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Stability by averaging of linear discrete-time systems*

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Abstract— **Recently, a constructive approach to averaging-based stability was proposed for linear continuous-time systems with small parameter $\varepsilon > 0$ and rapidly-varying almost periodic coefficients. The present paper extends this approach to discrete-time linear systems with rapidly-varying periodic coefficients. We consider linear systems with state delays, where results on the stability via averaging are missing. Differently from the continuous-time, our linear matrix inequalities (LMIs) are feasible for any delay (i.e. the system is exponentially stable) provided ε is small enough. We introduce an efficient change of variables that leads to a perturbed averaged system, and employ Lyapunov analysis to derive LMIs for finding maximum values of the small parameter $\varepsilon > 0$ and delay that guarantee the exponential stability. Numerical example illustrates the effectiveness of the proposed approach.**

I. INTRODUCTION

Time-varying control systems with almost periodic coefficients arise in many modern engineering applications including satellite attitude and hypersonic vehicle flight control systems [1], [4], [5], [15], [16], [17]. Over the last few decades these systems received a lot of attention from the control community [8], [11],[12], [13]. One of the most efficient methods for stability analysis of such systems is the method of averaging [2], [11], [14]. The main idea behind the averaging method relies on the approximation of the solutions of a time-varying system by solutions of a corresponding averaged system. The exponential stability of the averaged system guarantees the asymptotic stability of the original time-varying system for small enough parameter $\varepsilon > 0$. However, one of the main disadvantages of the classical averaging method is the inability of providing an efficient quantitative upper bound on the small parameter ε for which stability of the original system is preserved.

Recently, a constructive time-delay approach has been introduced for the periodic averaging of continuous-time systems [7]. By averaging the system backwards in time, the system was transformed into a model with time-delays of the length of the small parameter. Stability of the transformed system was shown to imply the stability of the original system [7]. Then, direct Lyapunov-Krasovkii method was applied to obtain LMI-based conditions that guarantee input to state stability of the transformed system and provide an efficient upper bound on the small parameter ε. This time-delay approach was also employed in [3], [7] for averaging of systems with constant/time-varying delays. Moreover, input-to-state stability (ISS), L_2-gain analysis and stochastic extension of the time-delay approach were presented in [19]. In [18], the time-delay approach to averaging approach was extended to discrete-time systems, also it was extended to ISS analysis of the perturbed systems, as well as to obtain practical stability of discrete-time switched affine systems.

Recently, a novel constructive approach for linear continuous-time systems with rapidly-varying almost periodic coefficients was introduced. Differently from the time-delay approach, the method of [9] relies on a novel non-delayed transformation which yields simpler analysis and essentially less conservative results in the numerical examples. This approach is applicable to averaging of systems with both constant and time-varying delays, where for the discrete-time the results are missing.

Our objective in this paper is to extend the approach of [9] to the discrete-time systems, including systems with constant delays. Although the fundamental ideas are inspired by the continuous case [9], construction of the appropriate transformations and the subsequent Lyapunov analysis are not immediately extendable from the continuous framework, but rather require significant adaptation to the discrete-time case. Linear discrete-time delayed systems with periodic coefficients are considered. Differently from [18], we start with a new presentation of the system, where the system matrix is presented as a linear combination of constant matrices multiplied by scalar rapidly-varying terms with zero average. We then suggest a new discrete-time transformation of the rapidly-varying coefficient, and employ a direct Lyapunov method leading to stability conditions in the form of LMIs. The LMIs are accompanied by theoretical guarantees on their feasibility for small enough values of the system parameters. Furthermore, differently from the continuous-time delayed case, we present conditions that guarantee stability of the discrete-time system for arbitrary delay, provided ε is small enough. Numerical example illustrates the efficiency of the suggested method.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with norm $|\cdot|$, and $|\cdot|_1$ is l^1 norm, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices with the induced matrix norm $||\cdot||$, 0_n and I_n are the zero matrix and the identity matrix of order n, respectively. $\mathbb{Z}_+$ is the set of non-negative integers. The notation $P > 0$ for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. $col\{\}$ is a column array of column scalars/vectors. The sub-diagonal elements of a symmetric matrix are denoted by $*$, the superscript T denotes matrix

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transposition, and $\otimes$ denotes the Kronecker product. For $0 < P \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$, we write $|x|_P^2 = x^T P x$. For two integers p and q with $p \le q$, the notation $\mathbf{I}[p,q]$ refers to the set $\{p, p+1, ..., q\}$, we denote $|w|_{[a,b]} = \max_{s\in \mathbf{I}[a,b]} |w_s|$ and $O(\varepsilon)$ is the big O notation.

In the stability analysis below we will use the following:

Lemma 1. *(Jensen's inequality [6, Chapter 6]) Let $d \in \mathbb{Z}_+$ and $0 < R \in \mathbb{R}^{n\times n}$. For all $k \in \mathbb{Z}_+$ and any $x_i \in \mathbb{R}^n$, $i \in \mathbf{I}[k-d, k-1]$, the following inequality holds:*

$$\frac{1}{d}|\sum_{i=k-d}^{k-1}[x_{i+1}-x_i]|_R^2 \le \sum_{i=k-d}^{k-1}|x_{i+1}-x_i|_R^2. \quad (1)$$

II. Stability analysis via averaging of discrete-time systems

A. Problem formulation

Consider the discrete-time system:

$$x_{k+1} = [I + \varepsilon A(k)]x_k, \quad (2)$$

where $x_k \in \mathbb{R}^n$, $A(k) : \mathbb{Z}_+ \to \mathbb{R}^{n\times n}$, $\varepsilon > 0$ is a small parameter. System (2) (and further System (30)) can be regarded as the discretization of the continuous-time system (2.1) (system (3.1)) in [9].

Assumption 1. *The matrices $A(k)$, $k \in \mathbb{Z}_+$ satisfy*

$$A(k) = A_{av} + \sum_{i=1}^{N} a_i(k)A_i, \quad (3)$$

where A_{av} is a Hurwitz matrix and $\{a_i(k)\}_{i=1}^N$, $k \in \mathbb{Z}_+$ are T-periodic with zero average i.e.

$$\frac{1}{T}\sum_{i=k}^{k+T-1} a_j(i) = 0, \quad \forall k \ge 0, \quad \forall j \in \{1,2,...,N\}. \quad (4)$$

Using Assumption 1, we present (2) as

$$x_{k+1} - x_k = \varepsilon(A_{av} + \sum_{j=1}^{N} a_j(k)A_j)x_k. \quad (5)$$

B. System transformation

For each $j \in \{1,2,...,N\}$, let us introduce

$$\rho_j(k) =: -\frac{\varepsilon}{T}\sum_{i=k}^{k+T}(k+T-i)a_j(i). \quad (6)$$

Since a_j is a T-periodic function (hence bounded) by Assumption 1, one has $\rho_j = O(\varepsilon)$. Taking into account (4),

$$\rho_j(k+1) - \rho_j(k) = -\frac{\varepsilon}{T}\sum_{i=k+1}^{k+T}(k+T-i)a_j(i) \quad (7)$$
$$-\frac{\varepsilon}{T}\sum_{i=k+1}^{k+T}a_j(i) + \frac{\varepsilon}{T}\sum_{i=k}^{k+T}(k+T-i)a_j(i) = \varepsilon a_j(k).$$

for all $j \in \{1,2,...,N\}$ and $k \in \mathbb{Z}_+$.

Introduce the change of variables

$$z_k = x_k - \sum_{j=1}^{N}\rho_j(k)A_j x_k, \quad (8)$$

where, for simplicity, we henceforth assume $N = 2$.

Clearly for small enough ε the matrix $I_n - \sum_{j=1}^{2}\rho_j(k)A_j$ is invertible, i.e. transformation (8) is invertible. A sufficient condition for this is given by the following inequality:

$$\delta_2 := \frac{\sum_{j=1}^{2}\varepsilon T a_{j,M}||A_j||}{2} < 1 \quad (9)$$

where $a_{j,M} := \sup_{k\in\mathbb{Z}}|a_j(k)|$, $j = 1,2$. Indeed, we have

$$\sup_{k\in\mathbb{Z}_+}||\sum_{j=1}^{2}\rho_j(k)A_j|| \le \delta_2 < 1, \quad (10)$$

and by employing a Neumann series and equation (10), we obtain

$$\sup_{k\in\mathbb{Z}_+}\left\|\left(I_n - \sum_{j=1}^{2}\rho(k)A_i\right)^{-1}\right\| \le \delta_1 = (1-\delta_2)^{-1}. \quad (11)$$

Using equations (5), (7) and (8), we obtain

$$z_{k+1} - z_k = \varepsilon A_{av}z_k - \varepsilon\sum_{j=1}^{2}\sum_{m=1}^{2}A_jA_m\rho_j(k+1)a_m(k)x_k$$
$$+\varepsilon\sum_{j=1}^{2}\rho_j(k)A_{av}A_jx_k - \varepsilon\sum_{j=1}^{2}A_jA_{av}\rho_j(k+1)x_k. \quad (12)$$

Denoting

$$\begin{aligned}&\mathscr{A} = [A_1, A_2],\ \mathscr{A}_1 = [A_1A_1, A_1A_2, A_2A_1, A_2A_2],\\ &\mathscr{Y}_\rho^{(i)}(k) = col\{\rho_j(k+i-1)x_k\}_{j=1}^2,\ i = 1,2,\\ &\mathscr{Y}_{\rho,a}(k) = col\{\rho_1(k+1)a_1(k)x_k, \rho_1(k+1)a_2(k)x_k,\\ &\qquad \rho_2(k+1)a_1(k)x_k, \rho_2(k+1)a_2(k)x_k\},\end{aligned}$$

the transformation (8) and system (12) can be presented as

$$z_k = x_k - \mathscr{A}\mathscr{Y}_\rho^{(1)}(k), \quad (13)$$
$$z_{k+1} - z_k = \varepsilon A_{av}z_k + \varepsilon A_{av}\mathscr{A}\mathscr{Y}_\rho^{(1)}(k)$$
$$-\varepsilon\mathscr{A}(I_2\otimes A_{av})\mathscr{Y}_\rho^{(2)}(k) - \varepsilon\mathscr{A}_1\mathscr{Y}_{\rho,a}(k). \quad (14)$$

Note that since $\rho_j = O(\varepsilon)$, (14) is of the form

$$z_{k+1} - z_k = \varepsilon A_{av}z_k + O(\varepsilon^2).$$

Let

$$\begin{aligned}&H_\rho = col\{h_\rho^{(1)}, h_\rho^{(2)}\},\\ &H_{\rho,a} = col\{h_{\rho,a}^{(1,1)}, h_{\rho,a}^{(1,2)}, h_{\rho,a}^{(2,1)}, h_{\rho,a}^{(2,2)}\},\end{aligned} \quad (15)$$

where $h_\rho^{(m)}$, $h_{\rho,a}^{(m,j)}$, $m,j = 1,2$ are bounds such that

$$\rho_m^2(k) \le h_\rho^{(m)},\ \rho_m^2(k+1)a_j^2(k) \le h_{\rho,a}^{(m.j)}, m,j = 1,2. \quad (16)$$

Since $\rho_j(k) = O(\varepsilon)$, one has in (15): $|H_\rho|_1 = O(\varepsilon^2)$ and $|H_{\rho,a}|_1 = O(\varepsilon^2)$. Then, for all positive diagonal matrices $\Lambda_\rho^{(1)}, \Lambda_\rho^{(2)} \in \mathbb{R}^{2\times 2}$ and $\Lambda_{\rho,a} \in \mathbb{R}^{4\times 4}$ the following hold:

$$\begin{aligned}&\mathscr{Y}_\rho^{(1)}(k)^T(\Lambda_\rho^{(1)}\otimes I_n)\mathscr{Y}_\rho^{(1)}(k) \le |\Lambda_\rho^{(1)}H_\rho|_1|x_k|^2,\\ &\mathscr{Y}_\rho^{(2)}(k)^T(\Lambda_\rho^{(2)}\otimes I_n)\mathscr{Y}_\rho^{(2)}(k) \le |\Lambda_\rho^{(2)}H_\rho|_1|x_k|^2,\\ &\mathscr{Y}_{\rho,a}^T(k)(\Lambda_{\rho,a}\otimes I_n)\mathscr{Y}_{\rho,a}(k) \le |\Lambda_{\rho,a}H_{\rho,a}|_1|x_k|^2.\end{aligned} \quad (17)$$

The matrices $\Lambda_\rho^{(1)}, \Lambda_\rho^{(2)}$ and $\Lambda_{\rho,a}$ will be decision variables in the LMIs derived below (see (27), (28)).

C. Lyapunov analysis

For stability analysis of (14) subject to (13), we introduce the Lyapunov function

$$V(k)=|z_k|_P^2,\ P>0 \tag{18}$$

and a decay rate $\alpha:=1-\varepsilon\theta$, where $0\le\theta<1/\varepsilon$. Denote

$$Q_\theta(\varepsilon):=A_{av}^TP+PA_{av}+\theta P+\varepsilon A_{av}^TPA_{av}. \tag{19}$$

Using (14), we obtain

$$\begin{aligned}
&V(k+1)-\alpha V(k)=|z_k|^2_{\varepsilon Q_\theta}+\varepsilon^2\mathscr{Y}_{\rho,a}^T(k)\mathscr{A}_1^TP\mathscr{A}_1\mathscr{Y}_{\rho,a}(k)\\
&+\varepsilon^2\mathscr{Y}_\rho^{(1),T}(k)(A_{av}\mathscr{A})^TP(A_{av}\mathscr{A})\mathscr{Y}_\rho^{(1)}(k)\\
&+\varepsilon^2\mathscr{Y}_\rho^{(2),T}(k)(I_2\otimes A_{av})^T\mathscr{A}^TP\mathscr{A}(I_2\otimes A_{av})\mathscr{Y}_\rho^{(2)}(k)\\
&+2\varepsilon z_k^T(I+\varepsilon A_{av})^TP(A_{av}\mathscr{A})\mathscr{Y}_\rho^{(1)}(k)\\
&-2\varepsilon z_k^T(I+\varepsilon A_{av})^TP\mathscr{A}(I_2\otimes A_{av})\mathscr{Y}_\rho^{(2)}(k)\\
&-2\varepsilon z_k^T(I+\varepsilon A_{av})^TP\mathscr{A}_1\mathscr{Y}_{\rho,a}(k)\\
&-2\varepsilon^2\mathscr{Y}_\rho^{(1),T}(k)(A_{av}\mathscr{A})^TP\mathscr{A}(I_2\otimes A_{av})\mathscr{Y}_\rho^{(2)}(k)\\
&-2\varepsilon^2\mathscr{Y}_\rho^{(1),T}(k)(A_{av}\mathscr{A})^TP\mathscr{A}_1\mathscr{Y}_{\rho,a}(k)\\
&+2\varepsilon^2\mathscr{Y}_\rho^{(2),T}(k)(I_2\otimes A_{av})^T\mathscr{A}^TP\mathscr{A}_1\mathscr{Y}_{\rho,a}(k).
\end{aligned} \tag{20}$$

Using (13), we present (20) as a quadratic in x_k via:

$$\begin{aligned}
|z_k|^2_{\varepsilon Q_\theta(\varepsilon)}&=|x_k|^2_{\varepsilon Q_\theta(\varepsilon)}-2x_k^T\varepsilon Q_\theta(\varepsilon)\mathscr{A}\mathscr{Y}_\rho^{(1)}(k)\\
&+\varepsilon\mathscr{Y}_\rho^{(1),T}(k)\mathscr{A}^TQ_\theta(\varepsilon)\mathscr{A}\mathscr{Y}_\rho^{(1)}(k),
\end{aligned} \tag{21}$$

$$\begin{aligned}
&2\varepsilon z_k^T(I+\varepsilon A_{av})^TP(A_{av}\mathscr{A})\mathscr{Y}_\rho^{(1)}(k)=\\
&-\varepsilon\mathscr{Y}_\rho^{(1),T}(k)\mathscr{A}^T(I+\varepsilon A_{av})^TP(A_{av}\mathscr{A})\mathscr{Y}_\rho^{(1)}(k)\\
&-\varepsilon\mathscr{Y}_\rho^{(1),T}(k)(A_{av}\mathscr{A})P(I+\varepsilon A_{av})\mathscr{A}\mathscr{Y}_\rho^{(1)}(k)\\
&+2\varepsilon x_k^T(I+\varepsilon A_{av})^TP(A_{av}\mathscr{A})\mathscr{Y}_\rho^{(1)}(k),
\end{aligned} \tag{22}$$

$$\begin{aligned}
&-2\varepsilon z_k^T(I+\varepsilon A_{av})^TP\mathscr{A}(I_2\otimes A_{av})\mathscr{Y}_\rho^{(2)}(k)=\\
&-2\varepsilon x_k^T(I+\varepsilon A_{av})^TP\mathscr{A}(I_2\otimes A_{av})\mathscr{Y}_\rho^{(2)}(k)\\
&+2\varepsilon\mathscr{Y}_\rho^{(1),T}(k)\mathscr{A}^T(I+\varepsilon A_{av})^TP\mathscr{A}(I_2\otimes A_{av})\mathscr{Y}_\rho^{(2)}(k),
\end{aligned} \tag{23}$$

$$\begin{aligned}
&-2\varepsilon z_k^T(I+\varepsilon A_{av})^TP\mathscr{A}_1\mathscr{Y}_{\rho,a}(k)=\\
&-2\varepsilon x_k^T(I+\varepsilon A_{av})^TP\mathscr{A}_1\mathscr{Y}_{\rho,a}(k)\\
&+2\varepsilon\mathscr{Y}_\rho^{(1),T}(k)\mathscr{A}^T(I+\varepsilon A_{av})^TP\mathscr{A}_1\mathscr{Y}_{\rho,a}(k).
\end{aligned} \tag{24}$$

Let

$$\begin{aligned}
&\eta(k)=col\{x_k,\mathscr{Y}_\rho^{(1)}(k),\mathscr{Y}_\rho^{(2)}(k),\mathscr{Y}_{\rho,a}(k)\},\\
&W_1:=|\Lambda_\rho^{(1)}H_\rho|_1|x_k|^2-\mathscr{Y}_\rho^{(1),T}(k)(\Lambda_\rho^{(1)}\otimes I_n)\mathscr{Y}_\rho^{(1)}(k),\\
&W_2:=|\Lambda_\rho^{(2)}H_\rho|_1|x_k|^2-\mathscr{Y}_\rho^{(2),T}(k)(\Lambda_\rho^{(2)}\otimes I_n)\mathscr{Y}_\rho^{(2)}(k),\\
&W_3:=|\Lambda_{\rho,a}H_{\rho,a}|_1|x_k|^2-\mathscr{Y}_{\rho,a}^T(k)(\Lambda_{\rho,a}\otimes I_n)\mathscr{Y}_{\rho,a}(k).
\end{aligned} \tag{25}$$

Then, (17) implies that $W_m\ge 0$ for all $m=1,2,3$. Using (17)-(25) and the S-procedure [6], we arrive at

$$\begin{aligned}
V(k+1)-\alpha V(k)&\le V(k+1)-\alpha V(k)+\varepsilon\textstyle\sum_{m=1}^3 W_m\\
&\le\varepsilon\eta^T(k)\Phi_\varepsilon\eta(k)\le 0,
\end{aligned} \tag{26}$$

Provided

$$\Phi_\varepsilon=\begin{bmatrix}\beta_1 & B\\ * & \Psi_\varepsilon\end{bmatrix}<0, \tag{27}$$

where

$$\begin{aligned}
&\Psi_\varepsilon=\begin{bmatrix}\phi_1 & \phi_2 & \phi_3\\ * & \phi_4 & \varepsilon(I_2\otimes A_{av})^T\mathscr{A}^TP\mathscr{A}_1\\ * & * & -(\Lambda_{\rho,a}\otimes I_n)+\varepsilon\mathscr{A}_1^TP\mathscr{A}_1\end{bmatrix},\\
&B=\begin{bmatrix}\beta_2 & \beta_3 & \beta_4\end{bmatrix},\\
&\beta_1=Q_\theta+\sum_{j=1}^2|\Lambda_\rho^{(j)}H_\rho|_1I_n+|\Lambda_{\rho,a}H_{\rho,a}|_1I_n,\\
&\beta_2=-Q_\theta\mathscr{A}+(I_n+\varepsilon A_{av})^TPA_{av}\mathscr{A},\\
&\beta_3=-(I_n+\varepsilon A_{av})^TP\mathscr{A}(I_2\otimes A_{av}),\\
&\beta_4=-(I_n+\varepsilon A_{av})^TP\mathscr{A}_1,\\
&\phi_1=-(\Lambda_\rho^{(1)}\otimes I_n)+\mathscr{A}^TQ_\theta(\varepsilon)\mathscr{A}\\
&\quad+\varepsilon(A_{av}\mathscr{A})^TP(A_{av}\mathscr{A})-(A_{av}\mathscr{A})^TP(I+\varepsilon A_{av})\mathscr{A}\\
&\quad-\mathscr{A}^T(I_n+\varepsilon A_{av})^TP(A_{av}\mathscr{A}),\\
&\phi_2=\mathscr{A}^T(I_n+\varepsilon A_{av})^TP\mathscr{A}(I_2\otimes A_{av})\\
&\quad-\varepsilon(A_{av}\mathscr{A})^TP\mathscr{A}(I_2\otimes A_{av}),\\
&\phi_3=\mathscr{A}^T(I_n+\varepsilon A_{av})^TP\mathscr{A}_1-\varepsilon(A_{av}\mathscr{A})^TP\mathscr{A}_1,\\
&\phi_4=-(\Lambda_\rho^{(2)}\otimes I_n)+\varepsilon(I_2\otimes A_{av})^T\mathscr{A}^TP\mathscr{A}(I_2\otimes A_{av}).
\end{aligned} \tag{28}$$

Summarizing, we arrive at:

Theorem 1. *Consider system* (2) *subject to Assumption 1 , let $H_\rho, H_{\rho,a}$ be defined by* (16)*. Given tuning parameters $\theta>0$ and $\varepsilon^*>0$ subject to $\theta\varepsilon^*<1$ and* (9) *with $\varepsilon=\varepsilon^*$. Let there exist $0<P\in\mathbb{R}^{n\times n}$, and diagonal positive matrices $\Lambda_\rho^{(1)},\Lambda_\rho^{(2)}\in\mathbb{R}^{2\times 2}$ and $\Lambda_{\rho,a}\in\mathbb{R}^{4\times 4}$ such that* (27) *with notation* (28) *holds for $\varepsilon=\varepsilon^*$. Then, system* (2) *is exponentially stable with a decay rate $\sqrt{1-\theta\varepsilon}$ for all $\varepsilon\in(0,\varepsilon^*]$, namely, there exists a $M>0$ such that for all $\varepsilon\in(0,\varepsilon^*]$, the solution of* (2) *initialized by $x_0\in\mathbb{R}^n$ satisfies*

$$|x_k|^2\le M(1-\theta\varepsilon)^k|x_0|^2,\ \forall k\in\mathbb{Z}_+. \tag{29}$$

Morover, if (9) *and* (27) *hold with $\varepsilon=\varepsilon^*$ and $\theta=0$, then* (2) *is exponentially stable for all $\varepsilon\in(0,\varepsilon^*]$. The inequalities* (9) *and* (27) *are always feasible for small enough ε and θ.*

Proof. Due to space limitations, the proof is omitted. □

III. Stability of the discrete-time systems with constant delays

A. Problem formulation

Using the new transformation, which is based on summation of the rapidly varying coefficients only (and does not include the state inside of the summation like in the time-delay approach [18]), we present stability conditions for discrete-time systems with delays. Consider the system

$$x_{k+1}=(I_n+\varepsilon A_0)x_k+\varepsilon A_D(k)x_{k-d},\ k\in\mathbb{Z}_+, \tag{30}$$

where d is a positive integer.

Assumption 2. *Assume that $A_D(k),\ k\in\mathbb{Z}_+$ is of the form:*

$$A_D(k)=A_d+\sum_{m=1}^{N_d}a_m(k)A_m, \tag{31}$$

where $A_{av}=:A_0+A_d$ is Hurwitz and $\{a_m(k)\}_{m=1}^{N_d}$ are T-periodic with the zero average (i.e. satisfy (4)*).*

B. System transformation

We modify the transformation (8) to account for the delay:

$$z_k = x_k - \sum_{m=1}^{N_d} \rho_m(k) A_m x_{k-d},\ k \geq d. \tag{32}$$

For simplicity of the presentation, we assume $N_d = 2$. Let

$$\xi_k = x_k - x_{k-d}, \quad k \in \mathbb{Z}_+.$$

By employing (4), (7) and (30) - (32) we obtain

$$\begin{aligned} z_{k+1} - z_k &= \varepsilon A_{av} z_k - \varepsilon A_d \xi_k \\ &+ \varepsilon \textstyle\sum_{m=1}^2 A_{av} A_m \rho_m(k) x_{k-d} \\ &- \varepsilon \textstyle\sum_{m=1}^2 A_m A_0 \rho_m(k+1) x_{k-d} \\ &- \varepsilon \textstyle\sum_{m=1}^2 A_m A_d \rho_m(k+1) x_{k-2d} \\ &- \varepsilon \textstyle\sum_{m=1}^2 \sum_{i=1}^2 A_m A_i \rho_m(k+1) a_i(k-d) x_{k-2d}, \end{aligned} \tag{33}$$

Since $\rho_j(k) = O(\varepsilon)$, equation(33) has the form

$$z_{k+1} - z_k = \varepsilon A_{av} z_k - \varepsilon A_d \xi_k + O(\varepsilon^2).$$

Denote

$$\begin{aligned} &\mathscr{Y}_{a,d}(k) = col\{a_j(k) x_{k-d}\}_{j=1}^2, \\ &\mathscr{Y}_{\rho,d}^{(m)}(k) = col\{\rho_j(k+m-1) x_{k-d}\}_{j=1}^2,\ m = 1,2, \\ &\mathscr{Y}_{\rho,d}(k) = col\{\mathscr{Y}_{\rho,d}^{(j)}(k)\}_{j=1}^2, \\ &\mathscr{Y}_{\rho,2d}(k) = col\{\rho_j(k+1) x_{k-2d}\}_{j=1}^2, \\ &\mathscr{Y}_{\rho,a,d}(k) = col\{\rho_1(k+1) a_1(k-d) x_{k-2d}, \\ &\rho_1(k+1) a_2(k-d) x_{k-2d}, \rho_2(k+1) a_1(k-d) x_{k-2d}, \\ &\qquad \rho_2(k+1) a_2(k-d) x_{k-2d}\}, \\ &\tilde{\mathscr{A}} = [\mathscr{A}, 0_{n\times n}, 0_{n\times n}], \mathscr{A}_{\rho,d} = [A_{av}\mathscr{A}, -\mathscr{A}(I_2 \otimes A_0)]. \end{aligned} \tag{34}$$

Then, (33) can be presented as

$$\begin{aligned} z_{k+1} - z_k &= \varepsilon A_{av} z_k - \varepsilon A_d \xi_k + \varepsilon \mathscr{A}_{\rho,d} \mathscr{Y}_{\rho,d}(k) \\ &- \varepsilon \mathscr{A}(I_2 \otimes A_d) \mathscr{Y}_{\rho,2d}(k) - \varepsilon \mathscr{A}_1 \mathscr{Y}_{\rho,a,d}(k), \end{aligned} \tag{35}$$

whereas due to (30), (32) we have

$$x_{k+1} - x_k = \varepsilon A_{av} x_k - \varepsilon A_d \xi_{k-d} + \varepsilon \mathscr{A} \mathscr{Y}_{a,d}(k), \tag{36}$$

$$z_k = x_k - \tilde{\mathscr{A}} \mathscr{Y}_{\rho,d}(k). \tag{37}$$

Let $H_a = col\{h_a^{(1)}, h_a^{(2)}\}$ and H_ρ, $H_{\rho,a}$ be as in (15), where $h_a^{(m)}$, $h_\rho^{(m)}$, $h_{\rho,a}^{(m,j)}$ satisfy, for all $k \in \mathbb{Z}_+$,

$$\begin{aligned} &a_m^2(k) \leq h_a^{(m)}, \quad \rho_m^2(k) \leq h_\rho^{(m)}, \\ &\rho_m^2(k+1) a_j^2(k-d) \leq h_{\rho,a}^{(m.j)},\ m, j \in \{1,2\}. \end{aligned} \tag{38}$$

Since $\rho_j(k) = O(\varepsilon)$, we have $|H_\rho|_1 = O(\varepsilon^2)$ and $|H_{\rho,a}|_1 = O(\varepsilon^2)$. Then, for any diagonal positive matrices $\Lambda_a, \Lambda_{\rho,2d} \in \mathbb{R}^{2\times 2}$ and $\Lambda_{\rho,a}, \Lambda_{\rho,d} \in \mathbb{R}^{4\times 4}$ the following holds:

$$\begin{aligned} &\mathscr{Y}_{a,d}^T(k)(\Lambda_a \otimes I_n) \mathscr{Y}_{a,d}(k) \leq |\Lambda_a H_a|_1 |x_{k-d}|^2, \\ &\mathscr{Y}_{\rho,d}^T(k)(\Lambda_{\rho,d} \otimes I_n) \mathscr{Y}_{\rho,d}(k) \leq |\Lambda_{\rho,d}(I_2 \otimes H_\rho)|_1 |x_{k-d}|^2, \\ &\mathscr{Y}_{\rho,2d}^T(k)(\Lambda_{\rho,2d} \otimes I_n) \mathscr{Y}_{\rho,2d}(k) \leq |\Lambda_{\rho,2d} H_\rho|_1 |x_{k-2d}|^2, \\ &\mathscr{Y}_{\rho,a,d}^T(k)(\Lambda_{\rho,a} \otimes I_n) \mathscr{Y}_{\rho,a,d}(k) \leq |\Lambda_{\rho,a} H_{\rho,a}|_1 |x_{k-2d}|^2. \end{aligned} \tag{39}$$

The matrices $\Lambda_a, \Lambda_{\rho,a}$, $\Lambda_{\rho,d}$ and $\Lambda_{\rho,2d}$ will be decision variables in the LMIs derived below (see (53), (54)).

C. Lyapunov analysis

For stability analysis of (35) subject to (37), introduce the Lyapunov function

$$V(k) = V_P(k) + \varepsilon \left(\sum_{m=1}^2 V_{S_m}(k) + V_R(k) \right), \tag{40}$$

$$V_P(k) = |z_k|_P^2,\ P > 0, \tag{41}$$

$$V_{S_m}(k) = \sum_{i=k-m\cdot d}^{k-1} \alpha^{k-i-1} |x_i|_{S_m}^2,\ m = 1,2,\ S_m > 0, \tag{42}$$

$$V_R(k) = d \sum_{i=-d}^{-1} \sum_{s=k+i}^{k-1} \alpha^{k-s-1} |x_{s+1} - x_s|_R^2,\ R > 0. \tag{43}$$

and a desired decay rate $\alpha := 1 - \varepsilon\theta$, where $0 \leq \theta < 1/\varepsilon$. Here V_{S_1} and V_R compensate x_{k-d}, whereas V_{S_2} compensates x_{k-2d} in the stability analysis. Then,

$$\begin{aligned} &V_P(k+1) - \alpha V_P(k) = \varepsilon |z_k|_{Q_\theta}^2 + \varepsilon^2 \xi_k^T A_d^T P A_d \xi_k \\ &+ \varepsilon^2 \mathscr{Y}_{\rho,d}(k)^T \mathscr{A}_{\rho,d}^T P \mathscr{A}_{\rho,d} \mathscr{Y}_{\rho,d}(k) \\ &+ \varepsilon^2 \mathscr{Y}_{\rho,a,d}(k)^T \mathscr{A}_1^T P \mathscr{A}_1 \mathscr{Y}_{\rho,a,d}(k) \\ &+ \varepsilon^2 \mathscr{Y}_{\rho,2d}(k)^T [\mathscr{A}(I_2 \otimes A_d)]^T P [\mathscr{A}(I_2 \otimes A_d)] \mathscr{Y}_{\rho,2d}(k) \\ &- 2\varepsilon z_k^T (I + \varepsilon A_{av})^T P A_d \xi_k \\ &+ 2\varepsilon z_k^T (I + \varepsilon A_{av})^T P \mathscr{A}_{\rho,d} \mathscr{Y}_{\rho,d}(k) \\ &- 2\varepsilon z_k^T (I + \varepsilon A_{av})^T P \mathscr{A}(I_2 \otimes A_d) \mathscr{Y}_{\rho,2d}(k) \\ &- 2\varepsilon z_k^T (I + \varepsilon A_{av})^T P \mathscr{A}_1 \mathscr{Y}_{\rho,a,d}(k) \\ &- 2\varepsilon^2 \xi_k^T A_d^T P \mathscr{A}_{\rho,d} \mathscr{Y}_{\rho,d}(k) + 2\varepsilon^2 \xi_k^T A_d^T P \mathscr{A}_1 \mathscr{Y}_{\rho,a,d}(k) \\ &+ 2\varepsilon^2 \xi_k^T A_d^T P \mathscr{A}(I_2 \otimes A_d) \mathscr{Y}_{\rho,2d}(k) \\ &- 2\varepsilon^2 \mathscr{Y}_{\rho,d}(k)^T \mathscr{A}_{\rho,d}^T P \mathscr{A}(I_2 \otimes A_d) \mathscr{Y}_{\rho,2d}(k) \\ &- 2\varepsilon^2 \mathscr{Y}_{\rho,d}(k)^T \mathscr{A}_{\rho,d}^T P \mathscr{A}_1 \mathscr{Y}_{\rho,a,d}(k) \\ &+ 2\varepsilon^2 \mathscr{Y}_{\rho,2d}(k)^T [\mathscr{A}(I_2 \otimes A_d)]^T P \mathscr{A}_1 \mathscr{Y}_{\rho,a,d}(k), \end{aligned} \tag{44}$$

where Q_θ is defined in (19). Substituting $z_k = x_k - \tilde{\mathscr{A}} \mathscr{Y}_{\rho,d}(k)$ in (44) , we get

$$|z_k|_{Q_\theta}^2 = |x_k|_{Q_\theta}^2 + |\mathscr{Y}_{\rho,d}(k)|_{\tilde{\mathscr{A}}^T Q_\theta \tilde{\mathscr{A}}}^2 - 2x_k^T Q_\theta \tilde{\mathscr{A}} \mathscr{Y}_{\rho,d}(k), \tag{45}$$

$$\begin{aligned} &- 2\varepsilon z_k^T (I_n + \varepsilon A_{av})^T P [A_d \xi_k - \mathscr{A}_{\rho,d} \mathscr{Y}_{\rho,d}(k) \\ &\qquad + \mathscr{A}(I_2 \otimes A_d) \mathscr{Y}_{\rho,2d}(k) + \mathscr{A}_1 \mathscr{Y}_{\rho,a,d}(k)] = \\ &- 2\varepsilon [x_k - \tilde{\mathscr{A}} \mathscr{Y}_{\rho,d}(k)]^T (I + \varepsilon A_{av})^T P \cdot [A_d \xi_k \\ &- \mathscr{A}_{\rho,d} \mathscr{Y}_{\rho,d}(k) + \mathscr{A}(I_2 \otimes A_d) \mathscr{Y}_{\rho,2d}(k) + \mathscr{A}_1 \mathscr{Y}_{\rho,a,d}(k)]. \end{aligned} \tag{46}$$

Along system (30), $V_{S_j}, j = 1,2$ are evaluated as

$$\begin{aligned} V_{S_1}(k+1) - \alpha V_{S_1}(k) &= (1 - \alpha^d) |x_k|_{S_1}^2 \\ &- \alpha^d |\xi_k|_{S_1}^2 + 2\alpha^d x_k^T S_1 \xi_k, \end{aligned} \tag{47}$$

$$V_{S_2}(k+1) - \alpha V_{S_2}(k) = |x_k|_{S_2}^2 - \alpha^{2d} |x_{k-2d}|_{S_2}^2. \tag{48}$$

Let

$$\begin{aligned} &\mathscr{L} = [A_{av}, -A_d, 0_{n\times 4n}, 0_{n\times 2n}, \mathscr{A}, 0_{n\times 4n}], \\ &\eta_k = col\{x_k, \xi_k, \mathscr{Y}_{\rho,d}(k), \mathscr{Y}_{\rho,2d}(k), \mathscr{Y}_{a,d}(k), \mathscr{Y}_{\rho,a,d}(k)\}, \end{aligned} \tag{49}$$

By Jensen's inequality (1), we obtain

$$V_R(k+1) - \alpha V_R(k) \leq \varepsilon^2 d^2 \eta_{k,d}^T \mathscr{L}^T R \mathscr{L} \eta_{k,d} - \alpha^d \xi_k^T R \xi_k. \tag{50}$$

Define

$$\begin{aligned}
&W := -\varepsilon\eta_k^T\Pi\eta_k + \varepsilon\lambda_{2d}|x_{k-2d}|^2 + \varepsilon\lambda_d|x_k-\xi_k|^2,\\
&\lambda_d := |\Lambda_a H_a|_1 + |\Lambda_{\rho,d}(I_2\otimes H_\rho)|_1,\\
&\lambda_{2d} := |\Lambda_{\rho,2d}H_\rho|_1 + |\Lambda_{\rho,a}H_{\rho,a}|_1, \qquad (51)\\
&\Pi = diag\{0_n,\Pi^{(1)}\},\\
&\Pi^{(1)} := diag\{0,\Lambda_{\rho,d},\Lambda_{\rho,2d},\Lambda_a,\Lambda_{\rho,a}\}\otimes I_n.
\end{aligned}$$

Then, (39) implies that $W \geq 0$. Using (39)-(51) and the S-procedure [6], we arrive at

$$\begin{aligned}
V(k+1)-\alpha V(k) &\leq V(k+1)-\alpha V(k)+W\\
&\leq \varepsilon\eta_k^T(\Theta_{\varepsilon,d}+\varepsilon^2 d^2\mathscr{L}^T R\mathscr{L})\eta_k\\
&\quad+\varepsilon x_{k-2d}^T(-\alpha^{2d}S_2+\lambda_{2d}I_n)x_{k-2d}\leq 0, \quad (52)
\end{aligned}$$

provided

$$\Theta_{\varepsilon,d}+\varepsilon^2 d^2\mathscr{L}^T R\mathscr{L}<0,\ -\alpha^{2d}S_2+\lambda_{2d}I_n<0, \qquad (53)$$

where

$$\begin{aligned}
&\Theta_{\varepsilon,d} = \begin{bmatrix}\beta_1 & B\\ * & \Psi_{\varepsilon,d}\end{bmatrix}, B = \begin{bmatrix}\beta_2 & \beta_3 & \beta_4 & 0_{n\times 2n} & \beta_5\end{bmatrix},\\
&\Psi_{\varepsilon,d} = -\Pi^{(1)} + \begin{bmatrix}\omega_1 & \omega_2 & \omega_3 & 0_{n\times 2n} & \varepsilon A_d^T P\mathscr{A}_1\\ * & \omega_4 & \omega_5 & 0_{4n\times 2n} & \omega_6\\ * & * & \omega_7 & 0_{2n\times 2n} & \omega_8\\ * & * & * & 0_{2n\times 2n} & 0_{2n\times 4n}\\ * & * & * & * & \varepsilon\mathscr{A}_1^T P\mathscr{A}_1\end{bmatrix},\\
&\beta_1 = Q_\theta+(1-\alpha^d)S_1+S_2+\lambda_d I_n,\ \alpha = 1-\theta\varepsilon,\\
&\beta_2 = \alpha^d S_1-(I_n+\varepsilon A_{av})^T PA_d-\lambda_d I_n,\\
&\beta_3 = -Q_\theta\tilde{\mathscr{A}}+(I_n+\varepsilon A_{av})^T P\mathscr{A}_{\rho,d},\\
&\beta_4 = -(I_n+\varepsilon A_{av})^T P\mathscr{A}(I_2\otimes A_d),\\
&\beta_5 = -(I_n+\varepsilon A_{av})^T P\mathscr{A}_1,\\
&\omega_1 = \varepsilon A_d^T PA_d-\alpha^d(S_1+R)+\lambda_d I_n,\\
&\omega_2 = A_d^T P(I_n+\varepsilon A_{av})\tilde{\mathscr{A}}-\varepsilon A_d^T P\mathscr{A}_{\rho,d}, \qquad (54)\\
&\omega_3 = \varepsilon A_d^T P\mathscr{A}(I_2\otimes A_d), \omega_4 = \varepsilon\mathscr{A}_{\rho,d}^T P\mathscr{A}_{\rho,d}+\tilde{\mathscr{A}}^T Q_\theta\tilde{\mathscr{A}}\\
&\quad -\tilde{\mathscr{A}}^T(I_n+\varepsilon A_{av})^T P\mathscr{A}_{\rho,d}-\mathscr{A}_{\rho,d}^T P(I_n+\varepsilon A_{av})\tilde{\mathscr{A}},\\
&\omega_5 = -\varepsilon\mathscr{A}_{\rho,d}^T P\mathscr{A}(I_2\otimes A_d)+\\
&\qquad\qquad \tilde{\mathscr{A}}^T(I_n+\varepsilon A_{av})^T P\mathscr{A}(I_2\otimes A_d),\\
&\omega_6 = -\varepsilon\mathscr{A}_{\rho,d}^T P\mathscr{A}_1+\tilde{\mathscr{A}}^T(I_n+\varepsilon A_{av})^T P\mathscr{A}_1,\\
&\omega_7 = \varepsilon[\mathscr{A}(I_2\otimes A_d)]^T P[\mathscr{A}(I_2\otimes A_d)].\\
&\omega_8 = \varepsilon[\mathscr{A}(I_2\otimes A_d)]^T P\mathscr{A}_1.
\end{aligned}$$

Summarizing, we arrive at:

Theorem 2. *Consider system* (30) *subject to Assumption 2, let* $H_a, H_\rho, H_{\rho,a}$, *be defined by* (38). *Given positive tuning parameters* θ, d^*, *and* ε^* *subject to* $\delta_2 < \alpha^{d^*}$ *and* $\theta\varepsilon^* < 1$. *Let there exist* $0 < P, S_1, S_2, R \in \mathbb{R}^{n\times n}$, *and diagonal positive matrices* $\Lambda_a, \Lambda_\rho^{(1)}, \Lambda_\rho^{(2)}, \Lambda_{\rho,2d} \in \mathbb{R}^{2\times 2}$ *and* $\Lambda_{\rho,a} \in \mathbb{R}^{4\times 4}$ *such that* (53) *with notations* (49), (51), (54) *holds with* $\varepsilon = \varepsilon^*$ *and* $d = d^*$. *Then system* (30) *is exponentially stable with a decay rate* $\sqrt{1-\theta\varepsilon}$ *for all* $\varepsilon \in (0,\varepsilon^*]$ *and* $0 \leq d \leq d^*$. *Namely, there exists* $M > 0$ *such that for all* $\varepsilon \in (0,\varepsilon^*]$ *and* $0 \leq d \leq d^*$, *the solution of* (30) *initialized at* $\{x_j\}_{j=-d}^0$ *satisfies*

$$|x_k|^2 \leq M\,|x|^2_{[-d,0]}\,(1-\theta\varepsilon)^{k-d},\ \forall d \leq k \in \mathbb{Z}_+. \qquad (55)$$

Moreover, if (53) *and* $\delta_2 < \alpha^d$ *hold with* $\varepsilon = \varepsilon^*$, $d = d^*$ *and* $\theta = 0$, *then* (30) *is exponentially stable for all* $\varepsilon \in (0,\varepsilon^*]$ *and* $0 \leq d \leq d^*$. *Also, given any d, the inequalities* (53) *and* $\delta_2 < \alpha^d$ *are always feasible for small enough* ε *and* θ.

Proof. The fact that feasibility of (53) and $\delta_2 < \alpha^d$ with ε^*, d^* implies feasibility for all $\varepsilon < \varepsilon^*$, $d < d^*$, follows by monotonicity of (53), and $\delta_2 < \alpha^d$ with respect to $\varepsilon < \varepsilon^*$, $d < d^*$ (i.e., as the small parameters decrease, the eigenvalues of (53) are non-increasing).

Feasibility of (53), and $\delta_2 < \alpha^d$ implies that for all $d \leq k \in \mathbb{Z}_+$,

$$\begin{aligned}
&V(k+1)-\alpha V(k)\leq 0 \Rightarrow V(k+1)\leq \alpha^{k+1-d}V(d),\\
&V(d) = |z_d|_P^2+\textstyle\sum_{m=1}^2\sum_{i=d-m\cdot d}^{d-1}\alpha^{d-i-1}|x_i|_{S_m}^2\\
&\quad +d\textstyle\sum_{i=-d}^{-1}\sum_{s=d+i}^{d-1}\alpha^{d-s-1}|x_{s+1}-x_s|_R^2.
\end{aligned} \qquad (56)$$

Also, $V(k) \geq \sigma_{min}(P)|z_k|^2$, for any $d \leq k \in \mathbb{Z}_+$. Thus, there exists some $M_1 > 0$ such that

$$|z_k|^2 \leq M_1\,|x|^2_{[-d,0]}\,\alpha^{k-d},\ d \leq k \in \mathbb{Z}_+. \qquad (57)$$

To conclude the same for the solution x_k of the system (30), for any $i \in \mathbb{Z}_+$, we denote $X_i = |x|^2_{[id,(i+1)d]}$. From (9), (33) and (57), we find that

$$X_{i+1} \leq M_2\alpha^{id}+\delta_2 X_i, i \in \mathbb{Z}_+,$$

where $M_2 = M_1\,|x|^2_{[-d,0]}$. Set $Y_1 = X_1$ and consider the linear difference equation

$$Y_{i+1} = M_2\alpha^{id}+\delta_2 Y_i,\ i \in \mathbb{Z}_+. \qquad (58)$$

By induction, we obtain $X_i \leq Y_i$ for all $i \in \mathbb{Z}_+$. Moreover, the solution of (58) is given by $Y_i = \mu_d\alpha^{(i-1)d}+\delta_2^{i-1}(X_1-\mu_d)$, $i \in \mathbb{Z}_+$, where $\mu_d = \frac{M_2\alpha}{\alpha-\delta_2}$. Note that Y_i is decreasing. Let $k \in \mathbb{Z}_+$ such that $k \in \mathbf{I}[id,(i+1)d]$. Then

$$\begin{aligned}
|x_k|^2 &\leq X_j \leq \delta_2^{i-1}(X_1-\mu_d)+\mu_d\alpha^{id-d}\\
&\leq \delta_2^{\frac{k-d}{d}}\frac{(X_1-\mu_d)}{\delta_2}+\mu_d\alpha^{-d}\alpha^{k-d}\\
&\leq \left(\frac{(X_1-\mu_d)}{\delta_2}+\mu_d\alpha^{-d}\right)\alpha^{k-d},
\end{aligned}$$

where the last inequality follows from $\delta_2 < \alpha^d$, which proves (55).

For LMI feasibility guarantees, choose $\Lambda_a = \Lambda_{\rho,2d} = \lambda_1 I_2$, $\Lambda_{\rho,a} = \Lambda_{\rho,d} = \lambda_1 I_4$, $R = \lambda_1 I_n$, $S_1 = \lambda_d I_n$, where $\lambda_1 > \lambda_d$, $S_2 = \lambda_2 I_n$ where $\lambda_2 = 2\lambda_{2d}$ (keep in mind that $\lambda_{2d} = O(\varepsilon^2)$). For $\theta = 0$ (so $\alpha = 1$), the inequality $-\alpha^{2d}S_2+\lambda_{2d}I_n < 0$ hold, and also for any d there is a small enough ε such that $\varepsilon^2 d^2\mathscr{L}^T R\mathscr{L}$ and $\delta_2 < \alpha^d$ (since $\delta_2 = O(\varepsilon)$) are small enough, while $\Theta_{\varepsilon,d}$ is independent of d. Therefore, by choosing $\theta = 0$, $d = 0$, (so $\alpha = 1$), it is enough to prove that $\Theta_{\varepsilon,d} < 0$. Also there is a $0 < P \in \mathbb{R}^n$ such that $\beta_1 < 0$ for small enough ε (see (19), Assumption 2). It is easily seen that $\Psi_{\varepsilon,d} < 0$ for large enough λ_1 and small enough $\varepsilon > 0$. Next, we apply Schur complement with respect to

$\Theta_{\varepsilon,d}$, whence $\Theta_{\varepsilon,d} < 0$ iff $\beta_1 - \frac{1}{\lambda_1}B(\lambda_1^{-1}\Psi_{\varepsilon,d})^{-1}B^T < 0$. Note that $-(\lambda_1^{-1}\Psi_{\varepsilon,d})^{-1}$ is bounded as $\lambda_1 \to \infty$ (converges to the identity matrix), whereas B and β_1 are independent of λ_1 implying the feasibility of $\Theta_{\varepsilon,d}$. Thus, for any large d, there exist small enough ε^* and θ such that feasibility is assured. $\square$

D. Numerical example

Example 2: (Stabilization by fast switching [10]) Consider (30) with $A_0 = 0$ and

$$A_D(k) = \begin{cases} A_1, \ k \in [100n\varepsilon, 100(n+0.4)\varepsilon), \\ A_2, \ k \in [100(n+0.4)\varepsilon, 100(n+1)\varepsilon), \end{cases} \quad (59)$$

where

$$A_1 = \begin{bmatrix} 0.1 & 0.3 \\ 0.6 & -0.2 \end{bmatrix}, A_2 = \begin{bmatrix} -0.13 & -0.16 \\ -0.33 & 0.03 \end{bmatrix}. \quad (60)$$

Given $k \in \mathbb{Z}_+$, (59) can be written as

$$A_D(k) = \chi_{[100n\varepsilon,100(n+0.4)\varepsilon)}(k)A_1 + [1 - \chi_{[100n\varepsilon,100(n+0.4)\varepsilon)}(k)]A_2,$$

where $\chi_{[100n\varepsilon,100(n+0.4)\varepsilon)}(k)$ is an indicator function. Here, $A_D(k)$ can be presented as (31) with

$$A_d = \begin{bmatrix} -0.038 & 0.024 \\ 0.042 & -0.062 \end{bmatrix},$$

A_1 and A_2 given by (60) and

$$a_1(k) = \begin{cases} 0.6, \ k \in [100n\varepsilon, 100(n+0.4)\varepsilon), \\ -0.4, \ k \in [100(n+0.4)\varepsilon, 100(n+1)\varepsilon), \end{cases},$$

$$a_2(k) = \begin{cases} -0.6, \ k \in [100n\varepsilon, 100(n+0.4)\varepsilon), \\ 0.4, \ k \in [100(n+0.4)\varepsilon, 100(n+1)\varepsilon), \end{cases}.$$

An explicit computation of $\rho_m(k)$, $m = 1, 2$ yields $|\rho_m(k)| \le \varepsilon h_\rho^{(m)} = 0.6\varepsilon$, $\forall k \in \mathbb{Z}_+$. We consider $\theta \in \{0, 0.01\}$, $\varepsilon = 0.05$. Note that with some simply calculations, it is easily shown that (9) holds for $\varepsilon = 0.05$. Verify the LMIs of Theorem 2 to obtain the maximal value d which preserves feasibility of the LMIs. We find the corresponding upper bounds d^* that guarantees the system's exponential stability for all $0 \le \varepsilon \le \varepsilon^*$:

$$\theta = 0, \ d^* = 28; \quad \theta = 0.01, \ d^* = 5.$$

We further provide simulations of system with a fixed $\varepsilon = 0.05$, $d = 28$ and an initial condition $x_i = [1, -0.5]^T$, $\forall i \in \mathbf{I}[-d, 0]$. The results are shown in Fig. 1. We see that the system state converges to the origin, which demonstrates the efficacy of the proposed method.

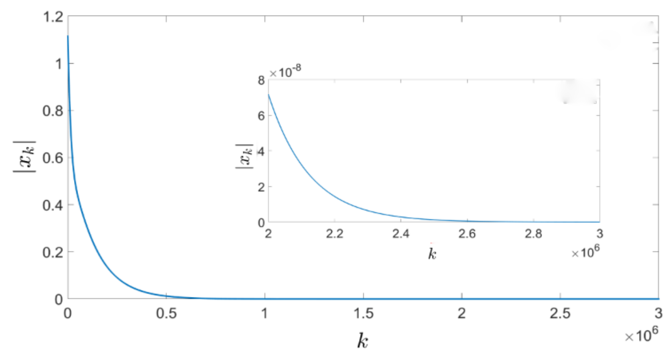

Fig. 1. The trajectory of the Euclidean norm of the system state x_i converges to the origin. Zoom plot of the graph is also provided.

IV. Conclusion

This paper presents a novel quantitative approach to averaging for stability of discrete-time systems with rapidly-varying periodic coefficients. By applying a novel system representation, state transformation and further employing a direct Lyapunov method, explicit LMI conditions for stability were derived. The LMIs provides upper bounds on the small parameters that preserve exponential stability of the original system. The method was extended to linear discrete systems with constant delay. Future work may include improvement of the method, its extension to time-varying delays and its control applications such as averaging-based control.

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