-loop controls given by the N- and NS-strategies for the initial conditions $N_0^\sharp$ and $N_0^\dagger$, respectively. Then for any $N_0 \in (N_0^\sharp, N_0^\dagger)$, the control

$$u(t) = \lambda u^\sharp(t) + (1-\lambda)u^\dagger(t), \quad \lambda = \frac{N_0^\dagger - N_0}{N_0^\dagger - N_0^\sharp}, \quad t \in [0,T]$$

is optimal.

Proof: Let $(S^\sharp, N^\sharp)$ and $(S^\dagger, N^\dagger)$ be the solutions given by the N- and NS-strategies for the initial condition $N_0^\sharp$ and $N_0^\dagger$ respectively. From equations (1), (2) one has

$$\begin{aligned} &\lambda\dot{S}^\sharp(t) + (1-\lambda)\dot{S}^\dagger(t) = \\ &\qquad -k_1\varphi(t) - k_1\left(\lambda K_R(S^\sharp(t)) + (1-\lambda)K_R(S^\dagger(t))\right) \\ &\qquad +k_1k_2\left(\lambda u^\sharp(t) + (1-\lambda)u^\dagger(t)\right), \\ &\lambda\dot{N}^\sharp(t) + (1-\lambda)\dot{N}^\dagger(t) = \\ &\qquad -k_3\varphi(t) + k_4 C_N^{in}\left(\lambda u^\sharp(t) + (1-\lambda)u^\dagger(t)\right) \end{aligned}$$

Note that one has $\lambda K_R(S^\sharp) + (1-\lambda)K_R(S^\dagger) = K_R(\lambda S^\sharp + (1-\lambda)S^\dagger)$ as K_R is an affine function in the domain E. Therefore

$$S(t) = \lambda S^\sharp(t) + (1-\lambda)S^\dagger(t), \ N(t) = \lambda N^\sharp(t) + (1-\lambda)N^\dagger(t)$$

defined for any $t \in [0,T]$ is the solution of (1), (2) for the control u and the initial condition $(1, N_0)$. As solutions $(S^\sharp, N^\sharp)$ and $(S^\dagger, N^\dagger)$ remain in the convex set E, we deduce that the solution (S,N) also, and the control u is admissible. One has $N^\sharp(T) = N^\dagger(T) = \eta_C S^\star$ (see Remarks 4.5 and 4.7). So we get $N(T) = \eta_C S^\star$. According to Lemma 4.1, we conclude that u is optimal. ■

Such a control is singular in the sense that it is not composed of arcs with extreme values of the control set (0 or $u_{\max}$), nor arcs on the boundary of the constraint set E.

Remark 4.9: Indeed, it can be shown that for $N_0 \in (N_0^\sharp, N_0^\dagger)$, there exists an infinity of singular controls such that the trajectory stays in E and hits the corner point $(S_S^\star, \eta_C S_S^\star)$ of E at the final time (with Bocop, we obtain an optimal control different to (4.8)). According to the remark to Lemma 4.1, all these controls are optimal.

V. Numerical illustrations

The former analysis has revealed the sub-domains on which the various strategies are optimal. This section gives an illustration with realist values of the parameters. We have considered a concave function

$$\varphi(t) = \frac{t(1+\alpha)}{t+\alpha}, \quad \alpha > 0$$

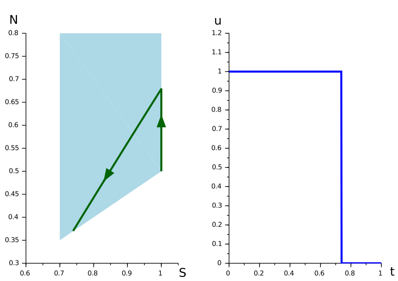
and chosen a plausible set of values of model parameters inspired by the literature. Then, we have computed the lower

k_1	k_2	k_3	k_4	$S^\star$	S_w	S_h	T	α	η_C
1	1	1.4	1	0.7	0.4	0.2	1	0.07	0.5

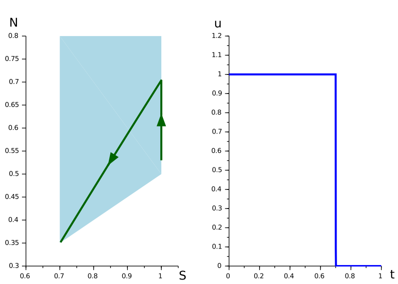
bounds (4), (5) on the operating parameters C_N^{in}, $u_{\max}$ for the set E to be viable: $\underline{C_N^{in}} = 1.4$, $\underline{u_{\max}} = 1$ and chosen the following values of operating parameters $C_N^{in} = 1.45$, $u_{\max} = 2$, which guarantee the set E to be viable (Proposition 3.2). Then, we have determined numerically the thresholds defined in Sections IV-A, IV-C, IV-C. Examples

$N_0^\sharp$	$N_0^\dagger$	$N_0^\flat$
0.547681	0.579387	0.641166

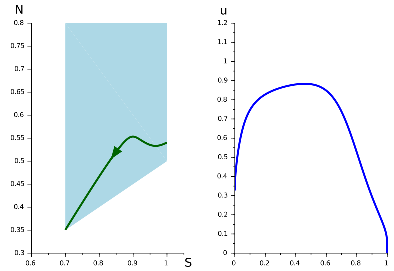
of trajectories and their controls using different strategies presented previously are given in the figures below:



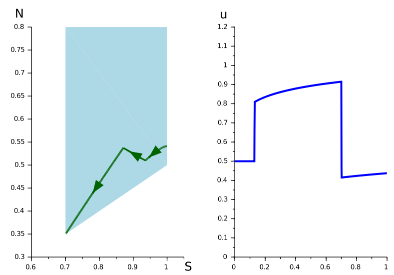
Optimal solution for $N_0 = \eta_C$



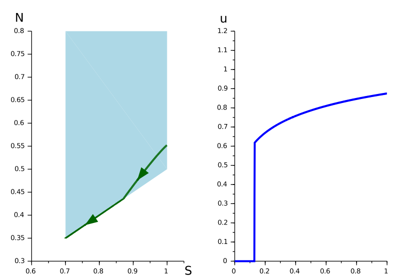
Optimal solution for $N_0 = N_0^\sharp$



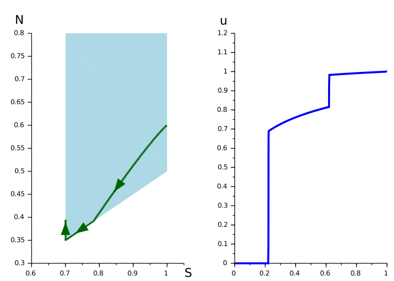
Optimal solution for $N_0 \in (N_0^\sharp, N_0^\dagger)$ found by Bocop



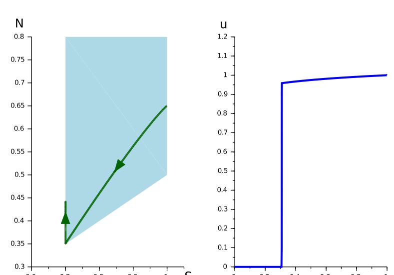
Optimal solution for $N_0 \in (N_0^\sharp, N_0^\dagger)$ with the control given by Proposition 4.8



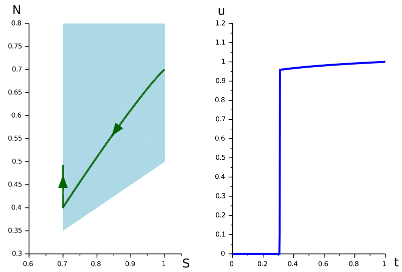
Optimal solution for $N_0 = N_0^\dagger$



Optimal solution for $N_0 \in (N_0^\dagger, N_0^\flat)$



Optimal solution for $N_0 = N_0^\flat$



Optimal solution for $N_0 > N_0^\flat$

Let us make some comments.

- When the initial quantity of nitrogen is high ($N_0 > N_0^\flat$), it is not surprising that the nitrogen constraint is not saturated (the S-strategy coincides with the optimal one already found for the model with no nitrogen stress [2]).

- For a lower value of initial nitrogen $N_0 \in (N_0^\dagger, N_0^\flat)$, the lack of irrigation conducts the system to face the nitrogen stress before the hydric one, and the NS-strategy is optimal.

- A surprising feature occurs when keeping the system at the edge of the nitrogen stress does not allow to reach the humidity threshold (the time horizon been reached before) for initial nitrogen content $N_0 < N_0^\sharp$. The NS-strategy is not optimal and the N-strategy is fundamentally different: it requires anticipating future needs since the beginning.

- Another non-intuitive feature occurs for $N_0 \in (N_0^\sharp, N_0^\dagger)$ for which the "corner" point $(S^\star, \eta_C S^\star)$ is reached exactly at the final time. The optimal control has to irrigate all the time with particular profiles (that are non-unique), avoiding the boundaries of E, unlike the other strategies. Note that the numerical software provides a continuous control while we proposed another optimal one that is discontinuous. In practice, the continuous one might be easier to apply.

This study reveals that the minimal residual nitrogen content in soil (i.e., at the final time) is obtained when the initial nitrogen content belongs to a particular interval of values. Therefore, having a low initial nitrogen content in the soil could paradoxically lead to more significant residual contents. This is explained by the larger water consumption needed to maintain maximal production.

VI. Conclusion

Considering the problem of minimal irrigation under maximal biomass production for a simplified crop model, we have given conditions for the optimal control with constraints to be feasible and depicted the different structures of the optimal strategy depending on the initial nitrogen content. These strategies can be implemented as simple feedback measuring online: the humidity level S if the limit of water stress is met before the harvesting time, or the nitrogen concentration N/S if the limit of the nitrogen stress is met before the harvesting time. In the other cases, the optimal control is determined as an open-loop control that can be computed using only the model's parameters. We have also exhibited a particular interval of values of the initial nitrogen content that conducts to the smallest residual nitrogen in the soil, which could serve as a theoretical target for the practitioners.

The perspectives of this preliminary work are to study nonviable cases and to characterize the viability kernel, that is, the set of initial conditions for which the optimization problem is feasible, and then to determine the optimal policy to be compared with the one found here. Another extension involves considering the possibility of nitrogen leaching, and so a modification of the considered model.

Acknowledgments

The authors are grateful to INRAE and INRIA Institutes for the PhD grant of the first author, as well as the support of the INRAE MATHNUM 'REUSE' and euro-mediterranean TREASURE networks.

References

[1] J.-P. Aubin. *Viability Theory*. Birkauser, 2009.
[2] K. Boumaza, N. Kalboussi, A. Rapaport, S. Roux, and C. Sinfort. Optimal control of a crop irrigation model under water scarcity. *Optimal Control Applications and Methods*, 42(6):1612–1631, 2021.
[3] B. Brogliato, A. Daniilidis, C. Lemaréchal, and V. Acary. On the equivalence between complementarity systems, projected systems and differential inclusions. *Systems & Control Letters*, 55(1):45–51, 2006.
[4] B. Cheviron, R.W. Vervoort, R. Albasha, R. Dairon, C. Le Priol, and J.-C. Mailhol. A framework to use crop models for multi-objective constrained optimization of irrigation strategies. *Environmental Modelling & Software*, 86:145–157, 2016.
[5] M.G. Dadjo, D. Efimov, J Harmand, A. Rapaport, and R Ushirobira. *An adaptive observer for time-varying nonlinear systems - application to a crop irrigation model*. 62nd IEEE Conference on Decision and Control, Dec 2023, Singapore, Singapore, 2023.
[6] M.G. Dadjo, A. Rapaport, J. Harmand, R. Ushirobira, and D. Efimov. Water-minimizing strategies under viability constraint for a crop fertirrigation model. Technical report, https://hal.inrae.fr/hal-04267286, October 2023.
[7] A. Haddon, A. Rapaport, S. Roux, and J. Harmand. Model based optimization of fertilization with treated wastewater reuse. *Advances in Water Resources*, 181(104561), 2023.
[8] N. Kalboussi, S. Roux, K. Boumaza, C. Sinfort, and A. Rapaport. About modeling and control strategies for scheduling crop irrigation. *IFAC-PapersOnLine*, 52(23):43–48, 2019.
[9] M. Kefi, N. Kalboussi, A. Rapaport, J. Harmand, and H. Gabtni. Model-based approach for treated wastewater reuse strategies focusing on water and its nitrogen content: A case study for olive growing farms in peri-urban areas of sousse, tunisia. *Water*, 15(4):755, 2023.
[10] S. Lopes, F. Fontes, R. Pereira, M.-R. de Pinho, and A. Goncalves. Optimal control applied to an irrigation planning problem. *Mathematical Problems in Engineering*, 5076879, 2016.
[11] O. Neto, A. Haddon, F. Aichouche, J. Harmand, M. Mulas, and F. Corona. Predictive control of activated sludge plants to supply nitrogen for optimal crop growth. *IFAC-PapersOnLine*, 54(3):200–205, 2021.
[12] N. Pelak, R. Revelli, and A. Porporato. A dynamical systems framework for crop models: Toward optimal fertilization and irrigation strategies under climatic variability. *Ecological Modelling*, 365:80–92, 2017.
[13] N. Schütze and G. Schmitz. Occasion: new planning tool for optimal climate change adaption strategies in irrigation. *Journal of Irrigation and Drainage Engineering*, 136(12):836–846, 2010.
[14] N. Schüze, M. de Paly, and U. Shamir. Novel simulation-based algorithms for optimal open-loop and closed-loop scheduling of deficit irrigation systems. *Journal of Hydroinformatics*, 14(1):136–151, 2011.
[15] U. Shani, Y. Tsur, and A. Zemel. Optimal dynamic irrigation schemes. *Optimal Control Applications and Methods*, 25(2):91–106, 2004.
[16] P. Steduto, T. Hsiao, D. Raes, and E. Fereres. AquaCrop - the FAO crop model to simulate yield response to water: I. concepts and underlying principles. *Agronomy Journal*, 101(3):426–437, 2009.
[17] B.J. Sunantara and J. Ramírez. Optimal stochastic multicrop seasonal and intraseasonal irrigation control. *Journal of Water Resources Planning and Management*, 123(1):39–48, 1997.
[18] Inria Saclay Team Commands. Bocop: an open source toolbox for optimal control. http://bocop.org, 2017.