2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

Robust Design of Phase-Locked Loops in Grid-Connected Power Converters

Riya Mathew, Juan G. Rueda-Escobedo and Johannes Schiffer

Abstract— **Phase-locked loop (PLL) algorithms are key elements for the successful integration of converter-interfaced renewable energy sources to the grid. Their main task is to estimate the phase angle of the terminal grid voltage with the aim to keep the converter output current synchronized to it. Yet, due to the increasing penetration of power-electronics-interfaced devices in power systems, the grid voltage signal becomes increasingly disturbed, making the reliable phase estimation a highly demanding task. To address this challenge, we present a robust design method based on matrix inequalities to tune the PLL gains, such that the estimation errors remain bounded in the presence of additive disturbances. Our design approach is formulated as a set of bilinear matrix inequalities (BMIs), which we then propose to solve using the P-K iteration method. This results in a convex problem to be solved at each step. Finally, the benefits of the proposed robust design are illustrated via a numerical example.**

I. INTRODUCTION

Power converters are central to integrating renewable energies into power systems [1], [2]. However, as they start substituting the conventional generation, problems associated with this transition like higher distortion in electrical signals and higher frequency volatility arise [3]–[5]. These phenomena diminish the performance of grid-connected power converters and pose a challenge to their continuous operation. As their participation in the grid increases, it is also necessary that power converters provide support to the grid. For this, their continuous operation, even under faulty conditions, is required [3], [6].

One of the challenges in the operation of power converters is keeping them synchronized with the grid in the presence of distorted electrical signals [7], [8]. To achieve this, different phase-locked loop (PLL) algorithms have been proposed through the years. Among them, one can find PLLs that address unbalances [9]–[11], harmonic content [8], [10] and fast frequency variations [12], [13]. The central idea of most of these developments is the modification of the PLL phase detector to enhance its response to one or more of the previous disturbances. However, no phase detector is perfect up to now, and thus all of them carry some sort of error [14].

R. Mathew is with the Control Systems and Network Control Technology Group, Brandenburg University of Technology Cottbus-Senftenberg (BTU C-S), 03046 Cottbus, Germany, riya.mathew@b-tu.de.

J. G. Rueda-Escobedo is with the División de Ciencias Básicas, Facultad de Ingeneiría, Universidad Nacional Autónoma de México (UNAM), 04510 Mexico City, Mexico, juan.rueda@fi.unam.edu.

J. Schiffer is with the Control Systems and Network Control Technology Group, Brandenburg University of Technology Cottbus-Senftenberg (BTU C-S), 03046 Cottbus, Germany and the Fraunhofer Research Institution for Energy Infrastructures and Geothermal Systems(IEG), 03046 Cottbus, Germany, schiffer@b-tu.de.

This work is partially supported by the project Reference Power Plant Lausitz (RefLau) 03EWR018C.

To further improve the performance of the different PLLs, instead of proposing new modifications to the phase detector, one can attempt to enhance the response of their loop filter. Conventionally, this filter has the transfer function of a proportional-integral (PI) controller [15] that corrects the estimates provided by the PLL, for which the information obtained from the phase detector is used. Tuning such controllers is challenging due to the nonlinear nature of PLLs. Furthermore, the presence of disturbances carried by the phase detector adds an extra complexity layer to the process. The correct tuning of the loop filter gains can reduce the impact of disturbances in the PLL performance and might help to keep power converters connected to the grid during faults. Yet, most of the design procedures available in the literature rely on small signal analysis and do not directly address the effect of disturbances [7], [16], [17].

To systematize the tuning process of PLLs in the presence of disturbances and to improve their reliability, in this work, we present a method based on matrix inequalities to tune the loop filter gains. Specifically, our contributions are:

1) We provide a set of bilinear matrix inequalities (BMIs), whose solution gives gains for the loop filter that ensure boundedness of the phase and frequency estimation errors in the presence of disturbances.
2) Based on the so called P-K iteration method [18], we provide an iterative algorithm that allows to search for solutions to the BMIs by solving only linear matrix inequalities (LMIs) in the process. This results in a convex problem to be solved at each step.

The method gives flexibility to the user in specifying the maximum size of the expected disturbance and the acceptable error. Then, the obtained gains with our algorithm guarantee the existence of a forward invariant set around the equilibrium, yielding bounded estimation errors. Differently from other approaches, we do not rely on a small signal analysis, meaning that the proposed method explicitly considers the nonlinearities present in the PLL dynamics. Therefore, this systematic design procedure for fine-tuning the PLL parameters indeed ensures a robust performance.

The paper is organized as follows. In Section II, a brief introduction to the PLL structure is provided, while In Section III the goal of this work is formulated. Later, in Section IV, our main contribution is presented, and its application is illustrated in Section V via a numerical example. Some final remarks are provided in Section VI. Finally, in the Appendix, the proofs of our claims can be found.

Notation. Along the note, $\mathbb{R}$ represents the set of real numbers and $\mathbb{R}_{>c}$ ($\mathbb{R}_{\geq c}$) the set of real numbers greater than (or equal to) $c \in \mathbb{R}$. Given a vector $v \in \mathbb{R}^n$, $\|v\| = (v^\top v)^{1/2}$ denotes its Euclidean norm. The $n \times n$ identity matrix is represented by I_n. Given two symmetric matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$, $A > B$ ($A \geq B$) means that $A - B$ is positive (semi)definite. Finally, the largest and smallest eigenvalues of a symmetric matrix $A \in \mathbb{R}^{n\times n}$ are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively.

II. PRELIMINARIES

A PLL is responsible for providing an estimate of the phase and frequency of a periodic electrical signal. A generic PLL consists of a *phase detector*, a *loop filter*, and a *controlled oscillator* [15, Sec. 4.2.2]. A block diagram of a typical PLL implementation is shown in Fig. 1. The phase detector produces an error signal that grows with and keeps the sign of the difference between the estimated phase angle and the actual one. The loop filter corrects the estimate produced by the controlled oscillator based on the information provided by the phase detector. Whereas the loop filter and the controlled oscillator are usually a PI controller and an integrator, the phase detector changes accordingly to the application. For instance, in single-phase applications, the error signal is obtained by multiplying the input signal $U(t)$ with the sine of the estimated phase angle $\hat{\phi}(t)$ [15, Sec. 4.2.2.2], i.e.,

$$\begin{aligned} U(t)\cdot \sin\big(\hat{\phi}(t)\big) &= A\,\cos\big(\phi(t)\big)\cdot \sin\big(\hat{\phi}(t)\big) \\ &= \frac{1}{2}A\sin\big(\phi(t)-\hat{\phi}(t)\big) + \frac{1}{2}A\sin\big(\phi(t)+\hat{\phi}(t)\big). \end{aligned}$$

Here, the first term carries the information about the phase estimation error, whereas the second is regarded as a high-frequency disturbance that the loop filter has to reject.

In three-phase applications, the standard Synchronous-Reference Frame (SRF) PLL uses the so called $dq0$-transformation as phase detector [15, Sec. 8.3]. The associated matrix to the transformation is given by

$$T_{dq0}\big(\hat{\phi}(t)\big) = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos\big(\hat{\phi}(t)\big) & \cos\big(\hat{\phi}(t)-\frac{2}{3}\pi\big) & \cos\big(\hat{\phi}(t)-\frac{4}{3}\pi\big) \\ -\sin\big(\hat{\phi}(t)\big) & -\sin\big(\hat{\phi}(t)-\frac{2}{3}\pi\big) & -\sin\big(\hat{\phi}(t)-\frac{4}{3}\pi\big) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

In such case, the output of the phase detector is

$$A\,\sin\big(\phi(t)-\hat{\phi}(t)\big)+\xi(t).$$

If the input signal is a symmetrical three-phase signal, one has $\xi(t) = 0$. If the input signal is unbalanced or distorted, $\xi(t)$ will appear as a disturbance for the PLL that the loop filter has to handle.

Although several variations of the SRF-PLL have been proposed to tackle the presence of unbalances and other disturbances found in three-phase systems (see for instance [10] and [11]), no phase detector is free of being disturbed by measurement noise, signal distortion, or other discrepancies with respect to the considered signal model, see [14].

Based on the previous discussion, we model the error signal provided by a phase detector for both, single-phase and three-phase signals, as

$$\psi(t) = -A\,\sin\big(\tilde{\phi}(t)\big)+\xi(t), \tag{1}$$

where $A \in \mathbb{R}_{>0}$ is a uncertain coefficient related to the amplitude of the input signal, $\tilde{\phi}(t) = \hat{\phi}(t) - \phi(t)$ is the difference between the estimate provided by the PLL $\hat{\phi}(t) \in \mathbb{R}$ and the actual signal's phase angle $\phi(t) \in \mathbb{R}$. Finally, $\xi(t) \in \mathbb{R}$ is an unknown, but bounded disturbance. In the following, we discuss how to tune the loop filter gains, i.e. the PI controller, to ensure that the estimate provided by the PLL remains bounded in the presence of $\xi(t)$.

III. PROBLEM STATEMENT

Consider a PLL implemented as shown in Fig. 1 with $\psi(t)$ as in (1). The loop filter dynamics together with the controlled oscillator can be represented by [14]:

$$\begin{bmatrix}\dot{\hat{\phi}}(t)\\ \dot{\hat{\omega}}(t)\end{bmatrix} = \begin{bmatrix}-k_P A & 1\\ -k_I A & 0\end{bmatrix}\begin{bmatrix}\sin\big(\tilde{\phi}(t)\big)\\ \hat{\omega}(t)\end{bmatrix} + \begin{bmatrix}k_P\\ k_I\end{bmatrix}\xi(t). \tag{2}$$

Here, the estimate of the signal's frequency is represented by $\hat{\omega}(t)$. The PLL parameters are the proportional gain $k_P \in \mathbb{R}_{>0}$ and the integral gain $k_I \in \mathbb{R}_{>0}$. Now, define the estimation error of the frequency as $\tilde{\omega}(t) = \hat{\omega}(t) - \omega$, with $\omega \in \mathbb{R}_{>0}$ being the constant signal's frequency satisfying $\dot{\phi}(t) = \omega$. The error dynamics induced by the PLL is

$$\begin{bmatrix}\dot{\tilde{\phi}}(t)\\ \dot{\tilde{\omega}}(t)\end{bmatrix} = \begin{bmatrix}-k_P A & 1\\ -k_I A & 0\end{bmatrix}\begin{bmatrix}\sin\big(\tilde{\phi}(t)\big)\\ \tilde{\omega}(t)\end{bmatrix} + \begin{bmatrix}k_P\\ k_I\end{bmatrix}\xi(t). \tag{3}$$

In the absence of $\xi(t)$, it is well known that (3) has an equilibrium set at $(\mathrm{mod}(\tilde{\phi}, 2\pi), \tilde{\omega}) = (0, 0)$, which is almost globally asymptotically stable [19]. If $\xi(t)$ is small, e.g., caused by measurement noise, one can expect that the estimation error remains bounded and close to one of the equilibria. However, the local stability caused by the periodicity and boundedness of the sine function makes the $\mathcal{L}_2$-gain of (3) only local. Therefore, rather than seeking its minimization as in a $\mathcal{H}_\infty$-filtering design, we aim for gains that ensure a bounded error in the presence of bounded disturbances. Consequently, we expect that any device relying on the PLL can operate during transient disturbances and recover faster thereafter. This objective is presented in the subsequent problem.

Problem 1. Consider the system (3) and let $A \in [A_{\min}, A_{\max}]$ and $|\xi(t)| \leq \bar{\xi}$ with $A_{\min} \in \mathbb{R}_{>0}$, $A_{\max} \in \mathbb{R}_{>0}$, and $\bar{\xi} \in \mathbb{R}_{\geq 0}$ being known constants. Let $D \subset \mathbb{R}^2$ be a compact domain containing the origin and no other nominal equilibrium of (3). Design k_P and k_I, such that

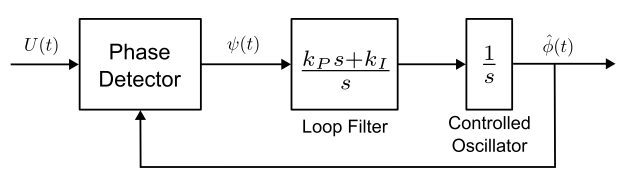


Fig. 1: Typical block diagram of a PLL.

Algorithm 1 *P-K* Iteration (Adapted from [18, Prop. 6])

Initialization: Set $\varepsilon \in (0, \pi/2)$, $\theta \in (0,1)$, $\alpha \in \mathbb{R}_{\geq 0}$, $j \leftarrow 1$, $\tau \leftarrow \texttt{true}$, choose P_0, such that
$$P_0 > \frac{\bar{\xi}^2}{\alpha\,\theta \sin^2(\varepsilon)} I_2,$$
with $\bar{\xi}$ given in Problem 1, and define a threshold $\sigma \in \mathbb{R}_{>0}$.
repeat
  step $j, 1$**:** Take $P \leftarrow P_{j-1}$ and solve the following LMI optimization problem in δ and K:
$$\begin{aligned} &\min \delta \\ &\text{subject to } P > \tfrac{\bar{\xi}^2}{\alpha\,\theta \sin^2(\varepsilon)} I_2, \\ &\qquad\qquad -Q_i \leq \delta\, I_3, \end{aligned} \tag{4}$$
  where the matrices Q_i are defined in (5b), matrices B_i, C, and F_i are defined in (5a), and $A_{\min}$ and $A_{\max}$ are given in Problem 1. Set $K_j \leftarrow K$.
  step $j, 2$**:** Take $K \leftarrow K_j$ and solve (4) in P and δ. Set $P_j \leftarrow P$ and $\delta_j \leftarrow \delta$.
  if $\delta_{j-1} - \delta_j < \sigma$ **or** $\delta_j < 0$ **then**
    $\tau \leftarrow \texttt{false}$
  else
    $j \leftarrow j + 1$
  end if
until τ
return $K \leftarrow K_j$ and $P \leftarrow P_j$.

$(\tilde{\phi}(t_0), \tilde{\omega}(t_0)) \in D$ implies that $(\tilde{\phi}(t), \tilde{\omega}(t)) \in D \; \forall t \geq t_0$, i.e., the set D is forward invariant with respect to (3).

The goal posed in Problem 1 is to design the PLL gains, in the presence of the uncertain coefficient A, to ensure the boundedness of the estimation error despite the impact of the bounded disturbance $\xi(t)$. Although the formulation only includes the origin, given the periodicity of the vector field of (3), any result for the origin can be transferred to any other nominal stable equilibrium of the system. Finally, note that no precise description of D is given. Later, however, D will be tied to the level sets of a Lyapunov function candidate and some design parameters. Thus, the shape of D will be dictated by the method of solution for Problem 1.

In the next section, we present a method to systematically address the design problem posed in this section and obtain suitable gains for the PLL's loop filter.

IV. MAIN RESULT

To solve Problem 1 the idea is to design k_P and k_I in a way that a forward invariant set D is created around the origin despite the presence of $\xi(t)$. First, we are going to specify the target D, for which the phase estimation error $\tilde{\phi}$ is limited to the interval $[-\varepsilon, \varepsilon]$, with $\varepsilon \in (0, \pi/2)$, and $\tilde{\omega}$ is asked to be bounded. Additionally, consider the following quadratic function:
$$V(\tilde{\phi}, \tilde{\omega}) = \chi^\top(\tilde{\phi}, \tilde{\omega}) P\, \chi(\tilde{\phi}, \tilde{\omega}), \tag{6}$$
where $\chi(\tilde{\phi}, \tilde{\omega}) = [\sin(\tilde{\phi}) \;\, \tilde{\omega}]^\top$ and $P \in \mathbb{R}^{2\times 2}$ is a positive definite matrix to be computed. Now, if one has that $|\tilde{\phi}| \leq \varepsilon$, it follows that $|\sin(\tilde{\phi})| \leq \sin(\varepsilon)$. The target forward invariant set D is defined as
$$D = \{\tilde{\phi}, \, \tilde{\omega} \in \mathbb{R} \,|\, V(\tilde{\phi}, \tilde{\omega}) < c^\star\}, \tag{7}$$
with
$$c^\star := \lambda_{\min}(P) \sin^2(\varepsilon). \tag{8}$$
Given P, the boundary of D described in (7) corresponds to the largest level set of V that is included in the ball $\|(\sin(\tilde{\phi}), \, \tilde{\omega})\| < \sin(\varepsilon)$ [20, pp. 317-318]. Thus, D describes a region around the origin that does not contain any other nominal equilibrium of (3) or zero of V. In the following, we provide a method to obtain k_P and k_I that renders D forward invariant, thus solving Problem 1.

Theorem 1. *Consider Problem 1. Fix the three parameters* $\varepsilon \in (0, \pi/2)$, $\theta \in (0, 1)$, *and* $\alpha \in \mathbb{R}_{>0}$, *and find matrices* $P \in \mathbb{R}^{2\times 2}$ *and* $K \in \mathbb{R}^{2\times 1}$, *such that*
$$P > \frac{\bar{\xi}^2}{\alpha\,\theta \sin^2(\varepsilon)} I_2, \quad Q_i \geq 0, \tag{9}$$
where the matrices Q_i *are defined in* (5b)*, the matrices* B_i*,* C*, and* F_i *are defined in* (5a)*, and* $A_{\min}$*,* $A_{\max}$*, and* $\bar{\xi}$ *are given in Problem 1.*

Take $[k_P \;\, k_I] = K^\top$ *for the loop filter in* (2)*. Then,* D *defined in* (7) *is a forward invariant set with respect to* (3)*. In addition, for* $\xi(t) \equiv 0$*, the origin of* (3) *is a locally asymptotically stable equilibrium point.*

In Theorem 1, the parameter ε allows the user to specify the tolerated error in $\tilde{\phi}$ and, indirectly, limits D. This is the case since the obtained gains ensure that $|\tilde{\phi}| \leq \varepsilon$. Additionally, ε is constrained to $(0, \pi/2)$ in order to prevent the inclusion of other zeros of V in D. This does not suppose a limitation since one would always like a small $\tilde{\phi}$. The other parameters, α and θ, also influence the solutions of (9). Specifically, reducing α and making θ close to one should facilitate solving (9). The first parameter, α, controls how *negative* is the derivative of V in D, whereas θ specifies which proportion of α is used to dominate the disturbance $\xi(t)$.

The difficulty in solving (9) appears in its bilinear nature, each Q_i contains products of P and K. Although these blocks resemble a structure output feedback design problem, methods such as [21] cannot be used to transform the BMIs into LMIs. This happens because K also appears in the off-diagonal blocks of each Q_i. Instead, we propose to use the so-called *P-K* iteration method [18, Sec. 4.1]. The idea of the method, adapted in Algorithm 1, is to fix one of the two variables to make (9) a set of LMIs and solve them in the other. Then, with the value obtained, solve for the LMIs in the first variable. This process is repeated until P and K satisfying (9) are obtained. Hence, Algorithm 1 allows us to search for solutions to (9) iteratively by solving only a convex problem in each step.

Now, solving (9) can be related to finding a solution for a *quadratic stabilization* problem in a system subject to a polytopic perturbation. In the specific case of (3), the polytope is used to describe the dependency of the state

$$F_0 = \begin{bmatrix} 0 & \cos(\varepsilon) \\ 0 & 0 \end{bmatrix},\ F_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B_0 = \begin{bmatrix} \cos(\varepsilon) & 0 \\ 0 & 1 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \tag{5a}$$

$$Q_0 = \begin{bmatrix} -PF_0 - F_0^\top P - \alpha P + A_{\min}\left(PB_0KC + C^\top K^\top B_0^\top P\right) & PB_0K \\ K^\top B_0^\top P & 1 \end{bmatrix},\ Q_1 = \begin{bmatrix} -PF_0 - F_0^\top P - \alpha P + A_{\max}\left(PB_0KC + C^\top K^\top B_0^\top P\right) & PB_0K \\ K^\top B_0^\top P & 1 \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} -PF_1 - F_1^\top P - \alpha P + A_{\min}\left(PB_1KC + C^\top K^\top B_1^\top P\right) & PB_1K \\ K^\top B_1^\top P & 1 \end{bmatrix},\ Q_3 = \begin{bmatrix} -PF_1 - F_1^\top P - \alpha P + A_{\max}\left(PB_1KC + C^\top K^\top B_1^\top P\right) & PB_1K \\ K^\top B_1^\top P & 1 \end{bmatrix}, \tag{5b}$$

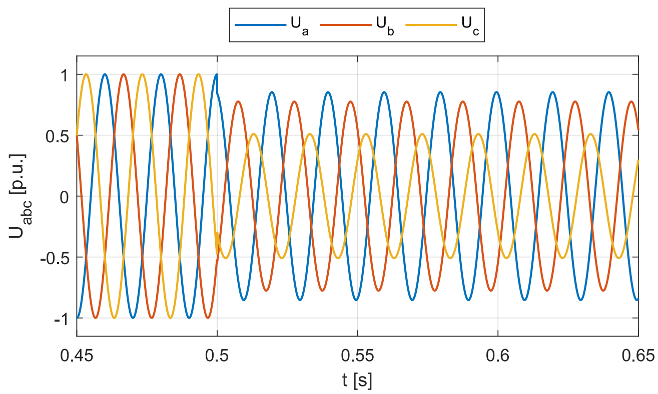


Fig. 2: Faulty three-phase signal used in the simulation example. The phase-to-phase fault appears at 0.5 [s].

matrices on $\tilde{\phi}$ and the uncertain coefficient A. Note, however, that, when ε approaches zero together with $A_{\min} \to A^\star$ and $A_{\max} \to A^\star$, for some constant $A^\star \in \mathbb{R}_{\geq 0}$, one has that $B_i \to I_2$ and the vertices Q_i converge to:

$$\begin{bmatrix} -PF_1 - F_1^\top P - \alpha P + A^\star\left(LC + C^\top L^\top\right) & L \\ L^\top & 1 \end{bmatrix},$$

with $L = PK$. Hence, close to the origin, (9) is equivalent to an LMI in P and L. Furthermore, if $\bar{\xi}$ is also reduced, the constraint on the smallest eigenvalue of P is relaxed, making (9) close to the design problem of an observer gain for the pair (F_1, C), which is observable. Thus, close to the origin and for small uncertainty and disturbance, the problem (9) is always feasible for some α and θ.

In the next section, we illustrate the application of Theorem 1 and Algorithm 1 to the robust tuning of a SRF-PLL, and the advantages of the resulting gains in the presence of line faults.

V. NUMERICAL EXAMPLE

We illustrate how the method developed in the previous section can be applied to the design of the loop filter of a SRF-PLL. For this, we assume that all signals are normalized and expressed in per unit. The coefficient A is limited between $A_{\min} = 0.7$ and $A_{\max} = 1.1$, i.e., it can decrease up to 30% and increase by a maximum of 10% of the nominal signal amplitude. For $\bar{\xi}$, we take 0.20, i.e., the disturbance can be up to 20% of the nominal signal amplitude. We set $\varepsilon = 2\pi/9$ ($40°$), which represents the tolerated error in $\tilde{\phi}$ that we desire, $\alpha = 1.1$ and $\theta = 0.8$. By solving (9) with Algorithm 1, we obtain the following K and P after 15 iterations:

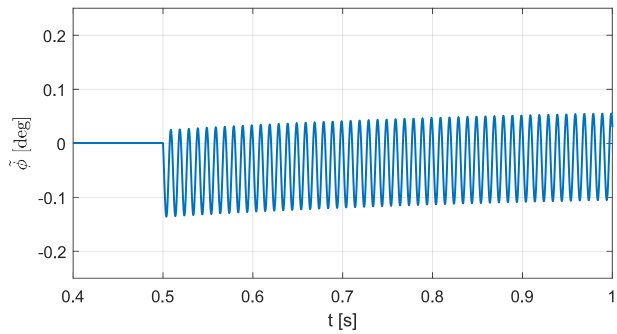
$$K = \begin{bmatrix} 3.5832 \\ 1.9421 \end{bmatrix}, \quad P = \begin{bmatrix} 0.3909 & -0.2772 \\ -0.2772 & 0.3837 \end{bmatrix}.$$

The substitution of K and P in Q_i in (5b) yields:

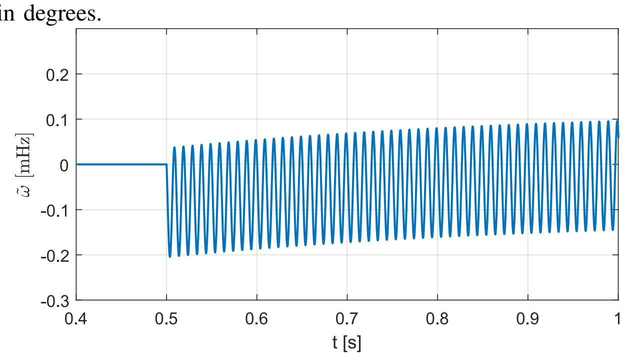
$$\lambda_{\min}(Q_0) = 0.0022, \quad \lambda_{\min}(Q_1) = 0.0025,$$
$$\lambda_{\min}(Q_2) = 0.0022, \quad \lambda_{\min}(Q_3) = 0.0399.$$

Furthermore, $c^\star$ in (8) results in 0.0455. Thus, $k_P = 3.5832$ and $k_I = 1.9421$ render D in (7) a forward invariant set with respect to (3).

To test the previous design, we consider a symmetric three-phase signal with initial amplitude of 1 [p.u.] and frequency of 50 [Hz]. We implement an SRF-PLL using the previously computed gains. At 0.5 [s], a phase-to-phase fault is introduced. The fault mimics the one described in



(a) Estimation error in the phase angle $\tilde{\phi}(t)$. The error is displayed in degrees.



(b) Estimation error in the frequency $\tilde{\omega}(t)$. The error is displayed in milihertz.

Fig. 3: Response of of the SRF-PLL with the designed gains in the presence of a line fault. As can be seen, despite the severe fault, the estimation error in both the phase and frequency remains bounded and quite small.

[15, p. 173]. After the fault, the three-phase signal consists of a positive sequence with amplitude of 0.70 [p.u.] and a negative sequence with amplitude of 0.20 [p.u.]. A plot of the signal is shown in Fig. 2, and the results of the estimation are presented in Fig. 3. As can be seen in Fig. 3a and Fig. 3b, despite the severe disturbance, both estimation errors remain bounded and are far from leaving the set D. From Fig. 3a, we note that the phase estimation error is not greater than $0.13°$ in magnitude, whereas the frequency estimation error does not exceed 0.2 [mHz]. Thus, the designed gains result in a good performance, and they endow the SRF-PLL loop filter of excellent disturbance attenuation properties.

VI. CONCLUSIONS

This study focuses on tuning PLLs in the presence of disturbances affecting the phase detector. The developed analysis takes into account the nonlinear nature of the PLL. It yields a set of BMIs, the solution of which provides gains that ensure a bounded estimation error. An algorithm based on the P-K iteration is introduced to solve the BMIs. With this approach, only convex problems have to be solved in each iteration. The resulting design ensures the robust operation of the PLL in the presence of disturbances, a feature illustrated in numerical simulations for the case of line faults.

Further research directions and applications for this work include: the joint analysis of disturbances caused by frequency variations and deviations from the signal's model; the minimization of the disturbance impact in the phase and frequency estimate; and the reduction of the conservatism induced by considering a joint quadratic Lyapunov function in the analysis. These directions are currently under exploration.

APPENDIX

Lemma 1. *Consider the nonlinear system* $\dot{x}(t) = f(x(t), \delta(t))$ *with* $x(t) \in \mathbb{R}^n$, $\delta(t) \in \mathbb{R}^m$ *being an exogenous input, and* $f(0,0) = 0$. *Let* $D \subset \mathbb{R}^n$ *be a given domain containing the origin, and* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *a differentiable function, positive definite in* D.

Let $c \in \mathbb{R}_{>0}$ *be such that* $V(x) = c$ *is the largest level set of* V *included in* D *and fix* $\theta \in (0,1)$. *Suppose there exist positive constants* α *and* $\bar{\delta}$, *such that* $\|\delta(t)\| \leq \bar{\delta}$, $\alpha c \theta > \bar{\delta}$, *and*

$$\frac{\partial V}{\partial x} \cdot f(x, \delta) \leq -\alpha V(x) + \bar{\delta}. \tag{10}$$

Then, if $V(x(t_0)) < c$, $V(x(t)) < c$ *for all* $t \geq t_0$.

Proof. From (10) we have that

$$\frac{\partial V}{\partial x} \cdot f(x, \delta) \leq -(1-\theta)\alpha V(x) < 0,$$

whenever $V(x) \geq \bar{\delta}/(\alpha\theta)$. Since $\alpha c \theta > \bar{\delta}$ by assumption it follows that $c > \bar{\delta}/(\alpha\theta)$. Hence, the derivative of V is negative definite in $\bar{\delta}/(\alpha\theta) \leq V(x) < c$, proving that $\mathcal{S} = \{x \in \mathbb{R}^n \,|\, V(x) < c\}$ is a forward invariant set for $\dot{x}(t) = f(x(t), \delta(t))$. □

Proof of Theorem 1. We take Lemma 1 as a base to show the existence of a forward invariant set with respect to (3). For that, consider D as in (7), and note that V, defined in (6), is positive in D and only has a zero in it (the origin). Following Lemma 1, we need to show that $\alpha\, c^\star\, \theta > \bar{\xi}^2$ and that $V(t)$ satisfies

$$\dot{V}(t) \leq -\alpha\, V(t) + \bar{\xi}^2, \tag{11}$$

with $K = [k_P \; k_I]^\top$ and P. Given the lower bound for P in (9) we have that $\lambda_{\min}(P) > \bar{\xi}^2/(\alpha\,\theta \sin^2(\varepsilon))$. Hence,

$$c^\star = \lambda_{\min}(P)\sin^2(\varepsilon) > \frac{\bar{\xi}^2}{\alpha\theta},$$

proving indeed that $\alpha\, c^\star\, \theta > \bar{\xi}^2$.

Now, we proceed to compute $\dot{V}(t)$, which results in

$$\dot{V}(t) = \chi^\top(t) P\, \dot{\chi}(t) + \dot{\chi}^\top(t) P\, \chi(t). \tag{13}$$

The time derivative of $\chi(t)$ is obtained as follows. Define $\chi_1(t) = \sin(\tilde{\phi}(t))$ and $\chi_2(t) = \tilde{\omega}(t)$, so that $\chi(t) = [\chi_1(t) \;\; \chi_2(t)]^\top$. Then, $\dot{\chi}_1(t)$ and $\dot{\chi}_2(t)$ are computed with (3) as

$$\dot{\chi}_1(t) = \frac{\partial \chi_1}{\partial \tilde{\phi}} \cdot \Big(-k_P A \sin(\tilde{\phi}(t)) + \tilde{\omega}(t) + k_P \xi(t) \Big),$$

$$\dot{\chi}_2(t) = \frac{\partial \chi_2}{\partial \tilde{\omega}} \cdot \Big(-k_I A \sin(\tilde{\phi}(t)) + k_I \xi(t) \Big).$$

This yields

$$\dot{\chi}(t) = F\big(\tilde{\phi}(t)\big)\chi(t) - A\, B\big(\tilde{\phi}(t)\big) K\, C\chi(t) + B\big(\tilde{\phi}(t)\big) K\, \xi(t), \tag{14}$$

where

$$F\big(\tilde{\phi}(t)\big) = \begin{bmatrix} 0 & \cos(\tilde{\phi}(t)) \\ 0 & 0 \end{bmatrix}, \; B\big(\tilde{\phi}(t)\big) = \begin{bmatrix} \cos(\tilde{\phi}(t)) & 0 \\ 0 & 1 \end{bmatrix}, \tag{15}$$

C is as in (5), and A the uncertain coefficient. By substituting (14) in (13), we obtain[1]

$$\dot{V}(t) = \chi^\top(t)\big(PF + F^\top P - A\big[PB\,K\,C + C^\top K^\top B^\top P\big]\big)\chi(t) + \chi^\top(t) PB\,K\xi(t) + \xi(t) K^\top B^\top P \chi(t). \tag{16}$$

Next, consider that by applying Young's inequality we obtain

$$\chi^\top(t) P\, BK\xi(t) + \xi(t) K^\top B^\top P \chi(t) \leq \chi^\top(t) PBKK^\top B^\top P \chi(t) + \xi^2(t),$$

and that $|\xi(t)| \leq \bar{\xi}$. Combining these inequalities with (16) yields

$$\dot{V}(t) \leq \chi^\top(t)\big(PF + F^\top P + PBKK^\top B^\top P\big)\chi(t) + \bar{\xi}^2 - \chi^\top(t)\big(APB\,K\,C + AC^\top K^\top B^\top P\big)\chi(t). \tag{17}$$

To apply Lemma 1 we need to show that the right-hand side of (17) satisfies (11). Since $V(\chi) = \chi^\top P \chi$, this is the case

[1] To keep the notation short, the dependency of the matrices on $\tilde{\phi}(t)$ is omitted in this part.

$$Q(\tilde{\phi}(t)) := \begin{bmatrix} -PF(\tilde{\phi}(t)) - F^\top(\tilde{\phi}(t))P - \alpha P + A\big[PB(\tilde{\phi}(t))KC + C^\top K^\top B^\top(\tilde{\phi}(t))P\big] & PB(\tilde{\phi}(t))K \\ K^\top B^\top(\tilde{\phi}(t))P & 1 \end{bmatrix}. \tag{12}$$

if

$$PF + F^\top P + \alpha P - APBKC - AC^\top K^\top B^\top P \\ + PBKK^\top B^\top P \leq 0,$$

which is obtained by adding αP to (17). From the Schur complement, the inequality above is equivalent to $Q(\tilde{\phi}(t)) \geq 0$, with $Q(\tilde{\phi}(t))$ defined in (12). Note that $Q(\tilde{\phi}(t))$ depends on $\tilde{\phi}(t)$ due to $F(\tilde{\phi}(t))$ and $B(\tilde{\phi}(t))$. To address the dependency of Q on $\tilde{\phi}$ and the uncertain coefficient A, we are going to describe it as a polytope.

For $|\tilde{\phi}| \leq \varepsilon$, one has that $\cos(\varepsilon) \leq \cos(\tilde{\phi}) \leq 1$. This together with $A_{\min} \leq A \leq A_{\max}$, yields the following polytopic description:

$$Q(\tilde{\phi}(t)) = \sum_{j=0}^{3} \gamma_j(t) Q_j,$$

with $\gamma_j(t) \geq 0$, $\sum_{j=0}^{3} \gamma_j(t) = 1$, Q_j given in (5b), and matrices B_i, C, and F_i given in (5a). Thus, four inequalities has to be checked.

Since K and P satisfy (9) by assumption, each Q_j is positive semidefinite, and thus $Q(\tilde{\phi}(t)) \geq 0$, implying (11). Hence, by Lemma 1, we conclude that D given in (7) is a forward invariant set with respect to (3). Furthermore, for $\xi(t) \equiv 0$, we have from (11) that $\dot{V}(t) \leq -\alpha V(t)$, showing that the origin is locally asymptotically stable. □

REFERENCES

[1] M. Hannan, M. H. Lipu, P. J. Ker, R. Begum, V. G. Agelidis, and F. Blaabjerg, "Power electronics contribution to renewable energy conversion addressing emission reduction: Applications, issues, and recommendations," *Applied Energy*, vol. 251, p. 113404, 2019.

[2] J. Lyons and V. Vlatkovic, "Power electronics and alternative energy generation," in *2004 IEEE 35th Annual Power Electronics Specialists Conference*, vol. 1, 2004, pp. 16–21 Vol.1.

[3] F. Milano, F. Dörfler, G. Hug, D. J. Hill, and G. Verbič, "Foundations and challenges of low-inertia systems," in *2018 Power Systems Computation Conference (PSCC)*, 2018, pp. 1–25.

[4] P. Tielens and D. Van Hertem, "The relevance of inertia in power systems," *Renewable and Sustainable Energy Reviews*, vol. 55, pp. 999–1009, 2016.

[5] K. S. Ratnam, K. Palanisamy, and G. Yang, "Future low-inertia power systems: Requirements, issues, and solutions - A review," *Renewable and Sustainable Energy Reviews*, vol. 124, p. 109773, 2020.

[6] M. G. Taul, X. Wang, P. Davari, and F. Blaabjerg, "An overview of assessment methods for synchronization stability of grid-connected converters under severe symmetrical grid faults," *IEEE Transactions on Power Electronics*, vol. 34, no. 10, pp. 9655–9670, 2019.

[7] Z. Ali, N. Christofides, L. Hadjidemetriou, E. Kyriakides, Y. Yang, and F. Blaabjerg, "Three-phase phase-locked loop synchronization algorithms for grid-connected renewable energy systems: A review," *Renewable and Sustainable Energy Reviews*, vol. 90, pp. 434–452, 2018.

[8] S. Golestan, J. M. Guerrero, and J. C. Vasquez, "Three-phase PLLs: A review of recent advances," *IEEE Transactions on Power Electronics*, vol. 32, no. 3, pp. 1894–1907, 2017.

[9] M. Karimi-Ghartemani and M. Iravani, "A method for synchronization of power electronic converters in polluted and variable-frequency environments," *IEEE Transactions on Power Systems*, vol. 19, no. 3, pp. 1263–1270, 2004.

[10] P. Rodríguez, R. Teodorescu, I. Candela, A. V. Timbus, M. Liserre, and F. Blaabjerg, "New positive-sequence voltage detector for grid synchronization of power converters under faulty grid conditions," in *2006 37th IEEE Power Electronics Specialists Conference*, 2006, pp. 1–7.

[11] P. Rodriguez, J. Pou, J. Bergas, J. I. Candela, R. P. Burgos, and D. Boroyevich, "Decoupled double synchronous reference frame PLL for power converters control," *IEEE Transactions on Power Electronics*, vol. 22, no. 2, pp. 584–592, 2007.

[12] J. G. Rueda-Escobedo, J. A. Moreno, and J. Schiffer, "Finite-time estimation of time-varying frequency signals in low-inertia power systems," in *2019 18th European Control Conference (ECC)*, 2019, pp. 2108–2114.

[13] ——, "Design and tuning of the super-twisting-based synchronous reference frame phase-locked-loop," in *2022 IEEE 61st Conference on Decision and Control (CDC)*, 2022, pp. 4300–4306.

[14] J. G. Rueda-Escobedo, S. Tang, and J. Schiffer, "A performance comparison of PLL implementations in low-inertia power systems using an observer-based framework," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 12 244–12 250, 2020, 21st IFAC World Congress.

[15] R. Teodorescu, M. Liserre, and P. Rodríguez, *Grid converters for photovoltaic and wind power systems*. John Wiley & Sons, 2011.

[16] S. Golestan, M. Monfared, and F. D. Freijedo, "Design-oriented study of advanced synchronous reference frame phase-locked loops," *IEEE Transactions on Power Electronics*, vol. 28, no. 2, pp. 765–778, 2013.

[17] L. Huang, H. Xin, Z. Li, P. Ju, H. Yuan, Z. Lan, and Z. Wang, "Grid-synchronization stability analysis and loop shaping for PLL-based power converters with different reactive power control," *IEEE Transactions on Smart Grid*, vol. 11, no. 1, pp. 501–516, 2020.

[18] M. S. Sadabadi and D. Peaucelle, "From static output feedback to structured robust static output feedback: A survey," *Annual Reviews in Control*, vol. 42, pp. 11–26, 2016.

[19] A. Rantzer, "Almost global stability of phase-locked loops," in *Proceedings of the 40th IEEE Conference on Decision and Control*, vol. 1, 2001, pp. 899–900 vol.1.

[20] H. K. Khalil, *Nonlinear Systems*. Prentice–Hall, 2002.

[21] C. Crusius and A. Trofino, "Sufficient LMI conditions for output feedback control problems," *IEEE Transactions on Automatic Control*, vol. 44, no. 5, pp. 1053–1057, 1999.