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# A Time-Delay Approach to Multi-variable Extremum Seeking with Measurement Noise

Xuefei Yang, Bowen Zhao and Emilia Fridman

*Abstract*— **For multi-variable static quadratic map, we present a time-delay approach to gradient-based extremum seeking (ES) with measurement noise, and provide a mean-square exponential ultimate boundedness (MSEUB) analysis. We consider the uncertain map where the Hessian matrix $H$ has a nominal known part and norm-bounded uncertainty, the extremum point belongs to a known ball, and the extremum value to a known interval. By applying a time-delay approach to the resulting stochastic ES system, we arrive at the neutral type time-delay system with stochastic perturbations. We further present the latter system as a retarded one and employ the variation of constants formula for the MSEUB analysis. Under the assumption that the upper bound of the 6th moment of the estimation error is a known arbitrarily large constant $L$, explicit condition in terms of simple scalar inequality depending on the bound $L$, tuning parameters and the intensity of measurement noise is established to guarantee the MSEUB analysis of the ES control systems. Example from the literature illustrates the efficiency of the new approach.**

## I. Introduction

ES is a model-free adaptive optimization control method which deals with dynamic problems where the input-output mapping relationship is unknown but its extremum values exist. ES control has attracted much attention and has been investigated from quite different aspects (see [1], [2], [3]). In the ES process, the measurements are usually noisy. For instance, the search path needs to be adjusted by measuring the output which is normally corrupted by measurement noise, this brings some difficulties in the stability analysis. To solve the difficulties caused by stochastic perturbations, some important results have been proposed for the research on the stochastic ES (see [4], [5], [6]). In [6], the authors studied the discrete-time ES with deterministic perturbations in the presence of stochastic noise via the stochastic approximation method. To guarantee the convergence of the algorithm, the boundedness of iteration sequence was assumed. The classical stochastic approximation methods and the general stochastic averaging theory provide the qualitative analysis only, and cannot quantitatively give the upper bounds on the parameters that preserve the stability.

Recently, a new constructive time-delay approach to the continuous-time averaging with efficient and quantitative bounds on the small parameter that ensures the stability was presented in [7]. Subsequently, this approach to averaging was successfully extended to discrete-time systems (see [8]) and applied for the quantitative stability analysis of deterministic ES algorithms by constructing appropriate Lyapunov-Krasovskii (L-K) functionals (see [9], [10]). Recently, a robust time-delay approach to ES by using the variation of constants formula was proposed in [11], which can greatly simplify the stability analysis and conditions compared to L-K method.

In this paper, we develop, for the first time, a time-delay approach to multi-variable ES with measurement noise via MSEUB analysis of the averaged system. We consider the uncertain map where the Hessian has a nominal known part and norm-bounded uncertainty, the extremum point belongs to a known ball, and the extremum value to a known interval. We first apply a time-delay transformation to the resulting stochastic ES system to get a neutral type time-delay system with stochastic perturbations, and then we further transform it to an averaged ODE stochastic perturbed model. Finally, we use the variation of constants formula to quantitatively analyze the MSEUB analysis of the averaged stochastic system (and thus of the original stochastic ES system). Assuming that the upper bound of the 6th moment of the estimation error is a known constant $L$ that can be arbitrarily large, explicit condition in terms of simple scalar inequality depending on the bound $L$, tuning parameters and the intensity of measurement noise is established to guarantee the MSEUB of the stochastic ES control systems. For the quantitative results restricted to 1D map, see the companion conference paper [12].

**Notation:** The notation $(\Omega, \mathscr{F}, \mathbb{P})$ refers to a complete probability space with its filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). The notation $B(t)$ refers to a (standard) one-dimensional Brownian motion defined on the probability space. The notations $|\cdot|$ and $\|\cdot\|$ refer to the usual Euclidean vector norm and the induced matrix 2 norm, respectively.

## II. A Time-Delay Approach to ES for Uncertain Map

Consider the multi-variable static map given by

$$\mathrm{d}y(t) = \left[Q^* + \tfrac{1}{2}\left(\theta(t)-\theta^*\right)^{\mathrm{T}} H \left(\theta(t)-\theta^*\right)\right]\mathrm{d}t + C\mathrm{d}B(t), \tag{1}$$

where $y(t) \in \mathbb{R}$ is the measurable output which is corrupted by Brownian motion $B(t) \in \mathbb{R}$, $\theta(t) \in \mathbb{R}^n$ is the vector input with initial value $\theta(0) \in \mathbb{R}^n$, $Q^* \in \mathbb{R}$, $\theta^* \in \mathbb{R}^n$. Here the

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Xuefei Yang and Bowen Zhao are with the Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin, China. yangxuefei@hit.edu.cn (X. Yang), 22b904058@stu.hit.edu.cn (B. Zhao). Emilia Fridman is with School of Electrical Engineering, Tel-Aviv University, Israel. emilia@tauex.tau.ac.il (E. Fridman).

scalar $C \geq 0$ is noise intensity, $H = H^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ is the Hessian matrix. Without loss of generality, we assume that $H > 0$ (or $H < 0$) meaning that the static map (1) has a minimum or a maximum value $Q^*$ at $\theta(t) = \theta^*$. Following [11], in order to derive efficient qualitative conditions, we assume that:

**A1** The extremum point $\theta^*$ to be sought is uncertain from a known interval $\theta_i^* \in [\underline{\theta}_i^*, \bar{\theta}_i^*]$, $i = 1,\ldots,n$ with $\sum_{i=1}^{n}\left(\bar{\theta}_i^* - \underline{\theta}_i^*\right)^2 = \sigma_0^2$.

**A2** The extremum value $Q^*$ is unknown, but it is subject to $|Q^*| \leqslant Q_{\mathrm{M}}^*$ with $Q_M^*$ being known.

**A3** The Hessian $H$ is uncertain and subject to $H = \bar{H} + \Delta H$ with $\bar{H} > 0$ (or $\bar{H} < 0$) being known and $\|\Delta H\| \leqslant \kappa$. Here $\kappa$ is a known scalar.

***Remark 1:*** In classical ES under stochastic noise (see [6]), $H$, $Q^*$ and $\theta^*$ in (1) are assumed to be unknown, where tuning parameters may be found from simulation only. Here we study a "grey box" model with Assumptions **A1-A3** and provide a quantitative analysis. There is a trade-off between the quantitative analysis with the plant information and the qualitative analysis without the model knowledge. For the "black box" model without Assumptions **A1-A3**, our method leads to qualitative results.

In order to derive less conservative results, following [13] we will first diagonalize the nominal part of the map by using an orthogonal transformation. Since $\bar{H} > 0$ (or $\bar{H} < 0$), there exists an orthogonal matrix $U \in \mathbb{R}^{n\times n}$ such that

$$U\bar{H}U^{\mathrm{T}} = \mathrm{diag}\left\{\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_n\right\} \triangleq \bar{\mathscr{H}}, \tag{2}$$

where $UU^{\mathrm{T}} = I$ and $\|U\| = 1$. Then we have

$$\mathscr{H} \triangleq UHU^{\mathrm{T}} = U\bar{H}U^{\mathrm{T}} + U\Delta HU^{\mathrm{T}} \triangleq \bar{\mathscr{H}} + \Delta\mathscr{H}. \tag{3}$$

From (2) and (3), (1) can be further reduced to

$$\mathrm{d}y(t) = \left[Q^* + \tfrac{1}{2}\left(\vartheta(t) - \vartheta^*\right)^{\mathrm{T}} \mathscr{H}\left(\vartheta(t) - \vartheta^*\right)\right]\mathrm{d}t + C\mathrm{d}B(t), \tag{4}$$

where

$$\vartheta(t) = U\theta(t),\ \vartheta^* = U\theta^*. \tag{5}$$

Define the real-time estimates $\hat{\theta}(t)$ and $\hat{\vartheta}(t)$ of $\theta^*$ and $\vartheta^*$, respectively, with the estimation errors:

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*,\ \tilde{\vartheta}(t) = \hat{\vartheta}(t) - \vartheta^*. \tag{6}$$

Design $\hat{\theta}(t) = U^{\mathrm{T}}\hat{\vartheta}(t)$, then from (5) and (6), we have

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta^* = U^{\mathrm{T}}\left(\hat{\vartheta}(t) - \vartheta^*\right) = U^{\mathrm{T}}\tilde{\vartheta}(t).$$

Since $U$ satisfies $\|U\| = 1$, we get

$$|\tilde{\theta}(t)| \leqslant \|U^{\mathrm{T}}\||\tilde{\vartheta}(t)| = |\tilde{\vartheta}(t)| \leqslant \|U\||\tilde{\theta}(t)| = |\tilde{\theta}(t)|,$$

namely, $|\tilde{\vartheta}(t)| = |\tilde{\theta}(t)|$. Then, it is sufficient to consider bounds on $\tilde{\vartheta}$-system.

For (4), the gradient-based ES algorithm is designed as

$$\vartheta(t) = \hat{\vartheta}(t) + S(t),\ \mathrm{d}\hat{\vartheta}(t) = KM(t)\cdot\mathrm{d}y(t) \tag{7}$$

with initial value $\hat{\vartheta}(0) \in \mathbb{R}^n$, where $S(t)$ and $M(t)$ are the dither signals satisfying

$$\begin{aligned} S(t) &= \left[a_1\sin(\omega_1 t), \cdots, a_n\sin(\omega_n t)\right]^{\mathrm{T}}, \\ M(t) &= \left[\tfrac{2}{a_1}\sin(\omega_1 t), \cdots, \tfrac{2}{a_n}\sin(\omega_n t)\right]^{\mathrm{T}}, \end{aligned} \tag{8}$$

in which $a_i$ are real numbers, $\omega_i \neq \omega_j \neq 0, i \neq j$ and $\omega_i/\omega_j$ are rational numbers. To be specific, we let

$$\omega_i = \tfrac{2\pi l_i}{\varepsilon},\ l_i \in \mathrm{N}_+,\ i = 1,\ldots,n. \tag{9}$$

The adaptation gain $K$ is chosen as

$$K = \mathrm{diag}\left\{k_1, k_2, \cdots, k_n\right\},\ i = 1,\ldots,n \tag{10}$$

such that $K\bar{\mathscr{H}}$ is Hurwitz (for instance, $K = kI_n$ with a scalar $k < 0$ or $> 0$). From (4), (6) and (7), we have

$$\begin{aligned} \mathrm{d}\tilde{\vartheta}(t) =& KM(t)Q^*\mathrm{d}t + \tfrac{1}{2}KM(t)\tilde{\vartheta}^{\mathrm{T}}(t)\mathscr{H}\tilde{\vartheta}(t)\mathrm{d}t \\ &+ KM(t)S^{\mathrm{T}}(t)\mathscr{H}\tilde{\vartheta}(t)\mathrm{d}t \\ &+ \tfrac{1}{2}KM(t)S^{\mathrm{T}}(t)\mathscr{H}S(t)\mathrm{d}t \\ &+ CKM(t)\mathrm{d}B(t),\ t \geqslant 0. \end{aligned} \tag{11}$$

In view of (3), system (11) can be rewritten as

$$\begin{aligned} \mathrm{d}\tilde{\vartheta}(t) =& KM(t)Q^*\mathrm{d}t + \tfrac{1}{2}KM(t)\tilde{\vartheta}^{\mathrm{T}}(t)\bar{\mathscr{H}}\tilde{\vartheta}(t)\mathrm{d}t \\ &+ KM(t)S^{\mathrm{T}}(t)\bar{\mathscr{H}}\tilde{\vartheta}(t)\mathrm{d}t \\ &+ \tfrac{1}{2}KM(t)S^{\mathrm{T}}(t)\bar{\mathscr{H}}S(t)\mathrm{d}t \\ &+ \omega(t)\mathrm{d}t + CKM(t)\mathrm{d}B(t),\ t \geqslant 0 \end{aligned} \tag{12}$$

with

$$\begin{aligned} \omega(t) = KM(t)&\left[\tfrac{1}{2}\tilde{\vartheta}^{\mathrm{T}}(t)\Delta\mathscr{H}\tilde{\vartheta}(t)\right. \\ &\left.+ S^{\mathrm{T}}(t)\Delta\mathscr{H}\tilde{\vartheta}(t) + \tfrac{1}{2}S^{\mathrm{T}}(t)\Delta\mathscr{H}S(t)\right]. \end{aligned} \tag{13}$$

For the initial value $\tilde{\vartheta}(0)$, solution of (12) is a stochastic process $\tilde{\vartheta}(t)$ with probability 1 satisfying (see [14])

$$\begin{aligned} \tilde{\vartheta}(t) = \tilde{\vartheta}(0) &+ \textstyle\int_0^t \left[KM(s)Q^*\right. \\ &\left.+ \tfrac{1}{2}KM(s)\tilde{\vartheta}^{\mathrm{T}}(s)\bar{\mathscr{H}}\tilde{\vartheta}(s)\right]\mathrm{d}s \\ &+ \textstyle\int_0^t KM(s)S^{\mathrm{T}}(s)\bar{\mathscr{H}}\tilde{\vartheta}(s)\mathrm{d}s \\ &+ \tfrac{1}{2}\textstyle\int_0^t KM(s)S^{\mathrm{T}}(s)\bar{\mathscr{H}}S(s)\mathrm{d}s \\ &+ \textstyle\int_0^t \omega(s)\mathrm{d}s + \int_0^t CKM(s)\mathrm{d}B(s),\ t \geqslant 0, \end{aligned} \tag{14}$$

where the last stochastic integral is in the sense of Itô type.

***Remark 2:*** Since $\theta(t) = U^{\mathrm{T}}\vartheta(t)$ and $\hat{\theta}(t) = U^{\mathrm{T}}\hat{\vartheta}(t)$, then from (7) it is easy to present the gradient-based ES algorithm for the original ES problem as follows:

$$\theta(t) = \hat{\theta}(t) + U^{\mathrm{T}}S(t),\ \mathrm{d}\hat{\theta}(t) = U^{\mathrm{T}}KM(t)\cdot\mathrm{d}y(t). \tag{15}$$

By using this algorithm and noting that $\tilde{\theta}(t) = U^{\mathrm{T}}\tilde{\vartheta}(t)$, we can obtain part (*ii*) in Theorem 1 below.

For the stability analysis of the ES system (12), inspired by [7], [12], we first apply the time-delay approach to averaging of (12). Let $\omega_i$ be defined in (9). Integrating from $t-\varepsilon$ to $t$ and dividing by $\varepsilon$ on both sides of equation (12), for $t \geqslant \varepsilon$ we have

$$\begin{aligned} \tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^t \mathrm{d}\tilde{\vartheta}(s) =& \tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^t KM(s)Q^*\mathrm{d}s \\ &+ \tfrac{1}{2\varepsilon}\textstyle\int_{t-\varepsilon}^t KM(s)\tilde{\vartheta}^{\mathrm{T}}(s)\bar{\mathscr{H}}\tilde{\vartheta}(s)\mathrm{d}s \\ &+ \tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^t KM(s)S^{\mathrm{T}}(s)\bar{\mathscr{H}}\tilde{\vartheta}(s)\mathrm{d}s \\ &+ \tfrac{1}{2\varepsilon}\textstyle\int_{t-\varepsilon}^t KM(s)S^{\mathrm{T}}(s)\bar{\mathscr{H}}S(s)\mathrm{d}s \\ &+ \tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^t \omega(s)\mathrm{d}s + \tfrac{1}{\varepsilon}\int_{t-\varepsilon}^t CKM(s)\mathrm{d}B(s). \end{aligned} \tag{16}$$

Denote

$$G(t) = \tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^t (s - t + \varepsilon)f(s)\mathrm{d}s,\ t \geqslant \varepsilon, \tag{17}$$

where $f$ is defined by

$$\begin{aligned} f(t) =& KM(t)\left[Q^* + \tfrac{1}{2}\tilde{\vartheta}^{\mathrm{T}}(t)\bar{\mathscr{H}}\tilde{\vartheta}(t)\right. \\ &\left.+ S^{\mathrm{T}}(t)\bar{\mathscr{H}}\tilde{\vartheta}(t) + \tfrac{1}{2}S^{\mathrm{T}}(t)\bar{\mathscr{H}}S(t)\right]. \end{aligned} \tag{18}$$

Then we can present the left-hand of (16) as

$$\begin{aligned} \tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^t \mathrm{d}\tilde{\vartheta}(s)\mathrm{d}t =& \mathrm{d}\left[\tilde{\vartheta}(t) - G(t)\right] \\ &+ \tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^t \omega(s)\mathrm{d}s\mathrm{d}t + \tfrac{1}{\varepsilon}\int_{t-\varepsilon}^t CKM(s)\mathrm{d}B(s)\mathrm{d}t \\ &- \omega(t)\mathrm{d}t - CKM(t)\mathrm{d}B(t),\ t \geqslant \varepsilon. \end{aligned} \tag{19}$$

For the first term on the right-hand side of (16), we get

$$\tfrac{1}{\varepsilon}\textstyle\int_{t-\varepsilon}^t KM(s)Q^*\mathrm{d}s = \tfrac{Q^*}{\varepsilon}K\mathrm{col}\left\{\tfrac{2}{a_i}\int_{t-\varepsilon}^t \sin\left(\tfrac{2\pi l_i}{\varepsilon}s\right)\mathrm{d}s\right\}_{i=1}^n = 0, \tag{20}$$

where $\int_{t-\varepsilon}^t \sin\left(\frac{2\pi l_i}{\varepsilon}s\right)\mathrm{d}s = 0$ $(i = 1,\ldots,n)$ is employed. Then system (12) can be further expressed as

$$\mathrm{d}\tilde{\vartheta}(t) = \left(f(t) + \omega(t)\right)\mathrm{d}t + CKM(t)\mathrm{d}B(t),\ t \geqslant 0. \tag{21}$$

By applying Itô formula (see Theorem 1.6.4 of [14]) and (21), we obtain

$$\begin{array}{l} \mathrm{d}\tilde{\vartheta}^{\mathrm{T}}(t)\bar{\mathscr{H}}\tilde{\vartheta}(t) = 2(f(t)+\omega(t))^{\mathrm{T}}\bar{\mathscr{H}}\tilde{\vartheta}(t)\mathrm{d}t \\ +C^2M^{\mathrm{T}}(t)K\bar{\mathscr{H}}KM(t)\mathrm{d}t + 2CM^{\mathrm{T}}(t)K\bar{\mathscr{H}}\tilde{\vartheta}(t)\mathrm{d}B(t) \end{array} \tag{22}$$

With (22) and $\int_{t-\varepsilon}^{t} KM(s)\mathrm{d}s = 0$, we can present the second term on the right-hand side of (16) as

$$\begin{array}{l} \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t} KM(s)\tilde{\vartheta}^{\mathrm{T}}(s)\bar{\mathscr{H}}\tilde{\vartheta}(s)\mathrm{d}s \\ = \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t} KM(s)\tilde{\vartheta}^{\mathrm{T}}(t)\bar{\mathscr{H}}\tilde{\vartheta}(t)\mathrm{d}s \\ -\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t} KM(s)\left[\tilde{\vartheta}^{\mathrm{T}}(t)\bar{\mathscr{H}}\tilde{\vartheta}(t) - \tilde{\vartheta}^{\mathrm{T}}(s)\bar{\mathscr{H}}\tilde{\vartheta}(s)\right]\mathrm{d}s \\ = -Y_1(t) - Y_2(t) - Y_3(t),\ t \geqslant \varepsilon, \end{array} \tag{23}$$

where

$$\begin{array}{l} Y_1(t) = \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\int_{s}^{t} KM(s)(f(\tau)+\omega(\tau))^{\mathrm{T}}\bar{\mathscr{H}}\tilde{\vartheta}(\tau)\mathrm{d}\tau\mathrm{d}s, \\ Y_2(t) = \frac{C}{\varepsilon}\int_{t-\varepsilon}^{t}\int_{s}^{t} KM(s)M^{\mathrm{T}}(\tau)K\bar{\mathscr{H}}\tilde{\vartheta}(\tau)\mathrm{d}B(\tau)\mathrm{d}s, \\ Y_3(t) = \frac{C^2}{2\varepsilon}\int_{t-\varepsilon}^{t}\int_{s}^{t} KM(s)M^{\mathrm{T}}(\tau)K\bar{\mathscr{H}}KM(\tau)\mathrm{d}\tau\mathrm{d}s. \end{array} \tag{24}$$

Note that $\int_{t-\varepsilon}^{t} M(s)S^{\mathrm{T}}(s)\mathrm{d}s = \varepsilon I_n$ since

$$\int_{t-\varepsilon}^{t}\frac{2a_j}{a_i}\sin\left(\frac{2\pi l_i}{\varepsilon}s\right)\sin\left(\frac{2\pi l_j}{\varepsilon}s\right)\mathrm{d}s = \begin{cases} \varepsilon,\ i=j, \\ 0,\ i\neq j. \end{cases}$$

Then by (21) we can calculate the third term on the right-hand side of (16) as

$$\begin{array}{l} \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} KM(s)S^{\mathrm{T}}(s)\bar{\mathscr{H}}\tilde{\vartheta}(s)\mathrm{d}s \\ = \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} KM(s)S^{\mathrm{T}}(s)\bar{\mathscr{H}}\tilde{\vartheta}(t)\mathrm{d}s \\ -\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} KM(s)S^{\mathrm{T}}(s)\bar{\mathscr{H}}\left[\tilde{\vartheta}(t)-\tilde{\vartheta}(s)\right]\mathrm{d}s \\ = K\bar{\mathscr{H}}\tilde{\vartheta}(t) - Y_4(t) - Y_5(t), \end{array} \tag{25}$$

where

$$\begin{array}{l} Y_4(t) = \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\int_{s}^{t} KM(s)S^{\mathrm{T}}(s)\bar{\mathscr{H}}(f(\tau)+\omega(\tau))\mathrm{d}\tau\mathrm{d}s, \\ Y_5(t) = \frac{C}{\varepsilon}\int_{t-\varepsilon}^{t}\int_{s}^{t} KM(s)S^{\mathrm{T}}(s)\bar{\mathscr{H}}KM(\tau)\mathrm{d}B(\tau)\mathrm{d}s. \end{array} \tag{26}$$

For the fourth term on the right-hand side of (16), we have

$$\begin{array}{l} \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t} KM(s)S^{\mathrm{T}}(s)\bar{\mathscr{H}}S(s)\mathrm{d}s \\ = \frac{1}{\varepsilon}K\mathrm{col}\left\{\sum_{i=1}^{n}\frac{a_i^2\bar{h}_i}{a_k}\int_{t-\varepsilon}^{t}\sin^2\left(\frac{2\pi l_i}{\varepsilon}s\right)\sin\left(\frac{2\pi l_k}{\varepsilon}s\right)\mathrm{d}s\right\}_{k=1}^{n} = 0, \end{array} \tag{27}$$

where $\int_{t-\varepsilon}^{t}\sin^2\left(\frac{2\pi l_i}{\varepsilon}s\right)\sin\left(\frac{2\pi l_k}{\varepsilon}s\right)\mathrm{d}s = 0$ has been used. Then employing (19), (20), (23), (25) and (27), we have

$$\begin{array}{l} \mathrm{d}[\tilde{\vartheta}(t) - G(t)] = K\bar{\mathscr{H}}\tilde{\vartheta}(t)\mathrm{d}t - \sum_{i=1}^{5} Y_i(t)\mathrm{d}t \\ +\omega(t)\mathrm{d}t + CKM(t)\mathrm{d}B(t),\ t\geqslant\varepsilon. \end{array} \tag{28}$$

The solution of system (16) is also a solution of the time-delay system (28). Thus, the stability of (16) can be guaranteed by the stability of (28). Finally, we set

$$z(t) = \tilde{\vartheta}(t) - G(t), \tag{29}$$

system (28) can be transformed to

$$\mathrm{d}z(t) = K\bar{\mathscr{H}}z(t)\mathrm{d}t + \bar{\omega}(t)\mathrm{d}t + CKM(t)\mathrm{d}B(t),\ t\geqslant\varepsilon \tag{30}$$

with

$$\bar{\omega}(t) = K\bar{\mathscr{H}}G(t) - \sum_{i=1}^{5} Y_i(t) + \omega(t). \tag{31}$$

If $\tilde{\vartheta}(t)$ (and thus $z(t)$) is of the order of $\mathscr{O}(1)$ in the mean square sense, the terms $G(t), Y_i(t)\,(i=1,3,4)$ defined in (17), (24) and (26), respectively, are of the order of $\mathscr{O}(\varepsilon^2)$, $\omega(t)$ defined in (13) is of the order of $\mathscr{O}(\kappa^2)$ and the terms $Y_i(t)\,(i=2,5)$ defined in (24) and (26), respectively, are of the order of $\mathscr{O}(C^2)$ in the mean square sense. Therefore, $\bar{\omega}(t)$ is of the order of $\mathscr{O}(\max\{\varepsilon^2,\kappa^2,C^2\})$ in the mean square sense. Thus, for small $\varepsilon>0$, $\kappa>0$ and $C>0$, system (30) can be regarded as a perturbation of the linear system $\mathrm{d}z(t) = K\bar{\mathscr{H}}z(t)\mathrm{d}t$, which is exponentially stable since $K\bar{\mathscr{H}}$ is Hurwitz. Via (29), the resulting bound on $|z|$ will lead to the bound on $\tilde{\vartheta}$ : $|\tilde{\vartheta}| \leq |z| + |G|$. The bound on $z$ will be found by utilizing the variation of constants formula to (30). For future use, we denote

$$\begin{array}{l} K_{\mathrm{M}} = \|K\| = \max_{i=1,\ldots,n}|k_i|,\ \bar{H}_{\mathrm{M}} = \left\|\bar{\mathscr{H}}\right\| = \max_{i=1,\ldots,n}\left|\bar{h}_i\right|, \\ \Theta_{\mathrm{m}} = \min_{i=1,\ldots,n}\left|k_i\bar{h}_i\right|,\ \Theta_{\mathrm{M}} = \max_{i=1,\ldots,n}\left|k_i\bar{h}_i\right|, \end{array} \tag{32}$$

and

$$\begin{array}{l} \Delta_f = 3\sum_{i=1}^{n}\frac{4k_i^2}{a_i^2}\left[\frac{L^{2/3}\bar{H}_{\mathrm{M}}^2}{4} + \sigma^2\bar{H}_{\mathrm{M}}^2\sum_{i=1}^{n}a_i^2\right. \\ \left. + \left(Q_{\mathrm{M}}^* + \frac{\bar{H}_{\mathrm{M}}}{2}\sum_{i=1}^{n}a_i^2\right)^2\right], \\ \Delta_\omega = \frac{3}{4}\sum_{i=1}^{n}\frac{4k_i^2}{a_i^2}\left[L^{2/3} + 4\sigma^2\sum_{i=1}^{n}a_i^2 + \left(\sum_{i=1}^{n}a_i^2\right)^2\right], \\ \Delta_{Y_1} = 2\left(\sum_{i=1}^{n}\frac{4k_i^2}{a_i^2}\right)\left(\sum_{i=1}^{n}\frac{4k_i^2\bar{h}_i^2}{a_i^2}\right)\left[4\sigma^2 {Q_{\mathrm{M}}^*}^2 + \left(\bar{H}_{\mathrm{M}}^2 + \frac{3\kappa^2}{4}\right)\right. \\ \left.\times\left(L + 4\sum_{i=1}^{n}a_i^2L^{2/3} + \sigma^2\left(\sum_{i=1}^{n}a_i^2\right)^2\right)\right], \\ \Delta_{Y_2} = \frac{\sigma^2}{2}\left(\sum_{i=1}^{n}\frac{4k_i^2}{a_i^2}\right)\left(\sum_{i=1}^{n}\frac{4k_i^2\bar{h}_i^2}{a_i^2}\right), \\ \Delta_{Y_3} = \frac{1}{16}\left(\sum_{i=1}^{n}\frac{4k_i^2}{a_i^2}\right)^2\left(\sum_{i=1}^{n}\frac{4k_i^2\bar{h}_i^2}{a_i^2}\right), \\ \Delta_{Y_4} = \frac{2\bar{H}_{\mathrm{M}}^2}{3}\sum_{i=1}^{n}\frac{4k_i^2}{a_i^2}\sum_{i=1}^{n}a_i^2\left(\Delta_f + \kappa^2\Delta_\omega\right), \\ \Delta_{Y_5} = \frac{1}{2}\left(\sum_{i=1}^{n}\frac{4k_i^2}{a_i^2}\right)\left(\sum_{i=1}^{n}\frac{4k_i^2\bar{h}_i^2}{a_i^2}\right)\sum_{i=1}^{n}a_i^2, \\ \bar{\Delta}(\varepsilon,C) = 2\varepsilon^2\Theta_{\mathrm{M}}^2\Delta_f + 6\kappa^2\Delta_\omega + 2\varepsilon^2\left(\Delta_{Y_1} + 3\Delta_{Y_4}\right) \\ + 6\varepsilon C^2\left(\Delta_{Y_2} + \varepsilon C^2\Delta_{Y_3} + \Delta_{Y_5}\right). \end{array} \tag{33}$$

*Theorem 1:* Let **A1-A3** be satisfied and $K\bar{\mathscr{H}}$ be Hurwitz, where $K$ is the adaptation gain given by (10). Suppose system (12) has a unique solution on $[0,\infty)$ and $\mathbb{E}|\tilde{\vartheta}(t)|^6 \leqslant L < \infty$, $t\geqslant 0$ with $L$ being an arbitrarily large known constant. Given $\sigma_0$ in **A1** and $\sigma$ satisfying $0<\sigma_0<\sigma<L^{1/6}$, consider the closed-loop system (12) with $\omega_i$ given by (9) and the initial condition $\mathbb{E}|\tilde{\vartheta}(0)|^2 \leqslant \sigma_0^2$. Given tuning parameters $a_i$ $(i=1,\ldots,n)$ and $\varepsilon^*, C^*>0$, let the following inequality holds:

$$\begin{array}{l} \Phi = 12\sigma_0^2 + 4\varepsilon^*\left[6\sigma\left(\Delta_f^{1/2} + \kappa\Delta_\omega^{1/2}\right) + 3C^{*2}\sum_{i=1}^{n}\frac{4k_i^2}{a_i^2}\right. \\ \left. + \frac{7\varepsilon^*\Delta_f}{6}\right] + \frac{6\bar{\Delta}(\varepsilon^*,C^*)}{\Theta_{\mathrm{m}}^2} + \frac{3C^{*2}}{\Theta_{\mathrm{m}}}\sum_{i=1}^{n}\frac{4k_i^2}{a_i^2} - \sigma^2 < 0, \end{array} \tag{34}$$

where $\Theta_{\mathrm{m}}$ is given by (32) and $\bar{\Delta}(\varepsilon,C)$, $\Delta_f$, $\Delta_\omega$ are given by (33). Then for all $\varepsilon\in(0,\varepsilon^*]$ and $C\in[0,C^*]$ , the following holds:

$(i)$ The solution of (12) satisfies

$$\begin{array}{l} \mathbb{E}|\tilde{\vartheta}(t)|^2 < \mathbb{E}|\tilde{\vartheta}(0)|^2 + 2\varepsilon\sigma\left(\Delta_f^{1/2} + \kappa\Delta_\omega^{1/2}\right) \\ + \varepsilon C^2\sum_{i=1}^{n}\frac{4k_i^2}{a_i^2} < \sigma^2,\ t\in[0,\varepsilon], \\ \mathbb{E}|\tilde{\vartheta}(t)|^2 < 12\mathrm{e}^{-2\Theta_{\mathrm{m}}(t-\varepsilon)}\mathbb{E}|\tilde{\vartheta}(0)|^2 + 4\varepsilon\mathrm{e}^{-2\Theta_{\mathrm{m}}(t-\varepsilon)} \\ \times\left[6\sigma\left(\Delta_f^{1/2} + \kappa\Delta_\omega^{1/2}\right) + 3C^2\sum_{i=1}^{n}\frac{4k_i^2}{a_i^2} + \varepsilon\Delta_f\right] \\ + \frac{6\bar{\Delta}(\varepsilon,C)}{\Theta_{\mathrm{m}}^2} + \frac{3C^2}{\Theta_{\mathrm{m}}}\sum_{i=1}^{n}\frac{4k_i^2}{a_i^2} + \frac{2\varepsilon^2\Delta_f}{3} < \sigma^2,\ t\geqslant\varepsilon, \end{array} \tag{35}$$

meaning that the ball

$$\mathbb{E}|\tilde{\vartheta}|^2 \leqslant \frac{6\bar{\Delta}(\varepsilon,C)}{\Theta_{\mathrm{m}}^2} + \frac{3C^2}{\Theta_{\mathrm{m}}}\sum_{i=1}^{n}\frac{4k_i^2}{a_i^2} + \frac{2\varepsilon^2\Delta_f}{3} \tag{36}$$

is exponential attractive with a decay rate $\delta = \Theta_m$.

($ii$) Consider $\hat{\theta}(t) = U^{\mathrm{T}}\hat{\vartheta}(t)$ and $\theta^* = U^{\mathrm{T}}\vartheta^*$, where $\hat{\vartheta}(t)$ is defined by (7). Then $\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*$ satisfies (35) and (36) with $\tilde{\vartheta}(t)$ replaced by $\tilde{\theta}(t)$.

*Proof 1:* See Appendix.

***Remark** 3:* Given parameters $L$, $\sigma_0$, $\sigma$, $a_i$ and $k_i$, (34) is always feasible for small enough $\varepsilon^*$ and $C^*$. To achieve the convergence of the ES algorithm and obtain the ultimate bound (UB) of $\tilde{\vartheta}(t)$ (also $\tilde{\theta}(t)$), as done in [15], [6] and [12], we introduced the condition that $\mathbb{E}|\tilde{\vartheta}(t)|^6$ is a priori bounded by a known constant $L$. The imposed bound $L$ is only required to be finite and can be arbitrarily large. In this sense, Theorem 1 guarantees semi-global convergence for small enough $\varepsilon^*$ and $C^*$. The introduced boundedness assumption is realistic for practical applications (see [15], [6]). In addition, Though that $\varepsilon^*$ and $C^*$ decrease as $L$ increases via (34), we can still achieve not too small $\varepsilon^*$ and $C^*$ by choosing suitable $a_i$, and $k_i$ ($i = 1, \ldots, n$) when $L$ is large enough. This has been illustrated in Example below and the one in [12].

***Remark** 4:* We give a discussion about the effect of tuning parameters on $\varepsilon^*$, $C^*$, the decay rate $\delta$ and UB. For simplicity, we let $k_i = k$ ($i = 1, \ldots, n$). For given $L$ (large enough), $a_i$ and $\sigma > \sigma_0 > 0$, it is clear that $\Phi$ in (34) is an increasing function w.r.t $|k|$, $\varepsilon^*$ and $C^*$. Therefore, $\varepsilon^*$ and $C^*$ decrease as $|k|$ increases. On the other hand, the decay rate $\delta = \Theta_m = \min_{i=1,\ldots,n} |k\bar{h}_i|$ increases as $|k|$ increases. So we can balance $\delta$, $\varepsilon^*$ and $C^*$ by adjusting the gain $K$. In addition, by using the similar arguments in Remark 3 of [11], we can find the relatively small UB for given large enough $L$ and the other available parameters.

## III. Example

Following [3], we apply our theory to the source seeking example in which the scalar output is subject to measurement noise (without delays). The noisy output function $y$ satisfies (1) with

$$Q^* = 1,\ H = \begin{bmatrix} -2 & -2 \\ -2 & -4 \end{bmatrix},\ \theta^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{37}$$

We can get an orthogonal matrix

$$U = \begin{bmatrix} 0.5257 & 0.8507 \\ -0.8507 & 0.5257 \end{bmatrix} \tag{38}$$

such that

$$\bar{H} = UHU^{\mathrm{T}} = \begin{bmatrix} -5.2361 & 0 \\ 0 & -0.7639 \end{bmatrix} \tag{39}$$

If $H$ is uncertain with $\|\Delta H\| \leqslant \kappa$, we choose $\kappa = 0.003$. We let $L = 100$ and select the tuning parameters of gradient-based ES algorithm (15) as

$$k_1 = 0.2 \cdot 10^{-3}, k_2 = 0.1 \cdot 10^{-2},\ a_1 = a_2 = 0.5. \tag{40}$$

To calculate UB, we choose $\rho = \beta = 10^{-3}$ following Remark 2 in [11]. The results that follow from Theorem 1 for known ($\kappa = 0$) and uncertain ($\kappa = 0.003$) $H$ are shown in Table I, in which UB corresponds to $\varepsilon = \varepsilon^*$.

Moreover, if $L = 10000$, we choose

$$k_1 = 0.2 \cdot 10^{-4}, k_2 = 0.1 \cdot 10^{-3},\ a_1 = a_2 = 0.5, \tag{41}$$

the results that follow from Theorem 1 for uncertain ($\kappa = 0.001$) $H$ are shown in Table II.

TABLE I

VALUES OF $\delta$, $C^*$, $\varepsilon^*$ AND UB IN VECTOR SYSTEMS UNDER (40)

| ES: sine wave | $\sigma_0$ | $\sigma$ | $\delta$ | $C^*$ | $\varepsilon^*$ | UB |
|---|---|---|---|---|---|---|
| Known $H$ | 0.5 | 1.84 | 0.0008 | 0.1 | 0.0366 | 0.2205 |
| Uncertain $H$ | 0.5 | 1.84 | 0.0008 | 0.1 | 0.0228 | 0.2851 |

TABLE II

VALUES OF $\delta$, $C^*$, $\varepsilon^*$ AND UB IN VECTOR SYSTEMS UNDER (41)

| ES: sine wave | $\sigma_0$ | $\sigma$ | $\delta$ | $C^*$ | $\varepsilon^*$ | UB |
|---|---|---|---|---|---|---|
| Uncertain $H$ | 0.5 | 1.85 | $0.8 \cdot 10^{-4}$ | 0.1 | 0.0190 | 0.3885 |

## IV. Conclusions

This paper developed a time-delay approach to multivariable ES corrupted by white noise for static quadratic maps. Given a known arbitrarily large bound $L$ on the 6th moment of estimation error, explicit condition in terms of simple inequality depending on the bound $L$, tuning parameters and noise intensity was established to guarantee the MSEUB of the ES control systems for uncertain static maps. The quantitative UB of seeking error was also presented. Future works may include the study on ES with stochastic perturbations for non-quadratic maps and dynamic maps.

## Appendix

Since $\hat{\theta}(t) = U^{\mathrm{T}}\hat{\vartheta}(t)$, it is not difficult to obtain part (ii) from part (i). Thus, we just need to prove part (i). The proof is divided into three parts. (A) First, under (42) and (43) below, we present the upper bound of $\mathbb{E}|\tilde{\vartheta}(t)|^2$ for $t \in [0, \varepsilon]$ as well as $\mathbb{E}|\bar{\omega}(s)|^2$ for $t \geq \varepsilon$; (B) Second, we show the practical stability of $z$-system in (30) (and thus $\tilde{\vartheta}$-system in (12)). (C) Third, we show the availability of (43) by contradiction.

*Proof of the part A.* From the assumption that $\mathbb{E}|\tilde{\vartheta}(t)|^6 \leqslant L < \infty,\ t \geqslant 0$ and Hölder$'s$ inequality, we have

$$\mathbb{E}|\tilde{\vartheta}(t)|^2 \leq L^{1/3},\ \mathbb{E}|\tilde{\vartheta}(t)|^4 \leq L^{2/3},\ t \geq 0. \tag{42}$$

We assume that

$$\mathbb{E}|\tilde{\vartheta}(t)|^2 < \sigma^2,\ t \geqslant 0 \tag{43}$$

with $0 < \sigma < L^{1/6}$. By **A2**, (8), (10) and (32), it follows from (18) that

$$\begin{aligned} |f(t)| &\leqslant \sqrt{\textstyle\sum_{i=1}^n \frac{4k_i^2}{a_i^2}} \left[ Q_M^* + \frac{\bar{H}_{\mathrm{M}}}{2} \left( |\tilde{\vartheta}(t)| + \sqrt{\textstyle\sum_{i=1}^n a_i^2} \right)^2 \right] \\ &= \sqrt{\textstyle\sum_{i=1}^n \frac{4k_i^2}{a_i^2}} \Big[ Q_M^* + \frac{\bar{H}_{\mathrm{M}}}{2} \big( |\tilde{\vartheta}(t)|^2 \\ &\quad + 2\sqrt{\textstyle\sum_{i=1}^n a_i^2} |\tilde{\vartheta}(t)| + \textstyle\sum_{i=1}^n a_i^2 \big) \Big],\ t \geqslant 0. \end{aligned} \tag{44}$$

Via (42)-(44), we further have

$$\begin{aligned} \mathbb{E}|f(t)|^2 &\leqslant \textstyle\sum_{i=1}^n \frac{4k_i^2}{a_i^2} \Big[ \frac{3\bar{H}_{\mathrm{M}}^2}{4} \mathbb{E}|\tilde{\vartheta}(t)|^4 \\ &\quad + 3\bar{H}_{\mathrm{M}}^2 \textstyle\sum_{i=1}^n a_i^2 \mathbb{E}|\tilde{\vartheta}(t)|^2 + 3\left( Q_{\mathrm{M}}^* + \frac{\bar{H}_{\mathrm{M}}}{2} \textstyle\sum_{i=1}^n a_i^2 \right)^2 \Big] \\ &< 3\textstyle\sum_{i=1}^n \frac{4k_i^2}{a_i^2} \Big[ \frac{L^{2/3}\bar{H}_{\mathrm{M}}^2}{4} + \sigma^2 \bar{H}_{\mathrm{M}}^2 \textstyle\sum_{i=1}^n a_i^2 \\ &\quad + \left( Q_{\mathrm{M}}^* + \frac{\bar{H}_{\mathrm{M}}}{2} \textstyle\sum_{i=1}^n a_i^2 \right)^2 \Big] = \Delta_f,\ t \geqslant 0, \end{aligned} \tag{45}$$

where $\Delta_f$ is given by (33). Noting from **A3**, (3) and $U^{\mathrm{T}}U = I$ that

$$\|\Delta\mathscr{H}\| = \sqrt{\lambda_{\max}(U\Delta HU^{\mathrm{T}}U\Delta HU^{\mathrm{T}})} = \|\Delta H\| \leqslant \kappa, \tag{46}$$

then via (8), (13) and (46), we get

$$
\begin{aligned}
|\omega(t)| &\leqslant |KM(t)|\left[\tfrac{1}{2}|\tilde{\vartheta}(t)|^2\|\Delta\mathscr{H}\| + |S(t)||\tilde{\vartheta}(t)|\|\Delta\mathscr{H}\| + \tfrac{1}{2}|S(t)|^2\|\Delta\mathscr{H}\|\right] \\
&\leqslant \frac{\kappa}{2}\sqrt{\sum_{i=1}^n \frac{4k_i^2}{a_i^2}}\left(|\tilde{\vartheta}(t)|^2 + 2\sqrt{\sum_{i=1}^n a_i^2}|\tilde{\vartheta}(t)| + \sum_{i=1}^n a_i^2\right),
\end{aligned} \tag{47}
$$

by which and (42)-(43), we further have for $t \geqslant 0$

$$
\begin{aligned}
\mathbb{E}|\omega(t)|^2 &\leqslant \frac{3\kappa^2}{4}\sum_{i=1}^n \frac{4k_i^2}{a_i^2}\left[\mathbb{E}|\tilde{\vartheta}(t)|^4 + 4\sum_{i=1}^n a_i^2\mathbb{E}|\tilde{\vartheta}(t)|^2 + \left(\sum_{i=1}^n a_i^2\right)^2\right] < \kappa^2\Delta_\omega,
\end{aligned} \tag{48}
$$

where $\Delta_\omega$ is given by (33). From Theorem 1.5.8 of [14], there holds

$$
\mathbb{E}\int_0^t M^{\mathrm{T}}(s)K\tilde{\vartheta}(s)\,\mathrm{d}B(s) = 0. \tag{49}
$$

By applying Itô formula (see Theorem 1.6.4 of [14]), Hölder's inequality, (43), (45), (48) and (49), we get the upper bound of the solution $\tilde{\vartheta}(t),\ t \in [0,\varepsilon]$ of (12) (also (21)) in the mean-square sense as follows

$$
\begin{aligned}
\mathbb{E}|\tilde{\vartheta}(t)|^2 &\leqslant \mathbb{E}|\tilde{\vartheta}(0)|^2 + 2\mathbb{E}\int_0^t |f(s)||\tilde{\vartheta}(s)|\mathrm{d}s + 2\mathbb{E}\int_0^t |\omega(s)||\tilde{\vartheta}(s)|\mathrm{d}s + C^2\mathbb{E}\int_0^t |KM(s)|^2\,\mathrm{d}s \\
&\leqslant \mathbb{E}|\tilde{\vartheta}(0)|^2 + 2\left[\mathbb{E}\int_0^t |f(s)|^2\,\mathrm{d}s\right]^{1/2}\left[\mathbb{E}\int_0^t|\tilde{\vartheta}(s)|^2\mathrm{d}s\right]^{1/2} \\
&\quad + 2\left[\mathbb{E}\int_0^t |\omega(s)|^2\,\mathrm{d}s\right]^{1/2}\left[\mathbb{E}\int_0^t|\tilde{\vartheta}(s)|^2\mathrm{d}s\right]^{1/2} + \varepsilon C^2\sum_{i=1}^n \frac{4k_i^2}{a_i^2} \\
&< \mathbb{E}|\tilde{\vartheta}(0)|^2 + 2\varepsilon\sigma\left(\Delta_f^{1/2} + \kappa\Delta_\omega^{1/2}\right) + \varepsilon C^2\sum_{i=1}^n \frac{4k_i^2}{a_i^2}.
\end{aligned} \tag{50}
$$

The first inequality in (35) follows from (50) since $\Phi < 0$ in (34) implies $\sigma_0^2 + 2\varepsilon^*\sigma(\Delta_f^{1/2} + \kappa\Delta_\omega^{1/2}) + \varepsilon^* C^{*2}\sum_{i=1}^n \frac{4k_i^2}{a_i^2} < \sigma^2$.

Next, we find the upper bound of $\mathbb{E}|\bar{\omega}(s)|^2$ for $t \geq \varepsilon$. From (17) and (44), we get

$$
\begin{aligned}
\mathbb{E}|G(t)|^2 &= \mathbb{E}\left|\frac{1}{\varepsilon}\int_{t-\varepsilon}^t (s-t+\varepsilon)f(s)\,\mathrm{d}s\right|^2 \\
&\leqslant \frac{1}{\varepsilon}\int_{t-\varepsilon}^t (s-t+\varepsilon)^2\mathbb{E}|f(s)|^2\,\mathrm{d}s < \frac{\varepsilon^2\Delta_f}{3},\ t \geqslant \varepsilon,
\end{aligned} \tag{51}
$$

then

$$
\mathbb{E}\left|K\bar{\mathscr{H}}G(t)\right|^2 \leqslant \left\|K\bar{\mathscr{H}}\right\|^2\mathbb{E}|G(t)|^2 < \frac{\varepsilon^2\Theta_{\mathrm{M}}^2\Delta_f}{3},\ t \geqslant \varepsilon. \tag{52}
$$

where $\Theta_M$ is given by (32). By (18), (32), (42), (43) and using Hölder's inequality, we obtain

$$
\begin{aligned}
&\mathbb{E}\left|\int_s^t KM(s)f^{\mathrm{T}}(\tau)\mathscr{H}\tilde{\vartheta}(\tau)\,\mathrm{d}\tau\right|^2 \\
&\leqslant 4\sum_{i=1}^n \frac{4k_i^2}{a_i^2}\mathbb{E}\Big[\left(\int_s^t Q_M^*\left|K\bar{\mathscr{H}}M(\tau)\right||\tilde{\vartheta}(\tau)|\mathrm{d}\tau\right)^2 + \frac{1}{4}\left(\int_s^t \left|K\bar{\mathscr{H}}M(\tau)\right|\left\|\mathscr{H}\right\||\tilde{\vartheta}(\tau)|^3\mathrm{d}\tau\right)^2 \\
&\quad + \left(\int_s^t |S(\tau)|\left|K\bar{\mathscr{H}}M(\tau)\right|\left\|\mathscr{H}\right\||\tilde{\vartheta}(\tau)|^2\mathrm{d}\tau\right)^2 + \frac{1}{4}\left(\int_s^t |S(\tau)|^2\left|K\bar{\mathscr{H}}M(\tau)\right|\left\|\mathscr{H}\right\||\tilde{\vartheta}(\tau)|\mathrm{d}\tau\right)^2\Big] \\
&\leqslant 4\left(\sum_{i=1}^n \frac{4k_i^2}{a_i^2}\right)\left(\sum_{i=1}^n \frac{4k_i^2\bar{h}_i^2}{a_i^2}\right)\Big[Q_{\mathrm{M}}^{*2}\mathbb{E}\left(\int_s^t|\tilde{\vartheta}(\tau)|\mathrm{d}\tau\right)^2 \\
&\quad + \frac{\bar{H}_{\mathrm{M}}^2}{4}\mathbb{E}\left(\int_s^t|\tilde{\vartheta}(\tau)|^3\mathrm{d}\tau\right)^2 + \bar{H}_{\mathrm{M}}^2\sum_{i=1}^n a_i^2\mathbb{E}\left(\int_s^t|\tilde{\vartheta}(\tau)|^2\mathrm{d}\tau\right)^2 + \frac{\bar{H}_{\mathrm{M}}^2}{4}\left(\sum_{i=1}^n a_i^2\right)^2\mathbb{E}\left(\int_s^t|\tilde{\vartheta}(\tau)|\mathrm{d}\tau\right)^2\Big] \\
&\leqslant 4\left(\sum_{i=1}^n \frac{4k_i^2}{a_i^2}\right)\left(\sum_{i=1}^n \frac{4k_i^2\bar{h}_i^2}{a_i^2}\right)(t-s) \\
&\quad \times\Big[Q_{\mathrm{M}}^{*2}\int_s^t\mathbb{E}|\tilde{\vartheta}(\tau)|^2\mathrm{d}\tau + \frac{\bar{H}_{\mathrm{M}}^2}{4}\int_s^t\mathbb{E}|\tilde{\vartheta}(\tau)|^6\mathrm{d}\tau + \bar{H}_{\mathrm{M}}^2\sum_{i=1}^n a_i^2 \\
&\quad \times\int_s^t\mathbb{E}|\tilde{\vartheta}(\tau)|^4\mathrm{d}\tau + \frac{\bar{H}_{\mathrm{M}}^2}{4}\left(\sum_{i=1}^n a_i^2\right)^2\int_s^t\mathbb{E}|\tilde{\vartheta}(\tau)|^2\mathrm{d}\tau\Big] \\
&< 4\left(\sum_{i=1}^n \frac{4k_i^2}{a_i^2}\right)\left(\sum_{i=1}^n \frac{4k_i^2\bar{h}_i^2}{a_i^2}\right)(t-s)^2\Big[\sigma^2 Q_{\mathrm{M}}^{*2} + \bar{H}_{\mathrm{M}}^2 \\
&\quad \times\left(\frac{L}{4} + \sum_{i=1}^n a_i^2L^{2/3} + \frac{\sigma^2}{4}\left(\sum_{i=1}^n a_i^2\right)^2\right)\Big],\ 0 \leqslant s \leqslant t.
\end{aligned} \tag{53}
$$

By (13), (32), (42), (43) and Hölder's inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left|\int_s^t KM(s)\omega^{\mathrm{T}}(\tau)\mathscr{H}\tilde{\vartheta}(\tau)\,\mathrm{d}\tau\right|^2 \\
&\leqslant 3\sum_{i=1}^n \frac{4k_i^2}{a_i^2}\Big[\frac{\kappa^2}{4}\sum_{i=1}^n \frac{4k_i^2\bar{h}_i^2}{a_i^2}\mathbb{E}\left(\int_s^t|\tilde{\vartheta}(\tau)|^3\mathrm{d}\tau\right)^2 + \kappa^2\sum_{i=1}^n a_i^2\sum_{i=1}^n \frac{4k_i^2\bar{h}_i^2}{a_i^2}\mathbb{E}\left(\int_s^t|\tilde{\vartheta}(\tau)|^2\mathrm{d}\tau\right)^2 \\
&\quad + \frac{\kappa^2}{4}\left(\sum_{i=1}^n a_i^2\right)^2\sum_{i=1}^n \frac{4k_i^2\bar{h}_i^2}{a_i^2}\mathbb{E}\left(\int_s^t|\tilde{\vartheta}(\tau)|\mathrm{d}\tau\right)^2\Big] \\
&\leqslant 3\kappa^2\left(\sum_{i=1}^n \frac{4k_i^2}{a_i^2}\right)\left(\sum_{i=1}^n \frac{4k_i^2\bar{h}_i^2}{a_i^2}\right)(t-s) \\
&\quad \times\Big[\frac{1}{4}\int_s^t\mathbb{E}|\tilde{\vartheta}(\tau)|^6\mathrm{d}\tau + \sum_{i=1}^n a_i^2\int_s^t\mathbb{E}|\tilde{\vartheta}(\tau)|^4\mathrm{d}\tau + \frac{1}{4}\left(\sum_{i=1}^n a_i^2\right)^2\int_s^t\mathbb{E}|\tilde{\vartheta}(\tau)|^2\mathrm{d}\tau\Big] \\
&< 3\kappa^2\left(\sum_{i=1}^n \frac{4k_i^2}{a_i^2}\right)\left(\sum_{i=1}^n \frac{4k_i^2\bar{h}_i^2}{a_i^2}\right)(t-s)^2 \\
&\quad \times\left[\frac{L}{4} + \sum_{i=1}^n a_i^2L^{2/3} + \frac{\sigma^2}{4}\left(\sum_{i=1}^n a_i^2\right)^2\right],\ 0 \leqslant s \leqslant t.
\end{aligned} \tag{54}
$$

Then via (53) and (54), we get

$$
\begin{aligned}
&\mathbb{E}\left|\int_s^t KM(s)(f(\tau)+\omega(\tau))^{\mathrm{T}}\mathscr{H}\tilde{\vartheta}(\tau)\,\mathrm{d}\tau\right|^2 \\
&\leqslant 2\mathbb{E}\left|\int_s^t KM(s)f^{\mathrm{T}}(\tau)\mathscr{H}\tilde{\vartheta}(\tau)\,\mathrm{d}\tau\right|^2 + 2\mathbb{E}\left|\int_s^t KM(s)\omega^{\mathrm{T}}(\tau)\mathscr{H}\tilde{\vartheta}(\tau)\,\mathrm{d}\tau\right|^2 \\
&< (t-s)^2\Delta_{Y_1},\ 0 \leqslant s \leqslant t,
\end{aligned} \tag{55}
$$

where $\Delta_{Y_1}$ is given by (33). From (24) and (55), we obtain

$$
\begin{aligned}
&\mathbb{E}|Y_1(t)|^2 \\
&\leqslant \frac{1}{\varepsilon}\int_{t-\varepsilon}^t\mathbb{E}\left|\int_s^t KM(s)(f(\tau)+\omega(\tau))^{\mathrm{T}}\bar{\mathscr{H}}\tilde{\vartheta}(\tau)\,\mathrm{d}\tau\right|^2\mathrm{d}s \\
&< \frac{\Delta_{Y_1}}{\varepsilon}\int_{t-\varepsilon}^t (t-s)^2\,\mathrm{d}s = \frac{\varepsilon^2}{3}\Delta_{Y_1},\ t \geqslant \varepsilon.
\end{aligned} \tag{56}
$$

Moreover, by using Itô isometry property (see Theorem 1.5.8 of [14]) and Hölder's inequality, we obtain from (8) and (24) that for $t \geqslant \varepsilon$

$$
\begin{aligned}
\mathbb{E}|Y_2(t)|^2 &\leqslant \frac{C^2}{\varepsilon}\int_{t-\varepsilon}^t\mathbb{E}\left|\int_s^t KM(s)M^{\mathrm{T}}(\tau)K\bar{\mathscr{H}}\tilde{\vartheta}(\tau)\,\mathrm{d}B(\tau)\right|^2\mathrm{d}s \\
&= \frac{C^2}{\varepsilon}\int_{t-\varepsilon}^t\mathbb{E}\int_s^t\left|KM(s)M^{\mathrm{T}}(\tau)K\bar{\mathscr{H}}\tilde{\vartheta}(\tau)\right|^2\mathrm{d}\tau\mathrm{d}s \\
&\leqslant \frac{C^2}{\varepsilon}\int_{t-\varepsilon}^t\int_s^t|KM(s)|^2\left|K\bar{\mathscr{H}}M(\tau)\right|^2\mathbb{E}|\tilde{\vartheta}(\tau)|^2\mathrm{d}\tau\mathrm{d}s < \varepsilon C^2\Delta_{Y_2}
\end{aligned} \tag{57}
$$

with $\Delta_{Y_2}$ given by (33), for $t \geqslant \varepsilon$

$$
\begin{aligned}
&\mathbb{E}|Y_3(t)|^2 \\
&\leqslant \frac{C^4}{4\varepsilon^2}\left(\int_{t-\varepsilon}^t\int_s^t|KM(s)||KM(\tau)|\left|K\bar{\mathscr{H}}M(\tau)\right|\mathrm{d}\tau\mathrm{d}s\right)^2 \\
&\leqslant \varepsilon^2C^4\Delta_{Y_3}
\end{aligned} \tag{58}
$$

with $\Delta_{Y_3}$ given by (33), for $t \geqslant \varepsilon$

$$
\begin{aligned}
\mathbb{E}|Y_4(t)|^2 &\leqslant \frac{1}{\varepsilon}\mathbb{E}\int_{t-\varepsilon}^t\left|\int_s^t KM(s)S^{\mathrm{T}}(s)\mathscr{H}(f(\tau)+\omega(\tau))\,\mathrm{d}\tau\right|^2\mathrm{d}s \\
&\leqslant \frac{1}{\varepsilon}\int_{t-\varepsilon}^t(t-s)\int_s^t|KM(s)|^2|S(s)|^2\left\|\mathscr{H}\right\|^2 \\
&\quad \times\mathbb{E}(|f(\tau)|+|\omega(\tau)|)^2\mathrm{d}\tau\mathrm{d}s < \varepsilon^2\Delta_{Y_4}
\end{aligned} \tag{59}
$$

with $\Delta_{Y_4}$ given by (33), where we have noted that (45) and (48), and for $t \geqslant \varepsilon$

$$
\begin{aligned}
\mathbb{E}|Y_5(t)|^2 &\leqslant \frac{C^2}{\varepsilon}\int_{t-\varepsilon}^t\mathbb{E}\left|\int_s^t KM(s)S^{\mathrm{T}}(s)\bar{\mathscr{H}}KM(\tau)\,\mathrm{d}B(\tau)\right|^2\mathrm{d}s \\
&= \frac{C^2}{\varepsilon}\int_{t-\varepsilon}^t\int_s^t\left|KM(s)S^{\mathrm{T}}(s)\bar{\mathscr{H}}KM(\tau)\right|^2\mathrm{d}\tau\mathrm{d}s \leqslant \varepsilon C^2\Delta_{Y_5}
\end{aligned} \tag{60}
$$

with $\Delta_{Y_5}$ given by (33). By Fubini's theorem and Theorem 1.5.8 in [14], we have

$$\mathbb{E}\left(Y_3^{\mathrm{T}}(t)Y_5(t)\right)=Y_3^{\mathrm{T}}(t)\mathbb{E}Y_5(t)=0, \tag{61}$$

by which and (58), (60) and (61), there holds

$$\begin{aligned}&\mathbb{E}\left|Y_3(t)+Y_5(t)\right|^2=\mathbb{E}\left|Y_3(t)\right|^2+\mathbb{E}\left|Y_5(t)\right|^2\\&\leqslant \varepsilon C^2\left(\varepsilon C^2\Delta_{Y_3}+\Delta_{Y_5}\right),\ t\geqslant\varepsilon.\end{aligned} \tag{62}$$

Finally, from (31), (48), (56)-(60), we get

$$\begin{aligned}&\mathbb{E}\left|\bar{\omega}(s)\right|^2\leqslant 6\mathbb{E}\left|K\bar{\mathscr{H}}G(t)\right|^2+6\textstyle\sum_{i=1}^2\mathbb{E}\left|Y_i(t)\right|^2\\&+6\mathbb{E}\left|Y_4(t)\right|^2+6\mathbb{E}\left|Y_3(t)+Y_5(t)\right|^2+6\mathbb{E}\left|\omega(t)\right|^2\\&<\bar{\Delta}(\varepsilon,C),\ t\geqslant\varepsilon,\end{aligned} \tag{63}$$

where $\bar{\Delta}(\varepsilon,C)$ is given by (33). In addition, via (29), (50) and (51), we obtain

$$\begin{aligned}&\mathbb{E}|z(\varepsilon)|^2\leqslant 2\mathbb{E}|\tilde{\vartheta}(\varepsilon)|^2+2\mathbb{E}|G(\varepsilon)|^2\\&<2\mathbb{E}|\tilde{\vartheta}(0)|^2+4\varepsilon\sigma\left(\Delta_f^{1/2}+\kappa\Delta_\omega^{1/2}\right)+2\varepsilon C^2\textstyle\sum_{i=1}^n\frac{4k_i^2}{a_i^2}+\frac{2\varepsilon^2\Delta_f}{3}.\end{aligned} \tag{64}$$

*Proof of the part B.* To make the second inequality in (35) hold, we use the variation of constants formula (see Theorem 3.3.1 of [14]) for (30) to obtain

$$\begin{aligned}z(t)=\;&\mathrm{e}^{K\bar{\mathscr{H}}(t-\varepsilon)}z(\varepsilon)+\textstyle\int_\varepsilon^t\mathrm{e}^{K\bar{\mathscr{H}}(t-s)}\bar{\omega}(s)\,\mathrm{d}s\\&+\textstyle\int_\varepsilon^t\mathrm{e}^{K\bar{\mathscr{H}}(t-s)}CKM(s)\,\mathrm{d}B(s),\ t\geqslant\varepsilon.\end{aligned} \tag{65}$$

Then, for $t\geqslant\varepsilon$

$$\begin{aligned}\mathbb{E}|z(t)|^2\leqslant\;&3\left\|\mathrm{e}^{K\bar{\mathscr{H}}(t-\varepsilon)}\right\|^2\mathbb{E}|z(\varepsilon)|^2+3\mathbb{E}\left|\textstyle\int_\varepsilon^t\left\|\mathrm{e}^{K\bar{\mathscr{H}}(t-s)}\right\|\right.\\&\left.\times|\bar{\omega}(s)|\,\mathrm{d}s\right|^2+3\mathbb{E}\left|\textstyle\int_\varepsilon^t\mathrm{e}^{K\bar{\mathscr{H}}(t-s)}CKM(s)\,\mathrm{d}B(s)\right|^2.\end{aligned} \tag{66}$$

Note that

$$\left\|\mathrm{e}^{K\bar{\mathscr{H}}t}\right\|=\left\|\mathrm{diag}\left\{\mathrm{e}^{k_1\bar{h}_1t},\ldots,\mathrm{e}^{k_n\bar{h}_nt}\right\}\right\|=\mathrm{e}^{-\Theta_{\mathrm{m}}t} \tag{67}$$

with $\Theta_m$ given by (32), then via (64) and (67) we have

$$\begin{aligned}&\left\|\mathrm{e}^{K\bar{\mathscr{H}}(t-\varepsilon)}\right\|^2\mathbb{E}|z(\varepsilon)|^2\leqslant 2\mathrm{e}^{-2\Theta_{\mathrm{m}}(t-\varepsilon)}\left[\mathbb{E}|\tilde{\vartheta}(0)|^2\right.\\&\left.+2\varepsilon\sigma\left(\Delta_f^{1/2}+\kappa\Delta_\omega^{1/2}\right)+\varepsilon C^2\textstyle\sum_{i=1}^n\frac{4k_i^2}{a_i^2}+\frac{\varepsilon^2\Delta_f}{3}\right].\end{aligned} \tag{68}$$

Via (63), (67) and Hölder$'s$ inequality, we get

$$\begin{aligned}&\mathbb{E}\left|\textstyle\int_\varepsilon^t\left\|\mathrm{e}^{K\bar{\mathscr{H}}(t-s)}\right\||\bar{\omega}(s)|\,\mathrm{d}s\right|^2\\&=\mathbb{E}\left|\textstyle\int_\varepsilon^t\mathrm{e}^{-\frac{\Theta_{\mathrm{m}}}{2}(t-s)}\mathrm{e}^{-\frac{\Theta_{\mathrm{m}}}{2}(t-s)}|\bar{\omega}(s)|\,\mathrm{d}s\right|^2\\&\leqslant\textstyle\int_\varepsilon^t\mathrm{e}^{-\Theta_{\mathrm{m}}(t-s)}\mathrm{d}s\int_\varepsilon^t\mathrm{e}^{-\Theta_{\mathrm{m}}(t-s)}\mathbb{E}|\bar{\omega}(s)|^2\mathrm{d}s<\frac{\bar{\Delta}(\varepsilon,C)}{\Theta_{\mathrm{m}}^2},\end{aligned} \tag{69}$$

and by (67) and Itô isometry property, we find

$$\begin{aligned}&\mathbb{E}\left|\textstyle\int_\varepsilon^t\mathrm{e}^{K\bar{\mathscr{H}}(t-s)}CKM(s)\,\mathrm{d}B(s)\right|^2\\&\leqslant\mathbb{E}\textstyle\int_\varepsilon^t\left\|\mathrm{e}^{K\bar{\mathscr{H}}(t-s)}\right\|^2|CKM(s)|^2\,\mathrm{d}s\leqslant\frac{C^2}{2\Theta_{\mathrm{m}}}\sum_{i=1}^n\frac{4k_i^2}{a_i^2}.\end{aligned} \tag{70}$$

Substituting (68)-(70) into (66), we arrive at

$$\begin{aligned}&\mathbb{E}|z(t)|^2<6\mathrm{e}^{-2\Theta_{\mathrm{m}}(t-\varepsilon)}\mathbb{E}|\tilde{\vartheta}(0)|^2+2\varepsilon\mathrm{e}^{-2\Theta_{\mathrm{m}}(t-\varepsilon)}\\&\times\left[6\sigma\left(\Delta_f^{1/2}+\kappa\Delta_\omega^{1/2}\right)+3C^2\textstyle\sum_{i=1}^n\frac{4k_i^2}{a_i^2}+\varepsilon\Delta_f\right]\\&+\frac{3\bar{\Delta}(\varepsilon,C)}{\Theta_{\mathrm{m}}^2}+\frac{3C^2}{2\Theta_{\mathrm{m}}}\textstyle\sum_{i=1}^n\frac{4k_i^2}{a_i^2},\ t\geqslant\varepsilon,\end{aligned}$$

by which, (29) and (51), we further have

$$\begin{aligned}&\mathbb{E}|\tilde{\vartheta}(t)|^2\leqslant 2\mathbb{E}|z(t)|^2+2\mathbb{E}|G(t)|^2\\&<12\mathrm{e}^{-2\Theta_{\mathrm{m}}(t-\varepsilon)}\mathbb{E}|\tilde{\vartheta}(0)|^2+4\varepsilon\mathrm{e}^{-2\Theta_{\mathrm{m}}(t-\varepsilon)}\\&\times\left[6\sigma\left(\Delta_f^{1/2}+\kappa\Delta_\omega^{1/2}\right)+3C^2\textstyle\sum_{i=1}^n\frac{4k_i^2}{a_i^2}+\varepsilon\Delta_f\right]\\&+\frac{6\bar{\Delta}(\varepsilon,C)}{\Theta_{\mathrm{m}}^2}+\frac{3C^2}{\Theta_{\mathrm{m}}}\textstyle\sum_{i=1}^n\frac{4k_i^2}{a_i^2}+\frac{2\varepsilon^2\Delta_f}{3},\ t\geqslant\varepsilon.\end{aligned} \tag{71}$$

The second inequality in (35) follows from (71) due to (34).

*Proof of the part C.* We show that the condition in (34) guarantees that the bound in (43) holds.

$(i)$ When $t\in[0,\varepsilon]$, since $\mathbb{E}|\tilde{\vartheta}(0)|^2\leqslant\sigma_0^2<\sigma^2$ and $\mathbb{E}|\tilde{\vartheta}(t)|^2$ is continuous in $t$, (43) holds for small enough $t>0$. We assume by contradiction that for some $t\in(0,\varepsilon]$ the formula (43) does not hold, namely, there exists the smallest $t^*$ $(0<t^*\leqslant\varepsilon)$ such that $\mathbb{E}|\tilde{\vartheta}(t^*)|^2=\sigma^2,\ \mathbb{E}|\tilde{\vartheta}(t)|^2<\sigma^2,\ t\in[0,t^*)$. Then by the same procedures for (44)-(49), we arrive at (50) in its non-strict version for $t\in[0,t^*]$. Furthermore, the feasibility of $\Phi<0$ in (34) ensures that $\mathbb{E}|\tilde{\vartheta}(t^*)|^2\leqslant\sigma_0^2+2\varepsilon^*\sigma(\Delta_f^{1/2}+\kappa\Delta_\omega^{1/2})+\varepsilon^*C^2\sum_{i=1}^n\frac{4k_i^2}{a_i^2}<\sigma^2$. This contradicts to $\mathbb{E}|\tilde{\vartheta}(t^*)|^2=\sigma^2$. Hence (43) holds for $t\in[0,\varepsilon]$.

$(ii)$ When $t\geqslant\varepsilon$, since $\mathbb{E}|\tilde{\vartheta}(\varepsilon)|^2<\sigma^2$ as shown in $(i)$ and $\mathbb{E}|\tilde{\vartheta}(t)|^2$ is continuous in time, (43) holds for some $t>\varepsilon$. We assume by contradiction that for some $t>\varepsilon$, (43) does not hold, namely, there exists the smallest time instance $t^*\in(\varepsilon,\infty)$ such that$\mathbb{E}|\tilde{\vartheta}(t^*)|^2=\sigma^2,\ \mathbb{E}|\tilde{\vartheta}(t)|^2<\sigma^2,\ t\in[\varepsilon,t^*)$. Thus $\mathbb{E}|\tilde{\vartheta}(t)|^2\leqslant\sigma^2$ holds for all $t\in[\varepsilon,t^*]$. Similar to the proof in parts A and B, we finally arrive at the non-strict version of (71) for $t\in[\varepsilon,t^*]$. Moreover, the feasibility of (34) ensures $\mathbb{E}|\tilde{\vartheta}(t^*)|^2<\sigma^2$. This contradicts to $\mathbb{E}|\tilde{\vartheta}(t^*)|^2=\sigma^2$. Hence $\mathbb{E}|\tilde{\vartheta}(t)|^2<\sigma^2$, for all $t\geqslant\varepsilon$. The proof is finished. ■

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