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Optimal placement and shape design of sensors via geometric criteria

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Abstract— **The optimal placement and design of sensors is commonly encountered in industrial and applied problems, such as urban planning and the supervision of temperature and pressure in gas networks. In essence, sensors are considered optimally designed when they ensure the highest level of observation for the specific phenomenon in question. Typically, this design process is guided by specific objectives and is subject to constraints commonly defined by an appropriate partial differential equation (PDE), taking into account the underlying physics of the process. In the present work, we focus on two independent study cases:**

- **The optimal shape design of a convex sensor.**
- **The optimal placement of a finite number of sensors inside a given region.**

Here, we address the problem in a purely geometric setting, without involving a specific PDE model. We consider a simple and natural geometric criterion of performance, based on distance functions. But, as we shall see, tackling it will require to employ geometric analysis methods.

I. Introduction

The optimal placement of sensors in various industrial applications is a critical component of modern monitoring and control systems. For example, in the context of gas networks, sensor placement is essential for ensuring the efficient distribution and management of gases like natural gas or hydrogen. Sensors are strategically located at various points within the network, such as pipelines, compressor stations, and distribution hubs, to monitor factors like pressure, temperature, flow rates, and gas composition. For instance, in a natural gas network, sensors placed near critical junctions can swiftly detect leaks or pressure fluctuations, allowing operators to take immediate corrective actions, reducing the risk of safety incidents and environmental damage. The data collected from these sensors can also be analyzed to optimize the network's performance and minimize energy losses, ultimately improving its reliability and cost-effectiveness.

The body of literature addressing optimal sensor placement is extensive, with techniques varying depending on the specific problem at hand and the underlying modeling assumptions. For example, in [9], the authors introduce an algorithm aimed at determining the optimal sensors to incorporate, ensuring compliance with diagnosis requirements related to fault detectability and isolability. They subsequently demonstrate the efficacy of their approach through an illustrative application to a model of an industrial valve, effectively showcasing the benefits and properties of their method. Another interesting example can be found in [6], where the authors investigate the optimal sensor placement in a water distribution network avoiding deliberate contaminant intrusion. Finally, we recommend consulting the books [2], [11] for further insight and practical examples.

In this note, we consider the problem of optimizing sensor design within a purely geometric framework, devoid specific PDE models. Our approach revolves around a straightforward yet effective performance metric based on distance functions. However, as we'll soon discover, navigating this challenge will necessitate the utilization of geometric analysis techniques and prompt consideration of classic approximations of distance functions via solutions of some suitable PDEs.

These problems can then be formulated in a shape optimization framework. Indeed, given a set $\Omega\subset\mathbb{R}^2$, and a mass fraction $c\in(0,|\Omega|)$, the problem can be mathematically formulated as follows:

$$\inf\{\sup_{x\in\Omega} d(x,\omega)\ |\ |\omega|=c \text{ and } \omega\subset\Omega\},$$

where $d(x,\omega):=\inf_{y\in\omega}\|x-y\|$ is the minimal distance from x to ω. In fact, the problem can be written in terms of the classical Hausdorff distance d^H. Indeed, when $\omega\subset\Omega$, one has

$$\sup_{x\in\Omega} d(x,\omega)=d^H(\omega,\Omega).$$

We are then interested in considering the following problem

$$\inf\{d^H(\omega,\Omega)\ |\ |\omega|=c \text{ and } \omega\subset\Omega\},\qquad (1)$$

where $c\in(0,|\Omega|)$.

By using a homogenization strategy, which consists in uniformly distributing the mass of the sensor over Ω (see Figure 1), we show that problem (1) does not admit a solution. Indeed, the infimum is equal to 0 and is asymptotically attained by a sequence of disconnected sets with an increasing number of connected components.

In order to make problem (3) non trivial, we chose to work with two classes of sensors:

- The class of convex sensors of a given area.
- The class of unions of N spherical sensors of a given radius.

II. The case of convex sensors

In this section, we focus on the optimal design of a convex sensor ω inside a given convex region Ω in such a way to

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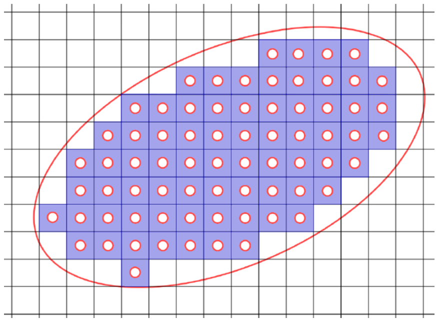

Fig. 1. The homogenization strategy.

minimize the maximal distance to all the points of Ω. The problem can then be written in the following form

$$\min\{d^H(\omega,\Omega) \mid \omega \text{ is convex} \subset \Omega \text{ such that } |\omega| = \alpha_0|\Omega|\}, \quad (2)$$

with $\alpha_0 \in (0,1)$. One may then consider to parameterize a convex sets K via its support functions h_K defined on $[0,2\pi)$ as follows:

$$h_K : [0,2\pi) \longmapsto \sup\left\{\left\langle \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, y\right\rangle \mid y \in K\right\}.$$

The support function is a classic tool that allows to parametrize convex sets. For more information, one may refer to [12, Chapter 2]. This parametrization is popular among convex geometers as it has some interesting properties:

- It allows to provide a simple criterion of the convexity of Ω. Indeed, Ω is convex if and only if $h_\Omega''+h_\Omega \geq 0$ in the sense of distributions, see for example [12, (2.60)].
- It is linear for the Minkowski sum and dilatation. Indeed, if Ω_1 and Ω_2 are two convex bodies and $\alpha,\beta > 0$, we have
$$h_{\alpha\Omega_1+\beta\Omega_2} = \alpha h_{\Omega_1} + \beta h_{\Omega_2},$$
see [12, Section 1.7.1].
- It allows to parametrize the inclusion in a simple way. Indeed, if Ω_1 and Ω_2 are two convex sets, we have
$$\Omega_1 \subset \Omega_2 \Longleftrightarrow h_{\Omega_1} \leq h_{\Omega_2}.$$
- It also provides elegant formulas for some geometric quantities. For example, the perimeter and the area of a convex body Ω are respectively given by
$$P(\Omega) = \int_0^{2\pi} h_\Omega(\theta)d\theta$$
and
$$|\Omega| = \frac{1}{2}\int_0^{2\pi}({h_\Omega'}^2 - h_\Omega^2)d\theta.$$
As for the Hausdorff distance between two convex bodies Ω_1 and Ω_2, it is given by
$$d^H(\Omega_1,\Omega_2) = \max_{\theta\in[0,2\pi]} |h_{\Omega_1}(\theta) - h_{\Omega_2}(\theta)|,$$
see [12, Lemma 1.8.14].

As explained in [8, Section 5], the support function can be used to provide a numerical scheme to solve problem (2) in particular and other shape optimization problems with convexity constraints in general. We refer to the following works for more examples and details [1], [3], [4]. Thus, problem (2) can be formulated in terms support functions as follows

$$\begin{cases} \inf\limits_{h\in H^1_{\text{per}}(0,2\pi)} \|h_\Omega - h\|_\infty, \\ h \leq h_\Omega, \\ h'' + h \geq 0, \\ \frac{1}{2}\int_0^{2\pi} h(h''+h)d\theta = \alpha_0|\Omega|, \end{cases} \quad (3)$$

where $H^1_{\text{per}}(0,2\pi)$ is the set of Sobolev H^1 functions that are 2π-periodic.

Problem (3) is then discretized in judicious way via truncated Fourier series of the involved (periodic) functions. Then, Matlab's routine `'fmincon'` is used to solve the obtained finite dimensional problem approximating problem (3).

In the following figures we present the results obtained for different shapes and different mass fractions $\alpha_0|\Omega|$, where $\alpha_0 \in \{0.01, 0.1, 0.4, 0.7\}$.

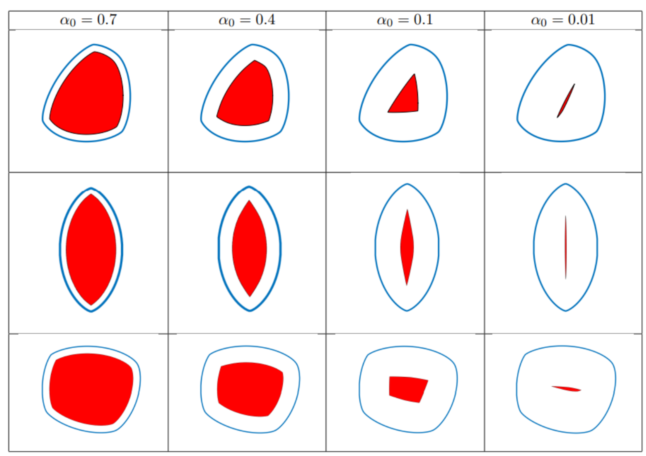

Fig. 2. Obtained optimal shapes for $\alpha_0 \in \{0.01, 0.1, 0.4, 0.7\}$ and different choices of Ω.

III. The case of N sensors

In this section, we consider the problem of minimizing the farthest distance to N spherical sensors of a given radius r. The problem is formulated as follows

$$\inf\{d^H(\cup_{i=1}^N B_i, \Omega) \mid \forall i \in \{1,\dots,N\}, \quad B_i \in \Omega\}, \quad (4)$$

where B_i are balls of radius r. We then consider (x_i) the centers of the balls and write (4) as a finite dimensional optimization problem

$$\inf\{f(x_1,\dots,x_N) \mid x_i,\dots,x_N \in \Omega_{-r}\}, \quad (5)$$

where $f(x_1,\dots,x_N) := \|d(\cdot,\cup B_i)\|_\infty = d^H(\cup_{i=1}^N B_i,\Omega)$ and $\Omega_{-r} := \{x\in\Omega \mid d(x,\partial\Omega) \geq r\}$.

Being non differentiable, the infinity norm $\|\cdot\|_\infty$ is in general not practical to consider when performing numerical optimization as one needs accurate formulas for the gradients of the objective function that one wants to optimize. We then chose to work with the following functional involving the p-norm $\|\cdot\|_p$, with $p > 0$ sufficiently large. We are then minimizing functions

$$f_p(x_1, \dots, x_N) := \|d(\cdot, \cup B_i)\|_p = \left(\int_\Omega d(x, \cup B_i)^p dx \right)^{1/p},$$

known as *average distance* functionals. We refer to [10] for a review on the average distance problems and relevant references.

In order to numerically tackle this problem, one has to compute the distance function the the balls B_i. The prevailing approach to distance computation is to solve the eikonal equation

$$|\nabla d| = 1$$

with Dirichlet boundary conditions $d = 0$ on the balls B_i.

Being non linear and hyperbolic, such equations present difficulties to be solved directly. Inspired by [5], we propose to use an approximation of distance functions via a classic PDE result of Varadhan [13]:

Theorem III.1. *Let Ω be an open subset of $\mathbb{R}^n$ and $\varepsilon > 0$, we consider the problem*

$$\begin{cases} w_\varepsilon - \varepsilon \Delta w_\varepsilon = 0 & \text{in } \Omega, \\ w_\varepsilon = 1 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

We have

$$\lim_{\varepsilon \to 0} -\sqrt{\varepsilon} \ln w_\varepsilon(x) = d(x, \partial\Omega) := \inf_{y \in \partial\Omega} \|x - y\|,$$

uniformly over compact subsets of Ω.

In Figure 3, we plot the approximation of the distance function to the boundary obtained via the result of Theorem III.1.

Let us now show how we can use the result of Theorem III.1 to construct an approximation of the objective function f defined in problem (5).

Proposition III.2. *Let us consider a fixed box D containing the set $\{x \in \mathbb{R}^n \mid d(x, \Omega) \le diam(\Omega)\}$, where $diam(\Omega)$ is the diameter of Ω. For $\varepsilon > 0$, we denote by w_ε the solution of the problem*

$$\begin{cases} w_\varepsilon - \varepsilon \Delta w_\varepsilon = 0 & \text{in } D \setminus \cup_{i=1}^N B_i \\ w_\varepsilon = 1 & \text{on } \partial \cup_{i=1}^N B_i \cup \partial D \end{cases}$$

The function $v_\varepsilon : x \longmapsto -\sqrt{\varepsilon} \ln w_\varepsilon(x)$ uniformly converges to

$$d(\cdot, \cup B_i) : x \longmapsto \inf_{y \in \cup_{i=1}^N B_i} \|x - y\|$$

on $\Omega \setminus \cup_{i=1}^N B_i$.

Proof: By Theorem III.1, the function v_ε uniformly converges to the function $d_{\omega \cup \partial B} : x \longmapsto \inf_{y \in \omega \cup \partial B} \|x - y\|$ on $\Omega \setminus \cup_{i=1}^N B_i$. Since the box D contains the set

$$\{x \in \mathbb{R}^n \mid d(x, \Omega) \le diam(\Omega)\},$$

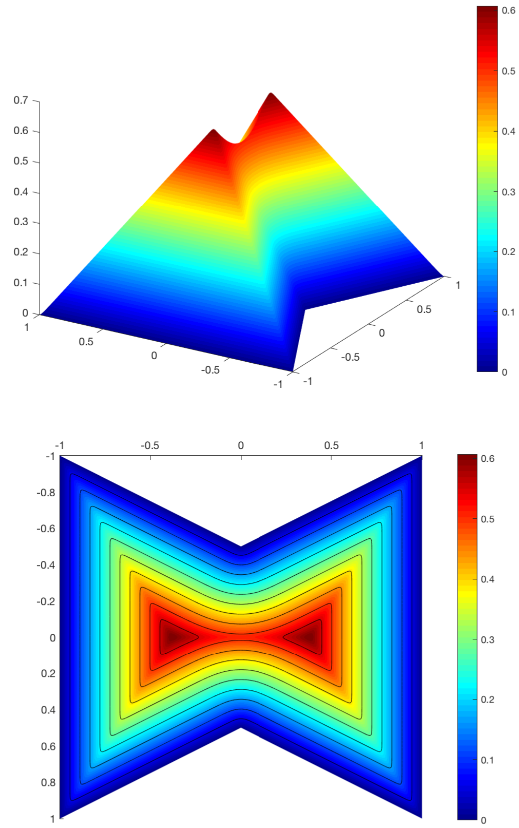

Fig. 3. Approximation of the distance function to the boundary via Varadhan's result of Theorem III.1, where we have used Matlab's toolbox `'PDEtool'` to solve problem (6), with $\varepsilon = 10^{-4}$.

we have for every $x \in \Omega \setminus \cup_{i=1}^N B_i$

$$d(x, \cup_{i=1}^N \partial B_i \cup \partial D) = d(x, \cup_{i=1}^N B_i),$$

because $d(x, \partial D) \ge diam(\Omega) \ge d(x, \cup_{i=1}^N B_i)$, where the first inequality is a consequence of the inclusion $\{x \in \mathbb{R}^n \mid d(x, \Omega) \le diam(\Omega)\} \subset D$ and the second one is a consequence of the inclusion $\cup_{i=1}^N B_i \subset \Omega$.

Now, that we have an approximation v_ε of the distance to the sensors B_i via Varadhan's result, we consider the following approximation of f:

$$f_{p,\varepsilon}(x_1, \dots, x_N) := \|v_\varepsilon\|_p = \left(\int_\Omega v_\varepsilon^p dx \right)^{1/p}$$

and will focus the numerical resolution of problems

$$\inf\{f_{p,\varepsilon}(x_1, \dots, x_N) \mid x_i, \dots, x_N \in \Omega_{-r}\}, \tag{7}$$

with $p > 0$. The first step is to compute the gradients of the function $f_{p,\varepsilon}$ via shape derivatives for perturbations corresponding to translations of the sensors B_i. To obtain practical formulas of the shape derivatives a judicious adjoint state is introduced. Once, the gradients are computed, we use them to perform a gradient descent in order to minimize the function $f_{p,\varepsilon}$ and thus find the optimal placement of the sensors B_i. More information is coming the paper in preparation [7].

In Figure 4, we present an example of the obtained optimal placement of $N \in \{1, 2, 3\}$ sensors.

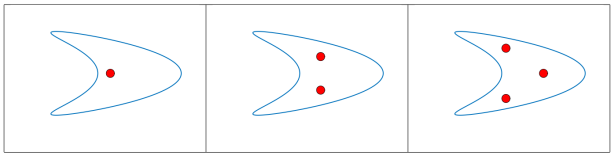

Fig. 4. Optimal placement of $N \in \{1, 2, 3\}$ sensors.

IV. CONCLUSION AND PERSPECTIVES

The problem of optimal shape and placement of sensors has been addressed in a purely geometric setting, independent of the physical process under consideration and in the absence of PDE restrictions. Problems are then recast in the context of of the optimization of the Hausdorff distance, but the use of Varadhan's approximation theorem naturally leads to consider optimization problems constrained by the Laplacian. This allows to apply the classical analytical and computational tools in PDE shape design.

In conclusion, mathematics provides analytical tools and frameworks necessary for modeling, optimizing, and analyzing sensor placement strategies within applied contexts. By leveraging mathematical principles such as optimization algorithms, graph theory, and statistical modeling, engineers and researchers can devise sophisticated sensor deployment schemes tailored to the specific needs and constraints of industrial systems. Hence, numerous avenues for further development and exploration remain, spanning both theoretical inquiry and practical application.

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