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Maximizing social power in multiple independent Friedkin-Johnsen models

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Abstract— **This paper investigates the problem of maximizing social power for a group of agents, who participate in multiple meetings described by independent Friedkin-Johnsen models. A strategic game is obtained, in which the action of each agent (or player) is her stubbornness over all the meetings, and the payoff is her social power on average. It is proved that, for all but some strategy profiles on the boundary of the feasible action set, each agent's best response is the solution of a convex optimization problem. Furthermore, even with the non-convexity on boundary profiles, if the underlying networks are given by a fixed complete graph, the game has a unique Nash equilibrium. For this case, the best response of each agent is analytically characterized, and is achieved in finite time by a proposed algorithm.**

I. INTRODUCTION

A convenient way to represent complex opinion dynamics processes is to assume that a group of agents interact with each other through meetings (e.g. face-to-face) and to model the opinion evolution during each such meeting through some multiagent opinion forming dynamical models like those of [6], [10], [14]. The result of these opinion forming processes at different meetings can then be combined together in different ways. One of the possible ways to do this combination makes use of a logic matrix to express the interdependence among the topics of the discussions going on at the different meetings, and combines it with the agents' interaction matrix in a tensor product fashion, see [11], [19]. Another is to consider the meetings as sequential, and to use a two-time-scale framework (a fast time scale for the single meeting dynamics, and a longer one for the sequence of meetings), see [17], [15], [24], [25], [26], [3]. In these two-time-scale models, the linking of consecutive meetings can be established according to different criteria. For example, in [15], the opinion evolution in each meeting is described by a DeGroot model, and in the sequence of DeGroot models the "self-appraisal" of the agents (i.e, the diagonal part of the interaction matrix) is updated. This mechanism is also applied to the Friedkin-Johnsen (FJ) model in [17], [24]. Alternatively, a concatenation rule is proposed in [25], [26], [3], in which the final opinions of each meeting become the initial opinions of the next meeting, and an FJ model is used for every meeting. A concrete application of the latter mechanism is to model the sequence of meetings occurring at the United Nations climate change conferences, see [3].

In many situations where opinion evolution takes place, and in particular in the discussion meeting setting we consider in this paper, it is natural to consider a game theoretical formulation in which, rather than passively observing the opinion evolution, the agents take actions to "win" the discussion, i.e., to impose their opinions on the other meeting participants. For instance, in the aforementioned UN climate conferences, the opinion forming process is in reality a negotiation process, in which each country tries to impose its own point of view on the other nations.

Since the opinion evolution itself encodes the impact of each agent on the whole group, an agent can try to choose an action strategically so as to maximize its utility. When the utility is based on the opinions themselves, many approaches have already been proposed in the literature, because this approach allows to view classic opinion dynamics models as best-response of strategic games [13], [4], [12], [8]. Related examples include [7], where two competing camps aim to drive the group opinions closer to their own stances, by allocating the resource put on each agent to change the associated edge weight in a two-stage FJ model. A similar problem is considered in [1], but for a single FJ model.

In [27], [2], the authors take a different approach, and consider as payoff each agent's "social power" accumulated through the concatenated FJ model [26]. Social power here refers to the influence of an agent on the final opinion outcome, and is defined as the column average of the solution matrix of the associated FJ model [17], [24]. The concept is strongly related to the centrality of an agent in the opinion dynamics process. The action of an agent in the game is the amount of stubbornness in each of the individual FJ models that form the concatenation. In an FJ model, in fact, the stubbornness profile of the agents influences the distribution of final social powers, hence an allocation of stubbornness that maximizes the social power in the concatenation of meetings can be sought.

In reality, for a complex negotiation process, the sequential meeting structure used in [27], [2] is just one of the possible ways to break the discussion into manageable steps. Another common structure is to have meetings that are disjoint (e.g. held in parallel), with each meeting discussing a specific issue, and with opinion outcomes on the different issues that

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are eventually assembled together in some way. An example is again given by the climate talks: every year several satellite meetings on different aspects of climate actions are arranged, whose resolutions constitute the basis for discussion at the annual plenary meeting (the so-called COP conference, see [3]). Taking this scenario into consideration, in this paper, we study the setting that a group of agents interacts in multiple disjoint meetings, with each meeting represented by an independent FJ model. The agents aim to maximize their average social power over the set of meetings by allocating a given budget of stubbornness in the FJ models. A strategic game is generated, with the utility of each player as her average social power over the FJ discussions.

The strategic game studied in this paper is characterized by several properties, which could be proved rigorously. First, for a single meeting (i.e., a single FJ model), the best strategy of each agent is to increase its stubbornness as much as possible (Theorem 1). Secondly, for multiple meetings, the cost function of each agent is convex over the feasible action set except for some boundary points (Proposition 1), which guarantees the existence of an Nash equilibrium (NE) in the social power game under the condition that all agents have positive stubbornness at each meeting (Theorem 2). If the underlying networks for the different FJ models all correspond to the same fixed graph, a NE can always be that all the agents allocate their budgets of stubbornness evenly over all the meetings (Theorem 3). Furthermore, if the fixed graph is fully connected and with uniform weights, even if the cost functions are non-convex at some boundary points, the aforementioned NE is unique for the game (Theorems 4 and 5). At last, for the fixed complete graph case, the best-response of each agent is characterized (Theorem 6), with the actions of all the other agents fixed. An algorithm is proposed to achieve the best-response in finite time (Algorithm 1).

While the idea of considering social power as utility follows from that of [27], the different setting investigated in this paper results into a different optimal policy for the social power game. In fact, keeping the FJ meetings independent (rather than concatenated) implies that the main feature of the solution of [27], namely the presence of an early mover advantage, no longer exists in the present social power game.

The paper is organized as follows: Section II gives some preliminary knowledge; the problem of interest is stated in Section III, while the main results are reported in Section IV. All the proofs are omitted for lack of space and will appear in an extended journal version of this paper.

Notations: All vectors are real column vectors and are denoted by bold lowercase letters $\mathbf{x}, \mathbf{y}, \dots$ The i-th entry of a vector $\mathbf{x}$ is denoted by $[\mathbf{x}]_i$ or, if no confusion arises, x_i. The symbol $\mathrm{diag}(\mathbf{x})$ represents the square matrix with diagonal entries equal to the entries of $\mathbf{x}$ and the others equal to 0. Matrices are denoted by the capital letters such as $A, B, \dots,$ of entries A_{ij} or $[A]_{ij}$. The identity matrix is denoted by I_n, with dimension sometimes omitted, depending on the context. The n-order vector and matrix with all entries being 0 or 1 are denoted $\mathbf{0}_n$ or $\mathbf{1}_n$, respectively with the dimensions omitted if there is no confusion. Let $\mathbf{e}_i$ be the vector in which the ith entry is 1 and all the others are 0. We use $[n]$ to represent the set $\{1, \dots, n\}$. Given a set $\mathcal{C}$, we use $|\mathcal{C}|$ to denote its cardinality. For a finite set of vectors $\mathbf{c}_i, i \in \mathcal{V}$, use $(\mathbf{c}_i)_{i\in\mathcal{V}}$ to denote $(\mathbf{c}_1^\top, \dots, \mathbf{c}_{|\mathcal{V}|}^\top)^\top$; for finitely many vector sets $\mathcal{C}_i, i \in \mathcal{V}$, their product is denoted by $\Pi_{i\in\mathcal{V}}\mathcal{C}_i$ or $(\mathcal{C}_i)_{i\in\mathcal{V}}$ (with the subscript sometimes omitted), i.e., $(\mathcal{C}_i) = \{(\mathbf{c}_i) : \mathbf{c}_i \in \mathcal{C}_i\}$. Given two square matrices $A, B \in \mathbb{R}^{n\times n}$, $A \succeq B$ means that $A - B$ is positive semi-definite; $A \geq B$ means that $A_{ij} \geq B_{ij}$ for all $i, j \in [n]$; A is called substochastic if $A \geq \mathbf{0}$ and $\mathbf{1} \geq \mathrm{A}\mathbf{1}$, and if the equality holds, A is called stochastic. Given a real number x, let $\lfloor x \rfloor$ be the nearest integer that is no larger than x.

II. Preliminaries

A. Strategic games

The following definitions are from [18].

Definition 1 (Strategic game) *Given*

- *a finite set of players* $\mathcal{V}$,
- *an action set* $\mathcal{A}_i \in \mathbb{R}^M$ *for each* $i \in \mathcal{V}$, *and*
- *a utility function* $u_i : \mathcal{A} \mapsto \mathbb{R}$ *for each* $i \in \mathcal{V}$, *with* $\mathcal{A} := \Pi_{j\in\mathcal{V}}\mathcal{A}_j$,

then the tuple $\langle \mathcal{V}, \mathcal{A}, (u_i)\rangle$ *is called a strategic game.*

Any $\mathbf{a} = (\mathbf{a}_i) \in \mathcal{A}$ is called an *action profile*, for which each $\mathbf{a}_i$ is an *action*. Given $i \in \mathcal{V}$, we use $\mathbf{a}_{-i} = (\mathbf{a}_j)_{j\in\mathcal{V}\setminus\{i\}}$ to denote the collection of actions of all agents but i. With a slight abuse of notation, the utility function $u_i(\mathbf{a})$ is sometimes denoted by $u_i(\mathbf{a}_i, \mathbf{a}_{-i})$ to emphasize its dependency on $\mathbf{a}_i$.

Let $\mathbf{u}(\mathbf{a}) = (u_1(\mathbf{a}), \dots, u_{|\mathcal{V}|}(\mathbf{a}))^\top$ be the vector of all utilities w.r.t. the profile $\mathbf{a}$. If the utility functions u_i are all differentiable, the pseudo-gradient mapping of the game $\langle \mathcal{V}, \mathcal{A}, (u_i)\rangle$ is defined as $\nabla_{\mathbf{a}}\mathbf{u}(\cdot) : \mathbb{R}^{M|\mathcal{V}|} \mapsto \mathbb{R}^{M|\mathcal{V}|}$, with

$$\nabla_{\mathbf{a}}\mathbf{u}(\bar{\mathbf{a}}) = (\nabla_{\mathbf{a}_i} u_i(\bar{\mathbf{a}}_i, \bar{\mathbf{a}}_{-i}))_{i\in\mathcal{V}}, \quad \forall \bar{\mathbf{a}} \in \mathcal{A}.$$

In a strategic game, one important concept is that of NE, which is a special action profile for which no player has any motivation to change its strategy.

Definition 2 (Nash equilibrium) *Given a strategic game* $\langle \mathcal{V}, \mathcal{A}, (u_i)\rangle$, *a profile* $\mathbf{a}^* = (\mathbf{a}_i^*)$ *is an NE if* $\mathbf{a}_i^* = \arg\max_{\mathbf{a}_i\in\mathcal{A}_i} u_i(\mathbf{a}_i, \mathbf{a}_{-i}^*)$.

The following definitions come from [23].

Given a mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ defined on a convex set Ω, it is called *monotone* if $\langle T\mathbf{x} - T\mathbf{y}, \mathbf{x} - \mathbf{y}\rangle \geq 0, \forall \mathbf{x}, \mathbf{y} \in \Omega$, and ($\sigma$-)*strongly monotone* if for some $\sigma > 0$, it holds

$$\langle T\mathbf{x} - T\mathbf{y}, \mathbf{x} - \mathbf{y}\rangle \geq \sigma\|\mathbf{x} - \mathbf{y}\|^2, \quad \forall \mathbf{x}, \mathbf{y} \in \Omega.$$

Given a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ defined on a convex set Ω, it is called *convex* if its gradient operator ∇f is monotone on Ω, and σ-*strongly* convex is ∇f is σ-strongly monotone on Ω.

B. FJ model

The FJ model is a DeGroot-like model for opinion dynamics in which some agents behave stubbornly, in the sense that they defend their positions while discussing with the other agents [10]. If n agents participate to a discussion, the FJ model has the following structure:

$$\mathbf{y}(t+1) = (I-\Theta)W\mathbf{y}(t) + \Theta\mathbf{y}(0), \quad t = 0, 1, \dots \quad (1)$$

where $\mathbf{y}$ is the n-dimensional opinion vector, W is a row-stochastic matrix, and $\Theta = \mathrm{diag}(\theta_1, \dots, \theta_n)$, with $\theta_i \in [0,1]$ representing the stubbornness of agent i. Stubbornness here means attachment of an agent to its own opinion, represented by the initial condition $\mathbf{y}(0)$ at the beginning of the discussion ($\theta_i = 0$ means agent i is not stubborn, $\theta_i = 1$ means a totally stubborn agent).

Let $\mathcal{G}_W$ be the graph associated to W in the FJ model.

Lemma 1 *[26] Assume $\theta_i \in [0,1]$ for all $i = 1, \dots, n$, and $\theta_i > 0$ for at least one i. If $\mathcal{G}_W$ is strongly connected, then*

(a) $(I-\Theta)W$ is Schur stable, i.e. $\rho((I-\Theta)W) < 1$,
(b) The matrix $\mathrm{V} = (I-(I-\Theta)W)^{-1}\Theta$ is stochastic,
(c) $\mathbf{y}(\infty) = \lim_{t\to+\infty}\mathbf{y}(t) = V\mathbf{y}(0)$.

The matrix V has the following special structure [26].

Lemma 2 *Suppose that all the conditions of Lemma 1 are satisfied. If only the first $u < n$ agents are stubborn (i.e., $\theta_i > 0$ for $i = 1, \dots, u$, and $\theta_i = 0$ for $i = u+1, \dots, n$), then $V = \left[\begin{array}{c|c} R & \mathbf{0}\end{array}\right]$, where R is an $n \times u$ matrix of all nonzero entries.*

III. Problem Formulation

Consider M independent discussions (or meetings), with the same group of participants, described by an agent set $\mathcal{V} = [n]$. For each of the discussions, say the mth discussion, the agents interact with their neighbors over a fixed graph $\mathcal{G}_m = (\mathcal{V}, \mathcal{E}_m, W_m)$, with $W_m = [w_{ij,m}]_{i,j\in\mathcal{V}} \in \mathbb{R}^{n\times n}_{\geq 0}$ the associated stochastic weight matrix, i.e., $(j,i) \in \mathcal{E}_m$ if and only if $w_{ij,m} > 0$. It's possible that $\mathcal{G}_m$ varies for $m \in [M]$. The opinions of all agents are collected in a vector $\mathbf{x}^m(t) = [x_{1,m}(t), x_{2,m}(t), \dots, x_{n,m}(t)]^\top$, with t as the time slot in the mth discussion. Based on the interactions, each agent changes his/her own opinion, and the overall opinion evolution is described by a FJ model, that is,

$$\mathbf{x}^m(t+1) = (I-\Theta_m)W_m\mathbf{x}^m(t) + \Theta_m\mathbf{x}^m(0),$$

where $\Theta_m = \mathrm{diag}(\theta_{1,m}, \theta_{2,m}, \dots, \theta_{n,m})$ represents the stubbornness of each agent to their initial opinions $\mathbf{x}^m(0)$, with $\theta_{i,m} \in [0,1]$ for all $i \in \mathcal{V}$ and $m \in [M]$. The following assumption is made throughout the paper, and is also widely used in literature [6], [20], [21].

Assumption 1 *For all $m \in [M]$, the graph $\mathcal{G}_m$ is strongly connected, with each node having a self loop, i.e., $\underline{w} := \min_{\substack{i\in\mathcal{V}\\ m\in[M]}} w_{ii,m} > 0$.*

Under Assumption 1, the opinion vector of each discussion m converges. The solution is written as

$$\mathbf{x}^m(\infty) := \lim_{t\to\infty}\mathbf{x}^m(t) = P_m\mathbf{x}^m(0).$$

If there exists some $\theta_{i,m} > 0$, Lemma 1 gives

$$P_m = (I-(I-\Theta_m)W_m)^{-1}\Theta_m, \quad (2)$$

Otherwise, the corresponding FJ model degrades to the DeGroot model, which gives $P_m = \mathbf{1}\mathbf{c}_m^\top$, with $\mathbf{c}_m$ as the normalized left eigenvector corresponding to the eigenvalue 1 of the weight matrix W_m. For both cases, P_m is a stochastic matrix that represents the influence of agents' initial opinions on the group's final opinions, and encodes the social power of each agent [15], [24]. In particular, in this paper we follow the definition of social power in [24]. Given an agent $i \in \mathcal{V}$, its social power in the mth discussion is defined as

$$\mathrm{sp}_{i,m}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}) = \frac{1}{n}\mathbf{1}^\top P_m\mathbf{e}_i. \quad (3)$$

We also assume that the agents can decide how much stubbornness to assign to each of the FJ models, in order to obtain the highest total social power over all the M discussions. To avoid trivial solutions, we assume that the total stubbornness of each agent i is upper bounded by a constant $K_i \in (0, M)$, i.e., $\sum_{m=1}^M \theta_{i,m} \leq K_i$. More in detail, the following strategic game is investigated:

- **Players**: all the agents are involved in all the M discussions, i.e., $\mathcal{V} = [n]$ for all $m \in [M]$;
- **Actions**: for each agent $i \in \mathcal{V}$, the feasible set of its actions is
$$\mathcal{A}_i = \{\boldsymbol{\theta}_i = (\theta_{i,1}, \theta_{i,2}, \dots, \theta_{i,M})^\top \text{ s.t.} \sum_{m=1}^M \theta_{i,m} \leq K_i \text{ and } \theta_{i,m} \in [0,1], \forall m \in [M]\}.$$
- **Utility functions**: the utility of each agent $i \in \mathcal{V}$ is the corresponding total social power, i.e.,
$$u_i(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}) = \sum_{m=1}^M \mathrm{sp}_{i,m}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}).$$

The strategic game $\langle \mathcal{V}, \mathcal{A}, (u_i)\rangle$ is called a *social power game*.

Problems of interest: For the social power game, we are interested in the existence and the location of the NE. Another related problem is to understand the best response of each agent i, that is, given the actions of the other agents, how should agent i maximizes his/her total social power, by carefully allocating the limited resource of stubbornness? Written in a mathematical form, the following optimization problem needs to be solved

$$\begin{aligned} \underset{\boldsymbol{\theta}_i}{\text{Minimize}} \quad & -\sum_{m=1}^M \mathrm{sp}_{i,m}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}) \\ \text{s.t.} \quad & \sum_{m=1}^M \theta_{i,m} \leq K_i, \\ & 0 \leq \theta_{i,m} \leq 1, \ \forall m \in [M]. \end{aligned} \quad (4)$$

Here $\boldsymbol{\theta}_{-i}$ is fixed, and $\mathrm{sp}_{i,m}$ is a function of $\boldsymbol{\theta}_i$, or more specifically, $\theta_{i,m}$.

IV. Main results

In this section, at first we deal with the social power optimization problem (4). Then the results are applied to investigate the properties of NE for the social power game $\langle \mathcal{V}, \mathcal{A}, (u_i) \rangle$.

A. Single meeting case

Before going to more general cases, consider the simplest case of $M = 1$. The straightforward intuition is that more stubbornness will result in a higher social power. The intuition is stated in the following Theorem 1 whose proof (as for all other results below) is omitted for lack of space.

For simplicity of notation, we omit m in the subscripts, and use θ_i instead of $\boldsymbol{\theta}_i$ to indicate that the action of agent i is a scalar.

Theorem 1 *Consider the social power optimization problem* (4) *with $M = 1$. Let Assumption 1 hold. If there exists $j \neq i$ such that $\theta_j > 0$, the solution to the problem* (4) *is $\theta_i = K_i$; otherwise any $\theta_i \in (0, K_i]$ is the solution of* (4)*.*

Remark 1 From Theorem 1, it is easy to see that the social power game $\langle \mathcal{V}, \mathcal{A}, (u_i) \rangle$ has a unique NE, i.e., $\boldsymbol{\theta}^* = (K_1, K_2, \ldots, K_n)$, regardless of the underlying network topology.

B. Multiple meetings: general graphs

For $M > 1$, it is hard to give an analytical solution to the problem (4). However, it can be shown to be a convex optimization problem, as stated in the following proposition.

Proposition 1 *Consider the social power optimization problem* (4) *with $M > 1$. Let Assumption 1 hold. If for all $m \in [M]$, there exists $j \neq i$ such that $\theta_{j,m} > 0$, then the cost function $-\sum_{m=1}^{M} \mathrm{sp}_{i,m}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i})$ is strongly convex in the argument $\boldsymbol{\theta}_i \in [0,1]^M$.*

TFrom Proposition 1, the social power game becomes an n-player convex game [22] if the following assumption holds.

Assumption 2 *There exists $\delta \in (0,1)$ such that for all $m \in [M]$, it holds $\theta_{i,m} \geq \delta$.*

Assumption 2 implies that for each meeting, all agents stubbornly support their own opinions.

Under Assumption 2, the social power game needs to be considered on the constrained action set $\mathcal{A}^\delta = (\mathcal{A}_i^\delta)$, with

$$\mathcal{A}_i^\delta = [\delta, 1]^M \cap \mathcal{A}_i.$$

According to Theorem 1 in [22], the existence of the NE can be obtained, as the following theorem shows.

Theorem 2 *Consider the social power game $\langle \mathcal{V}, \mathcal{A}^\delta, (u_i) \rangle$. Under Assumption 1, an NE exists for the game.*

In general, it is not clear if an NE exists for the social power game $\langle \mathcal{V}, \mathcal{A}, (u_i) \rangle$, even if the action set $\mathcal{A}$ can be approached by $\mathcal{A}^\delta$ as $\delta \to 0$. This is because the game is no longer a convex game, as implied by the following proposition.

Proposition 2 *Consider a profile $\boldsymbol{\theta}^0 \in \mathcal{A}$ such that $\theta_{i,m_0}^0 = 0, \forall i \in \mathcal{V}$ for a given $m_0 \in [M]$. For any $i \in \mathcal{V}$, the cost function $-u_i(\cdot, \boldsymbol{\theta}_{-i}^0)$ is not convex at $\boldsymbol{\theta}_i^0$.*

Even though convexity is missing, if the underlying graph is fixed, i.e., $W_1 = \cdots = W_M$, an NE for the game $\langle \mathcal{V}, \mathcal{A}, (u_i) \rangle$ can be that each agent allocates her stubbornness evenly over all the meetings.

Theorem 3 *Consider the social power game $\langle \mathcal{V}, \mathcal{A}, (u_i) \rangle$ with a fixed graph $\mathcal{G}$ for all $m \in [M]$. Let Assumption 1 hold. An NE is $\boldsymbol{\theta}^* = (\boldsymbol{\theta}_i^*)$, with $\boldsymbol{\theta}_i^* = \frac{K_i}{M}\mathbf{1}$ for all $i \in \mathcal{V}$.*

Remark 2 From Theorem 3, it is easy to see that if $\delta < \min_{i \in \mathcal{V}} \frac{K_i}{M}$, $\boldsymbol{\theta}^*$ is also an NE of the social power game $\langle \mathcal{V}, \mathcal{A}^\delta, (u_i) \rangle$ over a fixed graph.

Given a strategic game, we are also interested in whether the NE is unique or not, if it exists. For the social power game $\langle \mathcal{V}, \mathcal{A}, (u_i) \rangle$ with general and heterogeneous underlying topologies, the answer should be no, as indicated by the following example.

Example 1 Consider a 2-player game $\langle \mathcal{V}, \mathcal{A}, (u_i) \rangle$ with $\mathcal{V} = \{1, 2\}$. Let $K_1 = 3.4, K_2 = 0.45$ and $M = 5$. The weight matrices are

$$W_1 = W_2 = W_3 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix},$$
$$W_4 = W_5 = \begin{pmatrix} 0.95 & 0.05 \\ 0.95 & 0.05 \end{pmatrix}.$$

It can be verified that the two profiles $\boldsymbol{\theta}^*$ and $\bar{\boldsymbol{\theta}}^*$ are both NEs, where

$$\boldsymbol{\theta}_1^* = (1, 1, 1, 0.2, 0.2)^\top, \quad \boldsymbol{\theta}_2^* = (0.15, 0.15, 0.15, 0, 0)^\top;$$
$$\bar{\boldsymbol{\theta}}_1^* = (1, 1, 1, 0.19, 0.21)^\top, \quad \bar{\boldsymbol{\theta}}_2^* = (0.15, 0.15, 0.15, 0, 0)^\top.$$

In fact, all the profiles in the convex hull of $\boldsymbol{\theta}^*$ and $\bar{\boldsymbol{\theta}}^*$ are NEs.

Remark 3 For the NEs shown in Example 1, the agent 2 always chooses to be non-stubborn for the last two meetings. Intuitively, this is because the agent 1 has high influence in the social networks of the meetings $4-5$, which makes the agent 2 avoid competing for social power with her in the two meetings. Instead, the agent 2 prefer to being more stubborn in the meetings $1-3$. This tendency of the agent 2, in turn, gives the agent 1 more "freedom" to act in the meetings $4-5$, and results in multiple NEs.

The theoretical analysis of NE for the social power game $\langle \mathcal{V}, \mathcal{A}, (u_i) \rangle$ in general cases is challenging, as we are not able to use the standard results for convex games [22], [23], [9]. However, as will be shown in the next subsection, if the interaction network for each meeting (i.e., for each FJ model) is a fixed complete graph, the existence and uniqueness of NE can be fully addressed.

C. Multiple meetings: complete graph

Let the underlying graphs for all the meetings be a fixed complete graph, i.e., $W_m = \frac{1}{n}\mathbf{1}\mathbf{1}^\top, \forall m \in [M]$. For this case, the existence of NE of the social power game $\langle \mathcal{V}, \mathcal{A}, (u_i) \rangle$ has been given in the last subsection. In the following, at first, we will consider the constrained social power game $\langle \mathcal{V}, \mathcal{A}^\delta, (u_i) \rangle$ and show the uniqueness of NE.

Consider the pseudo-gradient $\nabla_{\boldsymbol{\theta}} u(\boldsymbol{\theta})$ of the constrained social power game $\langle \mathcal{V}, \mathcal{A}^\delta, (u_i) \rangle$. It is easy to see that

$$\nabla_{\boldsymbol{\theta}_i} u_i(\boldsymbol{\theta}) = (\frac{\partial \mathrm{sp}_{i,1}(\boldsymbol{\theta})}{\partial \theta_{i,1}}, \ldots, \frac{\partial \mathrm{sp}_{i,M}(\boldsymbol{\theta})}{\partial \theta_{i,M}})^\top, \quad \forall i \in \mathcal{V}.$$

The following lemma can be obtained.

Lemma 3 *For the constrained social power game $\langle \mathcal{V}, \mathcal{A}^\delta, (u_i) \rangle$ with $W_m = \frac{1}{n}\mathbf{1}\mathbf{1}^\top, \forall m \in [M]$, the opposite pseudo-gradient mapping -$\nabla_{\boldsymbol{\theta}} \mathbf{u}$ is σ-strongly monotone on $\mathcal{A}^\delta$ for some $\sigma > 0$.*

With the strong monotonicity of the pseudo-gradient mapping, we are able to apply the standard results for convex games [22], [23], [9] and immediately obtain the following theorem.

Theorem 4 *Consider the constrained social power game $\langle \mathcal{V}, \mathcal{A}^\delta, (u_i) \rangle$ with $W_m = \frac{1}{n}\mathbf{1}\mathbf{1}^\top, \forall m \in [M]$. If $\delta < \min_{i \in \mathcal{V}} \frac{K_i}{M}$, a unique NE $\boldsymbol{\theta}^* = (\boldsymbol{\theta}_i^*)$ exists, with $\boldsymbol{\theta}_i^* = \frac{K_i}{M}\mathbf{1}, \forall i \in \mathcal{V}$.*

Theorem 4 means that, if all agents need to stubbornly support their own opinions at each meeting and the interaction graphs are fully connected, the agents will add equal weights to all the meetings on the unique NE of the game.

Now we turn to the original social power game $\langle \mathcal{V}, \mathcal{A}, (u_i) \rangle$. Theorem 3 gives that the game adopts an NE $(\frac{K_i}{M}\mathbf{1})$ if the underlying networks are a fixed complete graph. Following the sensitivity analysis of variational inequalities similar to that of [5], the NE can also be shown to be unique.

Theorem 5 *Consider the social power game $\langle \mathcal{V}, \mathcal{A}, (u_i) \rangle$ with $W_m = \frac{1}{n}\mathbf{1}\mathbf{1}^\top, \forall m \in [M]$. A unique NE $\boldsymbol{\theta}^* = (\boldsymbol{\theta}_i^*)$ exists, with $\boldsymbol{\theta}_i^* = \frac{K_i}{M}\mathbf{1}, \forall i \in \mathcal{V}$.*

So far, the existence and uniqueness of NE has been addressed. To achieve the NE, a best-response dynamics is often used [22], [16]. For the social power game $\langle \mathcal{V}, \mathcal{A}, (u_i) \rangle$ with $W_m = \frac{1}{n}\mathbf{1}\mathbf{1}^\top, \forall m \in [M]$, given the actions of the other agents, the best response to each agent i is the solution of the optimization problem Eq. (4), or more explicitly, of the following problem

$$\begin{aligned} \underset{\boldsymbol{\theta}_i}{\text{Minimize}} \quad & \sum_{m=1}^{M} \frac{s_{i,m}}{\theta_{i,m} + s_{i,m}} \\ \text{s.t.} \quad & \sum_{m=1}^{M} \theta_{i,m} = K_i, \\ & 0 \le \theta_{i,m} \le 1, \ \forall m \in [M]. \end{aligned} \tag{5}$$

where $s_{i,m} = \sum_{j \neq i} \theta_{j,m}$, and the constraint $\sum_{m=1}^{M} \theta_{i,m} = K_i$ is due to Theorem 1. To solve (5), we write down the Lagrangian function and exploit the KKT conditions. The following theorem is then obtained.

Theorem 6 *Consider the social power optimization problem* (5) *with $s_{i,m} > 0$ for all $m \in [M]$. There exists a unique partition of the set $[M]$, $[M] = \mathcal{M}_0 \cup \mathcal{M}_* \cup \mathcal{M}_1$, such that*

1) if $m \in \mathcal{M}_0$, it holds $c \le \sqrt{s_{i,m}}$;
2) if $m \in \mathcal{M}_*$, it holds $\sqrt{s_{i,m}} < c < \sqrt{s_{i,m}} + \frac{1}{\sqrt{s_{i,m}}}$;
3) if $m \in \mathcal{M}_1$, it holds $\sqrt{s_{i,m}} + \frac{1}{\sqrt{s_{i,m}}} \le c$;

where c is defined as

$$c = \frac{\sum_{m \in \mathcal{M}_*} s_{i,m} + K_i - M_1}{\sum_{m \in \mathcal{M}_*} \sqrt{s_{i,m}}}, \tag{6}$$

with $M_1 = |\mathcal{M}_1|$. Moreover, the optimization problem (5) *admits a unique solution $\boldsymbol{\theta}_* \in [0,1]^M$, with*

$$\theta_m^* = \begin{cases} 0, & \text{if } m \in \mathcal{M}_0, \\ c\sqrt{s_{i,m}} - s_{i,m}, & \text{if } m \in \mathcal{M}_*, \\ 1, & \text{if } m \in \mathcal{M}_1. \end{cases} \tag{7}$$

Remark 4 Theorem 6 indicates that for a discussion event m, when $s_{i,m}$ is small enough, it will be optimal for agent i to take $\theta_{i,m} \in (0,1)$ instead of $\theta_{i,m} = 1$. This might be counter-intuitive. An interpretation is, when the total stubbornness of the other agents is small in a discussion, it is easy for agent i to obtain a high social power by being mildly stubborn. Therefore, instead of investing a lot of stubbornness resources at that discussion, it is better for agent i to put more stubbornness into other discussions to get more marginal social power increase.

Theorem 6 does not specify a clear form of $\mathcal{M}_0, \mathcal{M}_1$ and $\mathcal{M}_*$. However, it can be used to design an algorithm that achieves the accurate solution of the problem (5) in finite time. To do this, first sort the agents in $\mathcal{V}$ along an increasing order of $\sqrt{s_{i,m}} + \frac{1}{\sqrt{s_{i,m}}}$, say, $i_1, i_2, \ldots, i_M$, and sort the agents in $\mathcal{V}$ along a decreasing order of $\sqrt{s_{i,m}}$, say, $i^1, i^2, \ldots, i^M$. The algorithm is then written as Algorithm 1.

Remark 5 Due to the existence of $\boldsymbol{\theta}_*$, the outer loop (i.e., lines $4-14$) of Algorithm 1 must stop in finite time. In fact, for each m^*, the inner loop (i.e., lines $5-11$) will be executed for at most $M - m^* + 1$ steps. Therefore, the maximum number of executing times for the outer loop is

$$it_{\max} = \sum_{m^*=0}^{\lfloor K_i \rfloor} (M - m^* + 1) = (M + 1 - \frac{\lfloor K_i \rfloor}{2})(\lfloor K_i \rfloor + 1).$$

Algorithm 1 Best response algorithm for complete graph

Input: $\{s_{i,m}\}, (i^m), (i_m), K$
Output: $\boldsymbol{\theta}_* = (\theta_1^*, \theta_2^*, \ldots, \theta_M^*)^\top$

Initialization: $\mathcal{M}_1 \leftarrow \emptyset, \mathcal{M}_0 \leftarrow \emptyset, \mathcal{M}_* \leftarrow \mathcal{V}$
$m_* \leftarrow 0, m^* \leftarrow 0$
Compute c as in Eq. (6)
while 1)-3) in Theorem 6 are not satisfied **do**
    while $m^* + m_* < M$ **do**
        $m^* \leftarrow m^* + 1$
        $\mathcal{M}_0 \leftarrow \mathcal{M}_0 \cup \{i^{m^*}\}$
        if 1)-3) are satisfied **then**
            Break
        end if
    end while
    $\mathcal{M}_1 \leftarrow \mathcal{M}_1 \cup \{i_{m_*}\}$, $m_* \leftarrow m_* + 1$
    $m^* \leftarrow 0$, $\mathcal{M}_0 \leftarrow \emptyset$
end while
$\mathcal{M}_* \leftarrow \mathcal{V} \backslash (\mathcal{M}_0 \cup \mathcal{M}_1)$
$\boldsymbol{\theta}_*$ is taken as in (7)

V. Conclusion

The paper investigates the problem of social power maximization for a group of agents in multiple meetings, each represented by a FJ model. A social power game is proposed, for which it is proved that the cost function of each agent is convex over the feasible action set except for some boundary points. The existence and uniqueness of NE is proved for the case that the underlying networks are all a fixed complete graph. Concerning future work, a natural follow-up question is the following: For heterogeneous general underlying network topologies, is there any condition to guarantee that the game adopts a unique NE, and how does the network topology affect the formation of NE?

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