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Higher Order Iterative Learning Control of Discrete Linear Systems with Uncertain Parameters

Pavel Pakshin, Julia Emelianova, Eric Rogers and Krzysztof Galkowski

Abstract— **Iterative learning control emerged from the problem area of increasing the accuracy of finite-duration repetitive operations performed by robots, often termed trials. The ILC laws use past trial information to adjust the current trial's control signal. Most often, only data from the previous trial is used. A higher-order ILC law uses information from several previous trials. Recently, interest in these laws has increased in the literature with, in particular, robotic additive manufacturing problems. This paper develops a new higher-order ILC design for discrete linear uncertain systems that makes greater use of information generated over previous trials. An example using a model developed from measured frequency response data from a laboratory testbed illustrates the new design.**

I. INTRODUCTION

After the appearance of the first results, widely credited to [1], ILC quickly established itself as a research theme in theory/control design and applications. One starting point for the early literature is the survey papers [2], [3]. Currently, ILC laws are effectively used in additive manufacturing, in particular in high-precision multilayer laser deposition installations [4], [5] robotic-based rehabilitation of patients who have suffered a stroke [6], [7], in ventricular-support devices [8], and in numerous other applications. Some of these applications have seen supporting experimental and clinical trial results reported.

Speed of error convergence from trial to trial is a critical feature of an ILC design, i.e., given a reference trajectory, the error on each trial is the difference between the supplied reference trajectory and the output on this trial. Hence, the speed of convergence of the resulting error sequence is critical. Consequently, the use of optimization methods is well-established in this area. A very large number of the currently available ILC laws only make use of previous trial data to generate the control signal to compute the control input for the current trial. At the cost of storage, data from all previously completed trials is available. Such an ILC law is generally termed higher order, where 'higher order' means using data from a finite number, say $d > 1$, of previous trials to update the control law for the subsequent trial. If $d = 1$, the law is termed first order in this paper.

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Higher order ILC laws have been considered in previous research [9]–[18]. In this previous work, it has been demonstrated [10], [11] that higher-order ILC can provide a higher trial-to-trial error convergence rate than a first-order alternative, see also [12]–[16]. Moreover, in [12]–[14], it is argued that the acceleration effect of higher-order laws is achieved due to the effects of learning during several previous trials.

Other properties of higher-order ILC have also been studied in the literature, where [17] states that the essential motivation for using higher-order ILC is to reduce the impact of interference and noise. In [18], the optimality of the control system is considered in the sense of minimizing the trace of the covariance matrix of control errors in the presence of uncorrelated random disturbances. It is shown that a higher-order ILC does not reduce the minimum value compared to a first-order ILC. Thus, the conclusions of [17] and [18] contradict each other.

The results reported in [17] do not provide sufficiently complete evidence, and in [18], the ILC algorithm uses a discrete analog of the derivative of a raw random signal. It could be very problematic, especially in a physical implementation, due to numerical conditioning issues. This previous work did not consider the convergence rate of learning errors. Moreover, there has recently been active interest in developing and applying higher-order ILC algorithms in additive manufacturing problems [19], motivated by the features of the application area.

This paper develops a new method for higher-order ILC design for linear discrete-time uncertain systems. This law uses more information from a finite number of previous trials. Consider a particular sample point along the trial denoted by p,, then information from sample point $p \pm \lambda$ can be used, where $\lambda > 0$ is an integer. Such a control law is termed non-causal in the literature.

Control law design uses the stability theory for discrete linear repetitive processes where conditions for trial-to-trial error convergence of the learning error are developed in terms of the divergence properties of a vector Lyapunov function. This analysis leads to a linear matrix inequality (LMI) based design. The new design's performance is demonstrated using a model of one axis of a gantry robot (executing a 'pick and place' operation that is an example where ILC can be used to advantage) where the model used has been constructed from measured frequency response data. The paper concludes by discussing the possibilities of obtaining a result that quantifies the convergence of properties of high-order ILC laws relative to the first-order case.

II. Problem Specification

The systems considered are described by the following state space model in the ILC setting on trial k

$$\begin{aligned} x_k(p+1) &= A(\delta(p))x_k(p) + B(\delta(p))u_k(p), \\ y_k(p) &= Cx_k(p), \quad 0 \le p \le N-1, k = 1, \dots \end{aligned} \tag{1}$$

where $x_k(p) \in \mathbb{R}^n$ is the state vector, $u_k(p) \in \mathbb{R}$ and $y_k(p) \in \mathbb{R}$ are the input and trial profile, respectively. No loss of generality arises from assuming that the boundary conditions are $x_k(0) = 0$ and $y_0(p) = f(p)$, where $f(p)$ is known and specified a priori. The uncertainty is represented as

$$A(\delta(p)) = A + \sum_{j=1}^{l} \delta_j(p)A_j, \; B(\delta(p)) = B + \sum_{j=1}^{l} \delta_j(p)B_j, \tag{2}$$

where A and B are matrices of the nominal model; A_j and B_j, $(j = 1, 2, \dots, l)$ are known constant matrices of compatible dimensions; $\delta_j(p)$ are uncertain parameters. For brevity, the dependence of δ on p will be omitted from this point onwards.

The set of uncertain parameters is given by

$$\mathbf{D} = \{\delta = [\delta_1 \dots \delta_l]^{\mathrm{T}}, \; \delta_j \in [\underline{\delta}_j, \; \overline{\delta}_j], \; j = 1, 2, \dots, l\}$$

with the finite vertex set of 2^l elements:

$$\mathbf{D}_v = \{\delta = [\delta_1 \dots \delta_l]^{\mathrm{T}}, \; \delta_j \in \{\underline{\delta}_j, \; \overline{\delta}_j\} \; j = 1, 2, \dots, l\},$$

and the control law has the form

$$u_k(p) = \sum_{i=0}^{d} \tau_i v_{k-i}(p), \tag{3}$$

$$v_{k+1}(p) = v_k(p) + \Delta v_{k+1}(p), \; k = 1, 2, \dots \tag{4}$$

where $v_k = 0$, if $k \in [-d, 0]$, d is the number of previous trials, information from which is used in the current trial, Δv_{k+1} is update or correction taw for trial $k+1$, τ_i, $i \in [0, d]$ are weighting coefficients.

Assume that $CB(\delta) \neq 0$, $\delta \in \mathbf{D}$ and let $y_{ref}(p)$, $0 \le p \le N-1$, denote the specified reference trajectory for (1). Then the error on trial k is

$$e_k(p) = y_{ref}(p) - y_k(p). \tag{5}$$

The ILC design problem is to construct a sequence of trial inputs $\{u_k\}$ such that for $0 \le p \le N-1$, the following conditions on the error $e_k(p)$ and input $u_k(p)$ hold

$$|e_k(p)| \le \kappa \varrho^k, \; \kappa > 0, \; 0 < \varrho < 1, \tag{6}$$

$$\lim_{k \to \infty} |u_k(p)| = |u_\infty(p)| < \infty. \tag{7}$$

(In some parts of the literature, $u_\infty(p)$ is termed the learned control.) Moreover, the case when the first Markov parameter is zero can be considered as in, e.g., [20]. Similarly, the extension to multiple-input multiple-output examples is straightforward.

The analysis and design of this paper use an incremental model of the controlled dynamics, which results in a repetitive process description of the dynamics. Introduce the scalar variables $\breve{x}_{k,1}(p) = v_k(p)$, $\breve{x}_{k,2}(p) = v_{k-1}(p), \dots, \breve{x}_{k,d}(p) = v_{k-d+1}(p)$, $\breve{x}_{k,d+1}(p) = v_{k-d}(p)$ and vector $\breve{x}_k = [\breve{x}_{k,1} \dots \breve{x}_{k,d+1}]^{\mathrm{T}}$. Then by construction

$$\breve{x}_k(p) = A_d \breve{x}_{k-1}(p) + B_d v_k(p), \tag{8}$$

where

$$A_d = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, B_d = [1 \; 0 \; 0 \dots 0]^{\mathrm{T}}. \tag{9}$$

Using (8) the first equation in (1) can be written as

$$x_k(p+1) = A(\delta)x_k(p) + B(\delta)\theta \breve{x}_k(p), \tag{10}$$

where $\theta = [\tau_0 \; \tau_1 \dots \tau_d]$. Introduce, for the design purposes only,

$$\begin{aligned} \eta_k(p) &= x_k(p) - x_{k-1}(p), \\ \breve{\eta}_k(p) &= \breve{x}_k(p) - \breve{x}_{k-1}(p). \end{aligned} \tag{11}$$

(these vectors are the difference between the state vectors and trial profiles on two successive trials, and hence are termed incremental). Hence from (8) and (11) it follows that

$$\breve{\eta}_k(p) = A_d \breve{\eta}_{k-1}(p) + B_d \Delta v_k(p), \tag{12}$$

where $\Delta v_k(p) = v_k(p) - v_{k-1}(p)$.

Making use of (11), the state dynamics on any trial are described by

$$\begin{aligned} \eta_k(p+1) &= A(\delta)\eta_k(p) + B(\delta)\theta A_d \breve{\eta}_{k-1}(p) \\ &+ B(\delta)\theta^{\mathrm{T}} B_d \Delta v_k(p). \end{aligned} \tag{13}$$

Also, using (5) with $y_k(p) = Cx_k(p)$, the dynamics in terms of the incremental variables can be written in the form

$$\begin{aligned} \eta_k(p+1) &= A(\delta)\eta_k(p) + B(\delta)\theta A_d \breve{\eta}_{k-1}(p) \\ &+ B(\delta)\theta B_d \Delta v_k(p), \\ \breve{\eta}_k(p) &= A_d \breve{\eta}_{k-1}(p) + B_d \Delta v_k(p), \qquad (14) \\ \bar{e}_k(p) &= -CA(\delta)\eta_k(p) - CB(\delta)\theta A_d \breve{\eta}_{k-1}(p) \\ &+ \bar{e}_{k-1}(p) - CB(\delta)\theta B_d \Delta v_k(p), \qquad (15) \end{aligned}$$

where $\bar{e}_k(p) = e_k(p+1)$.

Consider the case when

$$\Delta v_k(p) = K_1 \eta_k(p) + K_2 \bar{e}_{k-1}(p), \tag{16}$$

where K_1 and K_2 are matrices of compatible dimensions to be designed. Then, using (14) and (16), the controlled dynamics are described by the state space model

$$\begin{aligned}\eta_k(p+1) &= (A(\delta)+B(\delta)\theta B_d K_1)\eta_k(p)\\ &+ B(\delta)\theta Ad\check{\eta}_{k-1}(p)+B(\delta)\theta B_d K_2\bar{e}_{k-1}(p),\\ \check{\eta}_k(p) &= B_d K_1\eta_k(p)+A_d\check{\eta}_{k-1}(p)\\ &+ B_d K_2\bar{e}_{k-1}(p),\\ \bar{e}_k(p) &= -C(A(\delta)+B(\delta)\theta B_d K_1)\eta_k(p)\\ &- CB(\delta)\theta A_d\check{\eta}_{k-1}(p)\\ &+ (1-CB(\delta)\theta B_d K_2)\bar{e}_{k-1}(p). \end{aligned} \quad (17)$$

The model (17) describes the dynamics of discrete linear repetitive processes [21]. Such processes are a distinct class of 2D systems where in the last model $\eta_k(p)$ is the state vector and $\bar{e}_k(p)$ is the trial profile. This paper uses the stability of these processes based on vector Lyapunov functions [22]. to obtain conditions for trial-to-trial error convergence of the controlled ILC dynamics and control law design to ensure this property.

III. Convergence analysis and control law design

Introduce the vector $\epsilon_k(p) = \left[\begin{array}{cc}\check{\eta}_k^{\mathrm{T}}(p) & \bar{e}_{k-1}(p)\end{array}\right]^{\mathrm{T}}$ and define the vector Lyapunov function on the trajectories of the system (17) as

$$V(\eta_k(p),\epsilon_k(p)) = \left[\begin{array}{c} V_1(\eta_k(p))\\ V_2(\epsilon_k(p))\end{array}\right], \quad (18)$$

where $V_1(\eta_k(p)) > 0, \ \eta_k(p) \neq 0, \ V_2(\epsilon_k(p)) > 0, \ \epsilon_k(p) \neq 0, \ V_1(0) = 0, \ V_2(0) = 0$. Define on the trajectories of the system (17) the following discrete counterpart of the divergence operator

$$\begin{aligned}\mathcal{D}V(\eta_k(p),\epsilon_k(p)) &= V_1(\eta_k(p+1))-V_1(\eta_k(p))\\ &+ V_2(\epsilon_{k+1}(p))-V_2(\epsilon_k(p)).\end{aligned} \quad (19)$$

(For brevity (19) will be termed the divergence from this point onwards.) Based on Theorem 1 in [22], the following result on the trial-to-trial error convergence for the controlled dynamics can be established (where $||\cdot||$ denotes the norm on the underlying function space).

Theorem 1: Suppose that there exist a vector Lyapunov function of the form (18) and positive scalars c_1, c_2 and c_3 such that on the trajectories of the system (17):

$$c_1||\eta_k(p)||^2 \leq V_1(\eta_k(p)) \leq c_2||\eta_k(p)||^2, \quad (20)$$

$$c_1||\epsilon_k(p)||^2 \leq V_2(\epsilon_k(p)) \leq c_2||\epsilon_k(p)||^2, \quad (21)$$

$$\mathcal{D}V(\eta_{k+1}(p),\epsilon_k(p)) \leq -c_3(||\eta_{k+1}(p)||^2+||\epsilon_k(p)||^2). \quad (22)$$

Then the ILC law given by (3), and (16) guarantees that the convergence conditions (6), and (7) hold for dynamics described by (17).

Proof: Using Theorem 1 from [22] in the case when (20) – (22) hold there exist $\alpha > 0$ and $0 < \lambda < 1$ such that

$$||\eta_k(p)||^2 + ||\epsilon_k(p)||^2 \leq \alpha\lambda^{k+p} \leq \alpha\lambda^k. \quad (23)$$

Therefore, $||\eta_k(p)|| \leq \varkappa\varrho^k$, where $\varkappa = \sqrt{\alpha}, \ \varrho = \sqrt{\lambda}$ and it follows that $|e_k(p)| \leq \varkappa\varrho^k$, and hence (6) holds. Next, since $\Delta v_k(p)$ is defined by the relation (16), using (4) and (23) gives

$$|v_{k+1}(p) \leq |v_k(p)| + \alpha_0\lambda^{\frac{k+p+1}{2}}, \quad (24)$$

where $\alpha_0 = \sqrt{2\alpha\max\{||K_1||, \ |K_2|\}}$. From (24) it follows that

$$|v_k(p)| \leq |v_0(p)| + \alpha_0\lambda^{\frac{p+1}{2}}\sum_{h=0}^{k-1}\lambda^{\frac{h}{2}}.$$

On the right side of the last inequality, there is a geometric progression converging as $k\to\infty$, hence the limit on the left side $|v_\infty(p)|$ for $k\to\infty$ exists and therefore

$$|v_\infty(p)| \leq |v_0(p)| + \frac{\alpha_0\lambda^{\frac{p+1}{2}}}{1-\lambda^{\frac{1}{2}}}.$$

Given (3), it follows that (7) holds. ■

Remark 1: The results in this paper are established using vector Lyapunov functions in explicit form, i.e., $V = [V_1 \ V_2]^T, V_1 > 0, V_2 > 0$ and (19). An alternative would be to use the scalar form $V_c = V_1 + V_2$ with $V_1 > 0$ and $V_2 > 0$, but the increment of this function along trajectories of (17) would have to be calculated in addition to the right-hand side (19). However, the divergence gives a more straightforward physical interpretation, which, similar to the usual Lyapunov function, can be interpreted as the generalized energy of the system.

Remark 2: Due to the finite trial length, trial-to-trial error convergence in ILC can occur even if, for linear dynamics, the state matrix is unstable. In such cases, one option is to design a stabilizing feedback control action and then apply the ILC law to the resulting dynamics, resulting in a two-step design procedure. Using the repetitive process/2D systems setting leads to a control law regulating the trials' dynamics. Also, using vector Lyapunov functions naturally extends to design for nonlinear dynamics.

Denote $\xi_k(p) = [\eta_k^{\mathrm{T}}(p) \ \check{\eta}_{k-1}^{\mathrm{T}}(p) \ e_{k-1}(p)]^{\mathrm{T}}$ and introduce the following matrices of compatible dimensions

$$\begin{aligned}\bar{A}(\delta) &= \left[\begin{array}{ccc} A(\delta) & B(\delta)\theta A_d & 0\\ 0 & A_d & 0\\ -CA(\delta) & -CB(\delta)\theta A_d & 1\end{array}\right],\\ \bar{B}(\delta) &= \left[\begin{array}{c} B(\delta)\theta B_d\\ B_d\\ -CB(\delta)\theta B_d\end{array}\right].\end{aligned}$$

Choose the entries of (18) as the quadratic forms

$$\begin{aligned}V_1(\eta_k(p)) &= \eta_k^{\mathrm{T}}(p)P_1\eta_k(p),\\ V_2(\epsilon_k(p)) &= \epsilon_k^T(p)P_2\epsilon_k(p),\end{aligned} \quad (25)$$

where $P_1 \succ 0$ and $P_2 \succ 0$. Calculating divergence along the trajectories of (17) gives

$$\begin{aligned}\mathcal{D}V(\eta_{k+1}(p),\epsilon_k(p)) &= \xi^{\mathrm{T}}[(\bar{A}(\delta)+\bar{B}(\delta)\bar{K}H)^{\mathrm{T}}P(\bar{A}(\delta)\\ &+ \bar{B}(\delta)\bar{K}H)-P]\xi, \ \delta\in\mathbf{D},\end{aligned}$$

where $P = \mathrm{diag}[P_1 \ \ P_2]$, $\bar{K} = [K_1 \ \ K_2]$, $H = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Assume also that the matrices P and K satisfy the bilinear matrix inequality

$$(\bar{A}(\delta) + \bar{B}(\delta)\bar{K}H)^{\mathrm{T}} P (\bar{A}(\delta) + \bar{B}(\delta)\bar{K}H) \\ - P + Q + K^{\mathrm{T}} R K \preceq 0, \ \delta \in \mathbf{D}, \quad (26)$$

where $Q \succ 0$ and $R \succ 0$ are weighting matrices that can (as one option) be chosen based on linear quadratic regulator theory recommendations. Then if (26) is solvable relative to $P \succ 0$ and K, conditions (20), (21) and (22) of Theorem 1 hold for the controlled dynamics. Using Schur's complement formulas and using the affine type of the inequality describing the uncertainties, (26) is reduced to the following set of LMIs:

$$\begin{bmatrix} X & (\bar{A}(\delta)X + \bar{B}(\delta)YH)^T & X & (YH)^T \\ \bar{A}(\delta)X + \bar{B}(\delta) & X & 0 & 0 \\ X & 0 & Q^{-1} & 0 \\ YH & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0, \delta \in \mathbf{D}_v, \quad (27)$$

where $X = \mathrm{diag}[X_1 \ X_2]$, $Y = KZ$ and Z is the solution of

$$HX = ZH. \quad (28)$$

Hence, the following theorem is established.

Theorem 2: Assume that for some matrix $Q \succ 0$ and scalar $R > 0$ the set of linear matrix inequalities (27) and (28) is solvable with respect to matrices $X = \mathrm{diag}[X_1 \ X_2] \succ 0$, Y and Z. Then conditions (20), (21) and (22) of Theorem 1 hold and the ILC law (3), (4), (16) ensures that the convergence conditions (6) and (7) hold for the controlled dynamics. The entries in the vector Lyapunov function (18) are given by (25) with $P_1 = X_1^{-1}$ and $P_2 = X_2^{-1}$. Moreover, the matrices in (16) are given by

$$K = [K_1 \ K_2] = YZ^{-1}. \quad (29)$$

Remark 3: When choosing entries in (25) as quadratic forms, $c_1 = \min\{\lambda_{\min}(P_1), \ \lambda_{\min}(P_2)\}, c_2 = \max\{\lambda_{\max}(P_1), \ \lambda_{\max}(P_2)\}$ and $c_3 = \lambda_{\min}(Q + K^{\mathrm{T}} R K)$, where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$, denote, respectively, the minimum and maximum eigenvalues of a matrix. It follows from [22] (Theorem 1) that in (6) κ depends on c_1 and c_2 and ϱ depends on c_3. Hence, selecting Q and R based on the recommendations of the LQR theory affects the rate of convergence of the learning error.

IV. Numerical Case Study

Consider the controlled movement of a manipulator along a horizontal axis orthogonal to the movement direction of the conveyor belt of a multi-axis portal robot test facility, where in [23] frequency response measurements have been used to develop and verify the following transfer function model as a starting point for control design

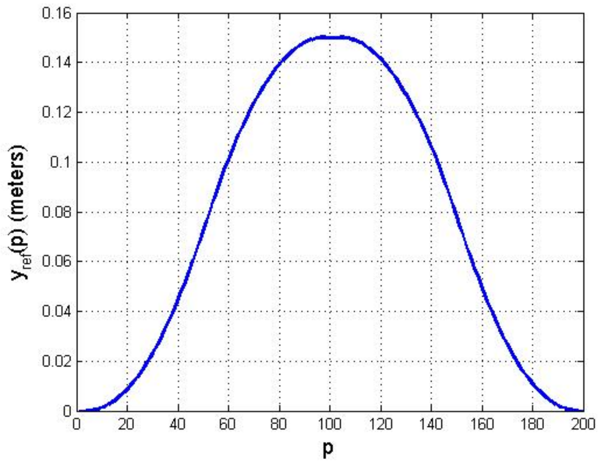

Fig. 1. The reference trajectory.

$$G(s) = \frac{23.736(s + 661.2)}{s(s^2 + 426.7s + 1.744 \cdot 10^5)}. \quad (30)$$

Moreover, Fig. 1 shows the reference trajectory for a trial length of 2 secs.

A sampling period of 0.01 secs was used to obtain a discrete state space model in [23]. Nominal values of the matrix A and vectors B and C for this model can be obtained using the standard MATLAB functions ss and c2d. The main focus in this section will be on the trial-to-trial error convergence rate, so only the gain from input to output is considered an uncertain parameter. This type of uncertainty is reflected only in the form of matrix $B(\delta(p))$ in the state space model. Namely

$$B(\delta(p)) = g(\delta(p))B_0,$$

where B_0 is the nominal matrix from the discrete approximation of (30) and $\underline{g} \leq g(\delta(p)) \leq \overline{g}$. Moreover, the particular case of $\underline{g} = 0.5$ and $\overline{g} = 2$ is considered in this section. For application, the control law of (3) and (4) is

$$\begin{aligned} \Delta v_k(p) &= K_1(x_k(p) - x_{k-1}(p)) \\ &\quad + K_2(y_{ref}(p+1) - Cx_{k-1}(p+1)), \\ v_k(p) &= \begin{cases} 0 & \text{if } k \in [-d \ 0], \\ v_{k-1} + \Delta v_k(p) & \text{if } k \geq 1, \end{cases} \\ u_k(p) &= \sum_{i=0}^{d} \tau_i v_{k-i}(p), \end{aligned} \quad (31)$$

where the second term on the right-hand side of this law uses the information at sample $p+1$ on the previous trial at sample p on the current trial. For $d = 0$ and $\tau_0 = 1$, these relations give the usual first-order ILC law; when $d = 1$, they give a second-order ILC algorithm, and so on.

As a first study into the relative performance of this higher-order law against first order, the weights τ_i are chosen on the premise that the previous trial has the most significant contribution to the control input for the subsequent trial. The others used in the control law follow the same principle, and starting with $\tau_0 = 1$ is feasible. The weights for the other previous trial signals used in the control law are then chosen as a decreasing sequence of fractions of τ_0.

A performance measure for the design is the root mean square error, denoted by $E(k)$, for each trial

$$\mathrm{E}(k) = \sqrt{\frac{1}{N}\sum_{p=0}^{N-1} |e_k(p)|^2}. \tag{32}$$

plotted against the trial number k, which should decrease monotonically from trial to trial. A practically-minded criterion is to count the number of trials until the error reduces to a specified percentage of the initial value.

The results in the remainder of this section are for the case when

$$Q = \mathrm{diag}[1\ 1\ 1\ 10\ 10\ 10\ 5\cdot 10^5],\ R = 10^{-3}.$$

This choice provides a tenfold reduction in $E(k)$ for the nominal system when applying first-order ILC in 10 trials. Consider also the case when $d = 2$ (a third-order law). Then, the special case when $\tau_0 = 1$ and $\tau_1 = \tau_2 = 0$ gives the first order law for which the matrices are

$$K_1 = [-15.5\ -12.7\ -5174.1],\ K_2 = 167.5.$$

Consider also the case when $\tau_1 = 0.8$ and $\tau_2 = 0$, for which

$$K_1 = [-6.8\ -5.3.\ -2186.0],\ K_2 = 174.1.$$

Fig.2 shows a comparison of these two designs. The increase in the trial-to-trial error convergence for the higher-order law is evident.

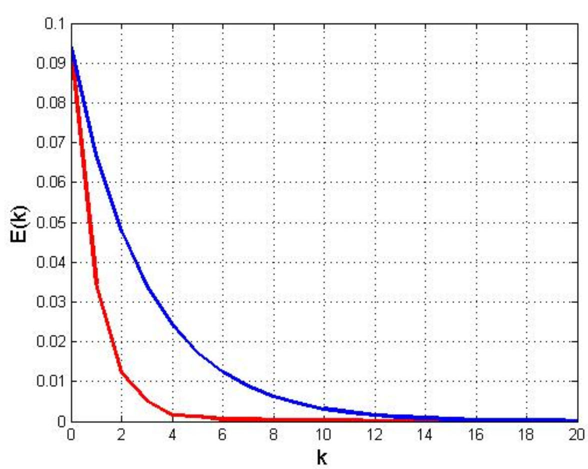

Fig. 2. $\mathrm{E}(k)$ progression with nominal gain ($g = 1$) for a first order (blue line) and a second order law (with $\tau_1 = 0.8$) (red line).

Calculations and simulations have concluded that for $\tau_0 = 1$, the third-order ILC law has good robustness properties. Fig. 3 shows that this law gives almost the same acceleration in the convergence of the learning error if the gain $g(\delta(p))$ takes a value on the lower bound of uncertainty. A typical progression of the control input is shown in Fig. 4

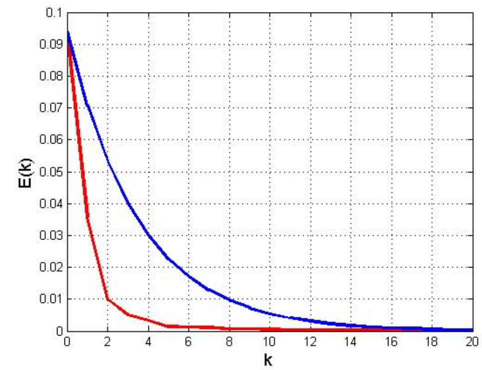

Fig. 3. $\mathrm{E}(k)$ progression with nominal gain ($g = 1$) for a first order law (blue line) and a third order law with gain on lower bound ($g = 0.5$) and $\tau_1 = 0.8$ $\tau_2 = 0.4$ (red line).

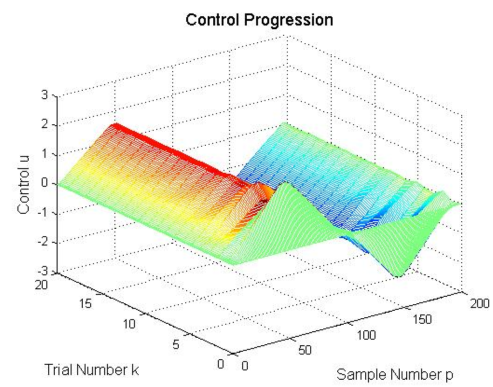

Fig. 4. Control progression for a third order law with $\tau_1 = 0.8$ $\tau_2 = 0.4$.

To compare the non-causal and causal law, in the latter case, the first equation in (31) is replaced by

$$\Delta v_k(p) = K_1(x_k(p) - x_{k-1}(p)) + K_2(y_{ref}(p) - Cx_k(p)).$$

Fig. 5 shows that the rate of trial-to-trial error convergence for the same K_1 and K_2 is faster for the non-causal law.

A logical next step is to seek to develop results that allow measurement of the trial-to-trial higher-order ILC against the standard case. One possible way forward is to investigate the Nesterov accelerated gradient method [24], [25].

V. Conclusions and Further Research

The paper has developed a new higher-order ILC law that is non-causal in the ILC sense. It makes more use of information from previous trials, where once complete, all such information is available to construct the input for the subsequent trial. Examples confirm that such a law can accelerate trial-to-trial error convergence, including when there is uncertainty associated with the dynamics.

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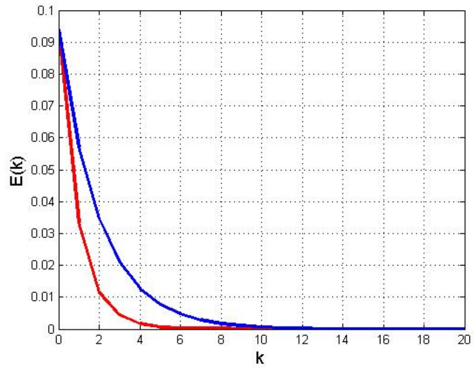

Fig. 5. $\mathrm{E}(k)$ progression for a causal third order law (blue line) and a non-causal law (red line).

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A significant area for future research is to develop measures that quantify the benefits of trial-to-trial error convergence of higher-order ILC laws. In other areas, multi-step methods in optimization theory have been investigated to speed up the convergence of iterative algorithms, e.g., [26], [27]. For example, two-step methods such as the heavy ball and conjugate gradient can significantly speed up the gradient method's convergence, e.g., [25]–[27]. Gradient-based algorithms have been extensively used in ILC design. Considering a trial in ILC as a step, further investigation could lead to significant progress, similar to the gradient method algorithm, see [24] for the non-ILC case. In the ILC case, [28] has reported some results but established a convergence rate slower than that for the non-ILC case.

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