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Robust Feedback Linearization for Full Relative Degree Input-Affine Nonlinear Systems

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***Abstract*— Feedback linearization generally implies exact knowledge of the dynamical model of a nonlinear input-affine process. This is an inherent limitation of the method in its current use in the literature, as models are only partially known, with the remaining unknown dynamics considered as uncertainties. The purpose of this paper is to consider the full relative degree case and determine conditions to ensure asymptotic stability for the entire family of processes by employing the nominal nonlinear model and an uncertain linear block. The resulting linear inverse additive uncertainty model always leads to an improper descriptor system which can be fit from frequency response data through a proposed algorithm. A numerical case study further illustrates this approach.**

I. Introduction

The problem of feedback linearization is well known in the literature, through the seminal works [1], [2], as a starting point. It implies an adequate change of coordinates such that the system is transposed into its normal form, having the zero dynamics and external dynamics as its two main subsystems. The main drawback is that it necessitates an exact model for the nonlinear process, which is hard to accurately determine experimentally or, simply, the dynamics vary in time. A historical overview with current challenges regarding the study of zero dynamics is provided in [3], while the need to manipulate the external dynamics leads to several control methods. In the context of feedback linearization and stabilization in the presence of uncertainties, there exist several emergent studies. A first approach was presented in [4], in which the authors present conditions for uniform stability of nonlinear systems using state feedback.

In the conference paper [5], later extended in [6], a robust feedback stabilization technique is presented for a class of input-affine nonlinear systems with matched uncertainties and input bound constraints. In a similar manner, the authors of [7] propose a method to robustify against parametric uncertainties. A common characteristic of the previous papers is that they technically employ three layers for the controller: the linearization diffeomorphism, the state feedback and one to counter the uncertainties (using different structures). In the case of multivariable systems, a recursive procedure to design a joint feedback-linearization coupled with a high-gain observer has been extensively developed in [8], allowing to recover the performance imposed on the nominal case.

A reinforcement learning approach used to compute the linearization diffeomorphism is presented in [9]. The authors propose a model-free technique using general function approximation architectures, with conditions under which the learning problem is strongly convex. A recent data-driven neural network-based solution is presented in [10]. Besides the nonlinear change of coordinates, the state equation matrices are also learned, with loss terms imposed to also ensure the pair's controllability. A signal-centric perspective is described in [11], also based on the data-driven paradigm, where nonlinear systems are continuously approximated by linear models in order to harvest the advantages of frequency response and robust control methods.

The main contributions of the current paper are: (i) a methodology for robust feedback linearization in the special case of systems where uncertainties maintain the full relative degree; (ii) a non-convex optimization problem formulation which manages to find the uncertainty model using an improper system (the first available solution, from our findings); (iii) to illustrate the proposed framework on the nonlinear mass-spring benchmark system. The uncertainty model is proved to lead to an inverse additive model, which allows the poles to cross from the stable to unstable regions by varying the uncertainty block $\|\Delta\|\leq 1$ [12], which is adequate for the studied case. An advantage of the proposed method compared to [5], [6], [7] is that our regulator architecture implies only the nonlinear coordinate change and a linear robust layer, compared to the three layers needed.

The rest of the paper is structured as follows: Sections II and III provide a theoretical overviews on the feedback linearization principle and linear uncertainty modelling, respectively. Section IV presents the proposed end-to-end approach to robustify the feedback linearization method in the face uncertainties, with an additional algorithm to fit stable and minimum-phase descriptor system models. Section V provides a practical illustration on a nonlinear mass-spring system, with some conclusions in VI.

Notations: Denote a scalar variable as $x \in \mathbb{R}$, column vectors $\mathbf{x} \in \mathbb{R}^n$, matrices $X \in \mathbb{R}^{m\times p}$. $\|\cdot\|$ will imply the ∞ norm of a matrix or system, depending on the argument type. Let $L_f h(\mathbf{x}) = \frac{\partial h}{\partial x} f(\mathbf{x})$ be the Lie derivative of h along the trajectory f, and $L_f^k h = L_f L_f^{k-1} h$, $L_f^0 h = h$. Let $\mathcal{V}\{\mathbf{x}\}$ be a neighborhood of a point $\mathbf{x} \in \mathbb{R}^n$.

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II. Feedback Linearization Background

For the purpose of this paper we consider a single-input and single-output (SISO) input-affine nonlinear system having the form:

$$(\Sigma_n) : \begin{cases} \dot{\mathbf{x}} & = f(\mathbf{x}) + g(\mathbf{x})u; \\ y & = h(\mathbf{x}), \end{cases} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, while maps $f, g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ must be smooth in their arguments.

Definition 1: The system (Σ_n) has the relative degree $r \leq n$ at a point $\overline{\mathbf{x}}$ if the following conditions hold:

(i) for all $\mathbf{x} \in \mathcal{V}\{\overline{\mathbf{x}}\}$ and $k < r-1$ we have $L_g L_f^k h(\mathbf{x}) = 0$;
(ii) $L_g L_f^{r-1} h(\overline{\mathbf{x}}) \neq 0$.

We assume that the nominal system (Σ_n) has the relative degree $r = n$ for each $\mathbf{x} \in \mathcal{D}_{\mathbf{x}}$. Then the following nonlinear transformation:

$$\Phi(\mathbf{x}) = \begin{pmatrix} \phi_1(\mathbf{x}) \\ \phi_2(\mathbf{x}) \\ \vdots \\ \phi_n(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} h(\mathbf{x}) \\ L_f h(\mathbf{x}) \\ \vdots \\ L_f^{n-1} h(\mathbf{x}) \end{pmatrix} \tag{2}$$

is a diffeomorphism [3]. In the new coordinates:

$$z_i = L_f^{i-1} h(\mathbf{x}),$$

and the system can be written as:

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \vdots \\ \dot{z}_{n-1} = z_n \\ \dot{z}_n = L_f^n h\left(\Phi^{-1}(\mathbf{z})\right) + L_g L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right) u \\ y = z_1 \end{cases} . \tag{3}$$

Now, the remaining nonlinearities are stored in the last state equation. To cancel this nonlinearity we consider the following state feedback control law:

$$u = \frac{1}{L_g L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right)} \left(v - L_f^n h\left(\Phi^{-1}(\mathbf{z})\right)\right). \tag{4}$$

The resulting inner closed-loop system becomes a chain of n integrators, and a good technique will be to consider an integrated linear-quadratic regulator (LQR). The main drawback of such a strategy is the requirement of having an exact model of the system. This paper proposes a solution in the presence of uncertainties which are assumed to not alter the relative degree, and without including zero dynamics.

III. Proposed Uncertainty Matching

In robust control, the process uncertainties can be classified in unstructured (used to describe residual dynamics) and parametric (to model inaccurate component characteristics) [12]. An arbitrary number of both types can be encompassed into a single block-diagonal structured uncertainty:

$$\begin{aligned} \boldsymbol{\Delta} = \{&\operatorname{diag}\left(\delta_1 I_{n_1}, \dots, \delta_s I_{n_s}, \Delta_1, \dots, \Delta_f\right), \\ &\delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j}, 1 \leq i \leq n_s, 1 \leq j \leq n_f\}, \end{aligned} \tag{5}$$

where blocks Δ_i are used for unstructured uncertainties, while blocks $\delta_i I_{n_i}$ are used for lumped parametric uncertainties. As further developed in Section IV-C, we desire to encompass all uncertainties from the block $\boldsymbol{\Delta}$ into a single uncertainty block, to obtain firm guarantees on the robust controller's performance.

One contribution of the current paper is to present a way to deal with uncertainties modelled as descriptor systems. There are only a few available solutions for such problems, so we briefly present a non-convex optimization problem starting from our recent paper [13], with a practical necessity for this approach sourced from [14]. This leads to the problem of finding a suitable uncertainty weighting function $W_\theta(s)$ for SISO models such that the set $\boldsymbol{\Delta}$ will contain a single full block, i.e. $n_s = 0$ and $n_f = 1$. As such, define:

$$W_\theta(s) = \frac{\beta_m s^m + \beta_{m-1} s^{m-1} + \dots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1} s^{n-1} + \dots + \alpha_1 s + \alpha_0} = \frac{\boldsymbol{\beta}(s)}{\boldsymbol{\alpha}(s)}, \tag{6}$$

with pair (n, m) as hyperparameters, with no assumptions about the relationship between n and m. The vector of unknown parameters is defined as:

$$\theta_n^m \equiv \theta = \begin{pmatrix} \beta_m & \dots & \beta_0 & \alpha_{n-1} & \dots & \alpha_0 \end{pmatrix}^\top \in \mathbb{R}^{m+n+1}. \tag{7}$$

Further denote $M^\star : \Omega \to \mathbb{R}_+$, $M^\star(\omega)$ as experimental lower bound magnitude measurements, for a given set $\Omega \subset \mathbb{R}_+$.

A least conservative fitting problem in an ℓ_1-norm sense can be defined based on the following minimization problem with nonlinear constraints:

$$\min_\theta L(\theta) = \sum_{\omega \in \Omega} \left| |W_\theta(j\omega)|^2 - M^\star(\omega)^2 \right| \tag{8}$$

$$\text{subject to } \begin{cases} |W_\theta(j\omega)| \geq M^\star(\omega), \ \omega \in \Omega \\ \boldsymbol{\alpha}(s), \ \boldsymbol{\beta}(s) \text{ Hurwitz} \\ W_\theta(j\omega) \text{ valid with respect to } M^\star(\omega) \end{cases} .$$

To solve problem (8), MATLAB's `fmincon` solver can be employed using the `interior-point` or `active-set` algorithms. The problem presents convergence guarantees due to the qualities of the subgradient method, ensured through an adequate definition of the sign function found in the gradient of the objective function and first constraint. The Hurwitz conditions for the polynomials $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be algorithmically verified using the Hurwitz matrix formulation, while the system validity condition in the sense of [15] has been shown in [13] to be implied by the lower bound conditions in the provided SISO case. As such, this approach can be used to fit improper systems also.

IV. Proposed Robust Feedback Linearization

In this section we present a set of theoretical results for the possibility to use a diffeomorphism which linearizes the nominal system (Σ_n) to also obtain a linearized uncertain model for the following continuous-time input-affine uncertain nonlinear system:

$$(\Sigma) : \begin{cases} \dot{\mathbf{x}} & = f(\mathbf{x}) + \Delta f(\mathbf{x}) + \left(g(\mathbf{x}) + \Delta g(\mathbf{x})\right) u; \\ y & = h(\mathbf{x}). \end{cases} \tag{9}$$

Theorem 1: If the nominal system (Σ_n) has relative degree n and $\Delta g \in \mathrm{Span}\{g\}$, then the relative degree of the uncertain system (Σ) is equal to n.

Proof: If the nominal system (Σ_n) has relative degree n, then:

$$L_g L_f^k h(\mathbf{x}) = 0, \quad k < n-1.$$

But, because $\Delta g \in \mathrm{Span}\{g\}$, we have

$$L_{g+\Delta g} L_f^k h(\mathbf{x}) = 0, \quad k < n-1,$$

so the uncertain system also has relative degree n. ■

The assumption of having $\Delta g \in \mathrm{Span}\{g\}$ encompasses a relevant class of uncertain systems available in the literature. Furthermore, it is less conservative than in [5], [6]. As stated before, we want to use the same diffeomorphism as in the case of the nominal system. The nominal system has the relative degree n, so we consider the following n functions as the coordinate change:

$$\phi_i(\mathbf{x}) = L_f^{i-1} h(\mathbf{x}), \quad i = \overline{1,n}, \tag{10}$$

the new coordinates being:

$$\mathbf{z} = \Phi(\mathbf{x}). \tag{11}$$

Considering the same linearizable input (4), the uncertain system (Σ) can be written as:

$$\dot{\mathbf{z}} = A\mathbf{z} + Bv + \tilde{f}(\mathbf{z}) + \tilde{g}(\mathbf{z})v, \tag{12}$$

where v is the command signal for the outer loop. We will next show that the terms $\tilde{f}$ and $\tilde{g}$, which admit polytopic approximations, can be included for two particular cases ($\Delta f, \Delta g \in \mathrm{Span}\{g\}$ and $\Delta g \in \mathrm{Span}\{g\}$, respectively) into an uncertainty expressed as an improper system. This approximation will then be used in robust synthesis, as in [16].

A. Case I: $\Delta f, \Delta g \in \mathrm{Span}\{g\}$

If $\Delta f, \Delta g \in \mathrm{Span}\{g\}$, according to Theorem 1, we can ensure that the uncertain system has full relative degree if the nominal system (Σ_n) has full relative degree. The following lemma presents the state-space realization after considering the coordinate transformation (11).

Lemma 1: The state-space realization of the uncertain system (Σ) in the new coordinates $\mathbf{z} = \Phi(\mathbf{x})$ is given by:

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \vdots \\ \dot{z}_{n-1} = z_n \\ \dot{z}_n = L_f^n h\left(\Phi^{-1}(\mathbf{z})\right) + L_{\Delta f} L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right) + \\ \qquad + \left(L_g L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right) + L_{\Delta g} L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right)\right) u. \end{cases} \tag{13}$$

Proof: For the first $n-1$ states we have:

$$\begin{aligned} \dot{z}_i &= \frac{\partial \phi_i}{\partial \mathbf{x}} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \\ &= \frac{\partial L_f^{i-1} h(\mathbf{x})}{\partial \mathbf{x}} \left(f(x) + \underbrace{\Delta f(\mathbf{x}) + (g(\mathbf{x}) + \Delta g(\mathbf{x}))\, u}_{\in \mathrm{Span}\{g\}} \right) = \\ &= L_f^i h(\mathbf{x}) = z_{i+1}. \end{aligned}$$

For the last state we have:

$$\begin{aligned} \dot{z}_n &= \frac{\partial \phi_n(\mathbf{x})}{\partial \mathbf{x}} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \\ &= \frac{\partial L_f^{n-1} h(\mathbf{x})}{\partial \mathbf{x}} \left(f(x) + \Delta f(\mathbf{x}) + (g(\mathbf{x}) + \Delta g(\mathbf{x}))\, u\right) = \\ &\overset{\mathbf{z}=\Phi(\mathbf{x})}{=} L_f^n h\left(\Phi^{-1}(\mathbf{z})\right) + L_{\Delta f} L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right) + \\ &\quad + \left(L_g L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right) + L_{\Delta g} L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right)\right) u. \end{aligned}$$

■

The main idea underlined in Lemma 1 consists in the possibility to write the system (Σ) in form (12) with:

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}; \tag{14a}$$

$$\tilde{f}(\mathbf{z}) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \tilde{f}_n \end{pmatrix}; \quad \tilde{g}(\mathbf{z}) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{L_{\Delta g} L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right)}{L_g L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right)} \end{pmatrix}, \tag{14b}$$

where:

$$\begin{aligned} \tilde{f}_n &= L_{\Delta f} L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right) - \\ &\quad - L_{\Delta g} L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right) \frac{L_f^n h\left(\Phi^{-1}(\mathbf{z})\right)}{L_g L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right)}. \end{aligned} \tag{15}$$

As such, the nominal system is represented by a chain of n integrators. Now, starting from this nominal linearized system around a given equilibrium point, we want to encompass the residual nonlinear terms as an uncertainty.

Theorem 2: The uncertainty which encompasses the residual nonlinearities from $\tilde{f}$ and $\tilde{g}$ considering an inverse additive uncertainty is modelled using an improper system having the zeros excess equal to the order of the system.

Proof: Considering $G_{\mathbf{\Delta}}$ obtained as a linearized representation around a given equilibrium point $(\overline{\mathbf{x}}, \overline{u})$:

$$\left(A + \left(\nabla_{\mathbf{z}} \tilde{f} + \nabla_{\mathbf{z}} \tilde{g} \overline{v}\right)\Big|_{\overline{\mathbf{z}}=\Phi(\overline{\mathbf{x}})}, B + \tilde{g}(\overline{z}), C_{\mathbf{z}}, 0 \right),$$

where $C_{\mathbf{z}} = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$, and the nominal system $G_n(s) = \frac{1}{s^n}$, we have:

$$G_{\mathbf{\Delta}}(s) = \frac{1+\beta}{s^n + \alpha_{n-1}s^{n-1} + \dots + \alpha_0},$$

so the inverse additive model is given by:

$$\begin{aligned} W(s)\Delta &= \frac{1}{G_{\mathbf{\Delta}}(s)} - \frac{1}{G_n(s)} = \\ &= \frac{s^n + \alpha_{n-1}s^{n-1} + \dots + \alpha_0}{1+\beta} - s^n = \\ &= -\frac{\beta}{1+\beta} s^n + \frac{\alpha_{n-1}s^{n-1} + \dots + \alpha_0}{1+\beta}, \end{aligned}$$

where $\|\Delta\| \leq 1$ and $W(s)$ is the uncertainty model, therefore the conclusion follows. ■

B. Case II: $\Delta g \in \operatorname{Span}\{g\}$

We now proceed to analyze the more general case of having $\Delta g \in \operatorname{Span}\{g\}$ which, according to Theorem 1, ensures that the uncertain system has the full relative degree in the context of having a nominal system with full relative degree. After the coordinate transformation (11) is performed, the state-space realization has the form presented in the following lemma.

Lemma 2: The state-space realization of the uncertain system (Σ) in the new coordinates $\mathbf{z} = \Phi(\mathbf{x})$ is given by:

$$\begin{cases} \dot{z}_1 = z_2 + L_{\Delta f} h\left(\Phi^{-1}(\mathbf{z})\right) \\ \dot{z}_2 = z_3 + L_{\Delta f} L_f h\left(\Phi^{-1}(\mathbf{z})\right) \\ \vdots \\ \dot{z}_{n-1} = z_n + L_{\Delta f} L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right) \\ \dot{z}_n = L_f^n h\left(\Phi^{-1}(\mathbf{z})\right) + L_{\Delta f} L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right) + \\ \qquad + \left(L_g L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right) + L_{\Delta g} L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right)\right) u. \end{cases} \tag{16}$$

Proof: For the first $n-1$ states we have:

$$\begin{aligned} \dot{z}_i &= \frac{\partial \phi_i}{\partial \mathbf{x}} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \\ &= \frac{\partial L_f^{i-1} h(\mathbf{x})}{\partial \mathbf{x}} \left(f(x) + \Delta f(\mathbf{x}) + \underbrace{(g(\mathbf{x}) + \Delta g(\mathbf{x}))}_{\in \operatorname{Span}\{g\}} u \right) = \\ &= L_f^i h(\mathbf{x}) + L_{\Delta f} L_f^{i-1} h(\mathbf{x}) = z_{i+1} + L_{\Delta f} L_f^{i-1} h\left(\Phi^{-1}(\mathbf{z})\right), \end{aligned}$$

while for the last state we have:

$$\begin{aligned} \dot{z}_n &= \frac{\partial \phi_n(\mathbf{x})}{\partial \mathbf{x}} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \\ &= \frac{\partial L_f^{n-1} h(\mathbf{x})}{\partial \mathbf{x}} \left(f(x) + \Delta f(\mathbf{x}) + (g(\mathbf{x}) + \Delta g(\mathbf{x})) u \right) = \\ &\overset{\mathbf{z}=\Phi(\mathbf{x})}{=} L_f^n h\left(\Phi^{-1}(\mathbf{z})\right) + L_{\Delta f} L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right) + \\ &+ \left(L_g L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right) + L_{\Delta g} L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right)\right) u. \end{aligned}$$

■

The main idea underlined by Lemma 2 consists in the possibility to write the system (Σ) in form (12) with the same A and B matrices as in (14a):

$$\tilde{f}(\mathbf{z}) = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_{n-1} \\ \tilde{f}_n \end{pmatrix} \quad \tilde{g}(\mathbf{z}) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{L_{\Delta g} L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right)}{L_g L_f^{n-1} h\left(\Phi^{-1}(\mathbf{z})\right)} \end{pmatrix} \tag{17}$$

where $\tilde{f}_n$ is the term (15), while:

$$\tilde{f}_i = L_{\Delta f} L_f^{i-1} h\left(\Phi^{-1}(\mathbf{z})\right), \quad i = \overline{1, n-1}. \tag{18}$$

As such, the nominal system is represented by a chain of n integrators. Now, starting from this nominal linearized system around a given equilibrium point, we want to include the residual nonlinear terms as an uncertainty.

Theorem 3: The uncertainty which encompasses the residual nonlinearities from $\tilde{f}$ and $\tilde{g}$, considering an inverse additive uncertainty, is modelled using an improper system having the zeros excess equal to the order of the system.

Proof: Considering $G_{\mathbf{\Delta}}$ obtained as a linearized representation around a given equilibrium point $(\overline{\mathbf{x}}, \overline{u})$:

$$\left(A + \left(\nabla_{\mathbf{z}} \tilde{f} + \nabla_{\mathbf{z}} \tilde{g} \overline{v} \right) \Big|_{\overline{\mathbf{z}}=\Phi(\overline{\mathbf{x}})}, B + \tilde{g}(\overline{z}), C_{\mathbf{z}}, 0 \right),$$

where $C_{\mathbf{z}} = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$, and the nominal system $G_n(s) = \frac{1}{s^n}$, we have:

$$G_{\mathbf{\Delta}}(s) = \frac{(1+\beta)(s^{n-2} + \gamma_{n-3} s^{n-3} + \cdots + \gamma_0)}{s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_0},$$

so the inverse additive model is given by:

$$\begin{aligned} W(s)\Delta &= \frac{1}{G_{\mathbf{\Delta}}(s)} - \frac{1}{G_n(s)} = \\ &= \frac{s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_0}{(1+\beta)(s^{n-2} + \gamma_{n-3} s^{n-3} + \cdots + \gamma_0)} - s^n, \end{aligned}$$

where $\|\Delta\| \leq 1$ and $W(s)$ is the uncertainty model, therefore the conclusion follows. ■

C. Robust Component

According to Theorems 2 and 3, the residual nonlinear dynamics resulting after the inner state feedback linearization command (4) can be encompassed into an inverse additive uncertainty whose model is an improper system having the excess of zeros equal to n. As such, we want to include a second robust component which manages to find a (possibly fixed-structure) controller which ensures RS and RP.

Therefore, the inner closed-loop system can be viewed as an upper linear fractional transform interconnection (ULFT) between the plant M, constructed based on the nominal system $G_n(s) = \frac{1}{s^n}$ and the uncertainty model $W(s)$, and the uncertainty block $\mathbf{\Delta}$, where:

$$M(s) = \begin{pmatrix} -G_n(s) W(s) & G_n(s) W(s) \\ -G_n(s) & G_n(s) \end{pmatrix}.$$

At this step, the uncertain system $G_{\mathbf{\Delta}} = \operatorname{ULFT}(M, \Delta)$ presents an input $u \in \mathbb{R}$ and an output $y \in \mathbb{R}$ representing the controller's interface. To impose the desired performances, an augmentation step should be performed, resulting an additional set of performance inputs $\mathbf{u_w} \in \mathbb{R}^{n_w}$ and performance outputs $\mathbf{y_z} \in \mathbb{R}^{n_z}$, obtaining an augmented plant $P(s)$, as in [12]. The most common augmentation technique is the so-called mixed-sensitivity loop-shaping, where three weighting functions are considered for the sensitivity, complementary sensitivity, and control effort, namely $W_S(s)$, $W_T(s)$, and $W_{KS}(s)$. The plant $P(s)$ becomes:

$$\begin{pmatrix} W_S(s) & 0 & 0 & I \\ -W_S(s) G_{\mathbf{\Delta}}(s) & W_{KS}(s) & W_T(s) G_{\mathbf{\Delta}}(s) & -G_{\mathbf{\Delta}}(s) \end{pmatrix}^\top. \tag{19}$$

The plant P presents a lower linear fractional transform interconnection (LLFT) with the controller K, the resulting closed-loop system being:

$$\mathbf{y_z} = \mathrm{LLFT}(P, K)\mathbf{u_w}. \tag{20}$$

For the nominal case (i.e. $\Delta = O$), there are several solutions, such as $\mathcal{H}_2$ or $\mathcal{H}_\infty$ synthesis, while for the uncertain system the most common method to encompass them is to consider the structured singular value $\mu_{\mathbf{\Delta}}(\mathrm{LLFT}(P, K))$ framework [17]. Moreover, if the controller K should have a fixed structure, described by a family $\mathcal{K}$, the resulting optimization problem can be written as:

$$\inf_{K \in \mathcal{K}} \mu_{\mathbf{\Delta}}(\mathrm{LLFT}(P, K)), \tag{21}$$

whose sub-optimal solution $K^\star$ should fulfill the condition $\mu_{\mathbf{\Delta}}(\mathrm{LLFT}(P, K^\star)) < 1$ to ensure robust stability (RS) and robust performance (RP). There are several approaches available in the literature to solve this problem, the most common being based on a D/G–K iteration using non-smooth optimization techniques [17]. However, these solutions manage to solve the (fixed-structure) μ-synthesis optimization problem (21) if the plant P is a proper or a strictly proper system. According to Theorems 2 and 3, the excess of poles against zeros is $-n$ for $W(s)$ and n for G_n, leading to a proper system $M(s)$, so the robust control techniques could thus be applied.

V. Numeric Example

The numerical example considered in the current paper is a nonlinear mass-spring system with a hardening spring and without friction, the state-space model being:

$$(\Sigma_{\mathrm{ms}}): \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k_1 x_1 - k_2 x_1^3 + m \cdot u \\ y = x_1 \end{cases}, \tag{22}$$

where parameters have their nominal values: $k_1^{(n)} = 1$, $k_2^{(n)} = 0.1$, $m^{(n)} = 1$, along with a tolerance of $\pm 10\%$ for each. The state-space model can be written as:

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} x_2 \\ -k_1^{(n)} x_1 - k_2^{(n)} x_1^3 \end{pmatrix}}_{f} + \underbrace{\begin{pmatrix} 0 \\ -\delta_{k_1} x_1 - \delta_{k_2} x_1^3 \end{pmatrix}}_{\Delta f} + \left(\underbrace{\begin{pmatrix} 0 \\ -m \end{pmatrix}}_{g} + \underbrace{\begin{pmatrix} 0 \\ -\delta_m \end{pmatrix}}_{\Delta g} \right) u, \tag{23}$$

so $\Delta f, \Delta g \in \mathrm{Span}\{g\}$. The relative degree of the system is $r = 2$ for each $\mathbf{x} \in \mathbb{R}^2$, so the diffeomorphism:

$$\mathbf{z} = \Phi(\mathbf{x}) = \begin{pmatrix} h(\mathbf{x}) \\ L_f h(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{24}$$

is a global diffeomorphism for the nominal system. The inner feedback which linearizes the nominal system is:

$$u = \frac{v + k_1^{(n)} x_1 + k_2^{(n)} x_1^3}{m^{(n)}}, \tag{25}$$

while the resulting uncertain system in $\mathbf{z}$ coordinates can be described as follows:

$$\dot{\mathbf{z}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{z} + Bv + \underbrace{\begin{pmatrix} 0 \\ \frac{\delta_m}{m^{(n)}} \end{pmatrix}}_{\tilde{g}(\mathbf{z})} v + \underbrace{\begin{pmatrix} 0 \\ \frac{\delta_m}{m^{(n)}} \cdot \left(k_1^{(n)} z_1 + k_2^{(n)} z_1^3 \right) - \delta_{k_2} z_1^3 - \delta_{k_1} z_1 \end{pmatrix}}_{\tilde{f}(\mathbf{z})}. \tag{26}$$

Considering the forced equilibrium point $(\overline{\mathbf{x}}, \overline{u}) = ((0,0)^\top, 0)$, the residual nonlinear terms $\tilde{f}, \tilde{g}$ could be encompassed as an inverse additive uncertainty. Considering the raw frequency data (cyan) from Figure 1, with the upper bound $M^\star$ (red), the improper transfer function model having the $\deg \boldsymbol{\alpha} = 1$ and $\deg \boldsymbol{\beta} = 3$ obtained with the solution proposed in Section III is:

$$W(s) = \frac{0.1080s^3 + 0.3072s^2 + 0.3204s + 0.2527}{s + 1.1770}. \tag{27}$$

For the augmentation we consider the following filters for the sensitivity and the complementary sensitivity:

$$W_S(s) = \frac{0.625s + 0.6}{s + 0.006}, \quad W_T(s) = \frac{s + 12}{0.01s + 24}, \tag{28}$$

while for the control effort we considered $W_{KS}(s) = \frac{1}{20}$. Using the `musyn` routine from MATLAB, after 10 D/G–K iterations we obtain the following PID controller as the solution of the fixed-structure mixed-sensitivity loop shaping μ-synthesis control problem:

$$K(s) = 1.1119 + \frac{0.1712}{s} + \frac{1.7543s}{0.1239s + 1}, \tag{29}$$

with an upper bound of the structured singular value:

$$\mu_{\mathbf{\Delta}}(\mathrm{LLFT}(P, K)) \leq 0.9968 < 1, \tag{30}$$

so robustness is ensured by the linear controller.

For the simulation discussion, we consider a set of 10 Monte Carlo simulations in which both initial conditions and model parameters vary. For the initial conditions we imposed

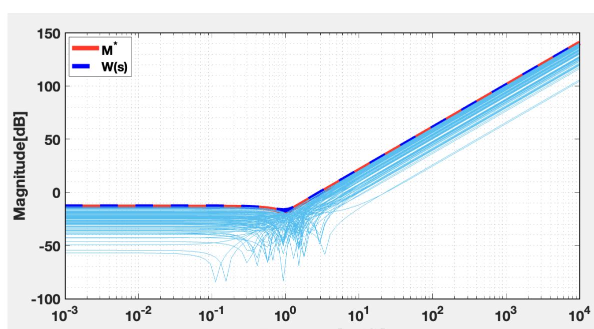

Fig. 1. The raw frequency data of the inverse additive uncertainty resulting from the nonlinear residual terms from (12) after the inner state feedback linearization (cyan), along with their maximum values (red) and the frequency response of the solution of the problem described in Section III (blue).

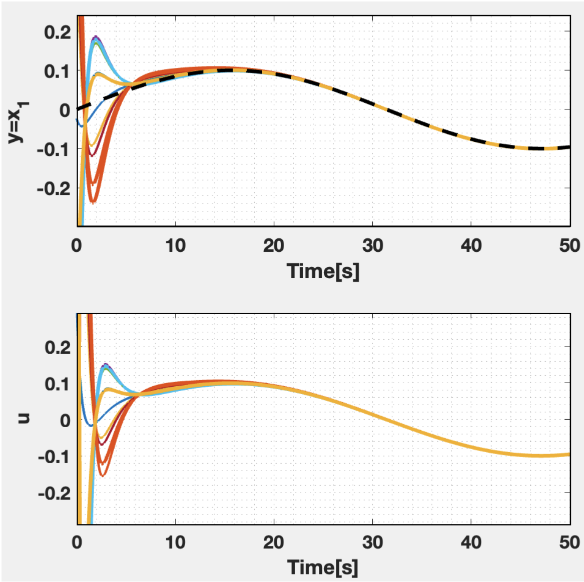

Fig. 2. First row: robust reference tracking of the resulting closed-loop system when the model's parameters and the initial conditions vary in a given range. Second row: the command signal which fulfills the imposed maximum value via the loop shaping augmentation step.

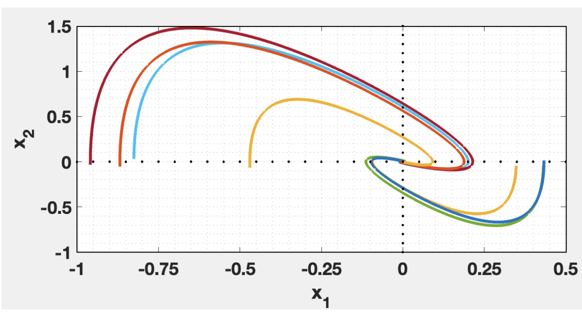

Fig. 3. Closed-loop trajectories starting from various initial conditions and considering different model parameter configurations.

$|x_1(0)| \leq 1$ and $|x_2(0)| \leq 0.1$, with the scope of following a sinusoidal trajectory, according to the idea of Duffing's equation [1]. The reference frequency is $0.1[rad/s]$, with the magnitude 0.1. Figure 2 illustrates the robustness of the proposed method against parameters variations and various initial conditions, having a consistent convergence rate of $10[s]$, while the command signal presented in the second row does not exceed the imposed bound limitations via the loop shaping procedure. In terms of the convergence to the equilibrium point, the phase portrait from Figure 3 shows that all trajectories manage to reach the equilibrium point irrespective of the given initial condition and the parameter configuration in the given range. Additionally, the rightmost pair of blue and green trajectories start from the same initial conditions, but the model parameters are different, with the finding that the convergence speeds remain invariant.

VI. Conclusions and Future Works

The paper provided a mechanism to extend the state feedback linearization technique for uncertain nonlinear systems which preserves the full relative degree by including the residual nonlinearities into an uncertainty described using descriptor systems, even if the polytopic approximation can be conservative. Additionally, a first iteration of a fitting mechanism for descriptor systems has been introduced. The main advantage against other relevant papers which deal with the same problem (e.g. [5], [6], [7]) consist in removing one control layer by considering only the feedback linearization law and the robust component.

As further research directions, we want to extend the proposed mechanism for systems with zero dynamics and variable relative degree by keeping the same structure of the controller. Additionally, a convex mechanism for the linear differential-algebraic systems of equations (DAEs) fitting problem will later replace the non-convex solution presented in Section III.

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