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Distributed Feedback Optimization of Networked Nonlinear Systems Using Relative Output Measurements*

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Abstract— **This paper investigates the distributed feedback optimization problem of nonlinear multi-agent systems. In such systems, each agent can measure the relative outputs between itself and its neighbors but lacks access to their absolute states and internal controller states. By combining distributed optimization and singular perturbation methods, a novel distributed controller design is presented, that relies solely on each agent's real-time gradient values of its local objective function and its relative output measurements to neighboring agents. The boundedness of the closed-loop signals and the convergence of the agent outputs to the minimizer of the total cost are proved rigorously. A numerical example is conducted to validate the effectiveness of the proposed approach.**

I. INTRODUCTION

In recent decades, research on multi-agent systems has rapidly expanded [1], [2]. In such systems, agents work together to facilitate the achievement of a common objective for the entire system through the collection of partial information and the exchange of information with neighboring agents.

Within the realm of multi-agent systems, consensus and distributed optimization are two popular topics. The consensus problem aims to bring the outputs of all agents to a common point using the exchanged information [3]. In contrast, distributed optimization focuses on a more challenging issue where the desired common point that all agents converge to is the minimizer of a total objective function. However, each agent can only access its corresponding local objective function, which constitutes a portion of the total objective function [4], [5]. To address the distributed optimization problem, researchers have developed various discrete-time [6] and continuous-time [7] distributed algorithms, generalizing the dynamics of each agent from first-order to higher-order models [8], [9].

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Another prominent topic in the field of multi-agent systems that has gained considerable attention in recent years is distributed feedback optimization. Feedback optimization addresses the challenge of guiding a physical system's output towards the solution of an optimization problem without prior knowledge of the explicit optimal solution [10], [11]. A key feature of feedback optimization is that the designed algorithm relies solely on the real-time value of the objective function's gradient at the plant's output, rather than the analytic expression of the gradient function. Distributed feedback optimization broadens the scope of feedback optimization to multi-agent systems, where agents are expected to reach a consensus on the minimizer of the total objective function by utilizing the real-time gradient values of local objective functions. Various distributed feedback optimization algorithms have been developed for power systems [12], [13], general linear systems [14], [15], [16], and nonlinear systems [17], [18].

This paper focuses on solving the distributed feedback optimization problem, where each agent can only measure the real-time gradient values of its local objective function and its relative outputs to neighboring agents. The motivation for this study comes from cooperative exploration tasks [19], where the gradient function of an environmental field is unknown and sharing state information across the network is more costly than measuring relative positions of nearest neighbors. Unlike existing consensus research [20], [21], this study also requires that the consensus point minimizes the total objective function. Additionally, unlike previous works focusing on agents with specific forms [22], [23], [24], this paper explores multi-agent systems with more general dynamics, offering increased applicability and flexibility.

This paper proposes a novel distributed feedback optimization algorithm by integrating distributed optimization results and the singular perturbation method [25]. In this algorithm, each agent only needs to measure the real-time gradient value of its local objective function and the relative outputs between itself and its neighbors. Notably, both the agents and the distributed feedback optimization algorithm are not required to have exponentially stable equilibria. As a result, existing stability analysis results for singularly perturbed systems cannot be directly applied. To address this issue, we rigorously analyze the solution of the closed-loop system with a specific initial condition using Barbalat's lemma. We show that if each agent's dynamics are output strictly passive, local objective functions exhibit strong convexity with Lipschitz continuous gradients, and the information exchange graph is strongly connected and weight-balanced, then suitable controller parameters can be chosen to ensure that all signals in the closed-loop system remain bounded and the outputs of all agents converge to the optimal solution of the total objective function. The paper also discusses the significance of the initial condition requirement and the generalization of the proposed method to time-varying digraphs.

The remainder of this paper is structured as follows: Section II outlines the problem under investigation. Section III presents the main result, which is verified through a numerical simulation in Section IV. Finally, Section V provides concluding remarks.

II. PROBLEM FORMULATION

Consider a heterogeneous multi-agent system with the dynamics of agent $i \in \mathcal{N} = \{1, 2, \ldots, N\}$ described by

$$\dot{x}_i = f_i(x_i, u_i), \tag{1}$$

$$y_i = g_i(x_i), \tag{2}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^m$, $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^m \to \mathbb{R}^{n_i}$ and $g_i : \mathbb{R}^{n_i} \to \mathbb{R}^m$ are the state, control input, output, dynamics and output map, respectively. The dimensions of the state and input, together with the dynamics and output map, can vary among agents. It is assumed that the functions f_i and g_i are locally Lipschitz.

This paper aims to design distributed controllers for the multi-agent system (1)–(2) such that all signals in the closed-loop system are bounded, and

$$\lim_{t\to\infty} y_i(t) = \arg\min_{s\in\mathbb{R}^m} \sum_{j\in\mathcal{N}} c_j(s), \tag{3}$$

for all $i \in \mathcal{N}$, where the local objective functions $c_i : \mathbb{R}^m \to \mathbb{R}$, $i \in \mathcal{N}$ are assumed to be continuously differentiable. In the distributed controllers, agent i uses the real-time gradient value $\nabla c_i(y_i)$ as feedback, rather than the gradient function ∇c_i. Furthermore, agent i is only able to measure the relative outputs $y_i - y_j$ with respect to its neighboring agents $j \in \mathcal{N}$. In this setting, it is not feasible to achieve the control objective (3) by first obtaining the optimal solution of the optimization problem offline and then adjusting the outputs of the agents to the optimal solution.

The information exchange topology of the multi-agent system is described by a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$. Each element of the node set $\mathcal{N} = \{1, 2, \ldots, N\}$ represents an agent. An edge (i, j) with $i, j \in \mathcal{N}$ of the edge set $\mathcal{E}$ means that agent j can receive information from agent i. The adjacency matrix $\mathcal{A} \in \mathbb{R}^{N\times N}$ consists of nonnegative entries a_{ij}s' which satisfy $a_{ji} = 1$ if $(i, j) \in \mathcal{E}$, and $a_{ji} = 0$ if $(i, j) \notin \mathcal{E}$. The digraph $\mathcal{G}$ is called strongly connected, if for any ordered pair of nodes $i, j \in \mathcal{N}$, there exists a path from i to j. Its Laplacian matrix L is a matrix of dimension $N \times N$ with the elements defined as $l_{ij} = -a_{ij}$ for $i, j \in \mathcal{N}, i \neq j$, and $l_{ii} = \sum_{j\in\mathcal{N}} a_{ij}$ for $i \in \mathcal{N}$. Clearly, $L1_N = 0$ holds. The graph is called weight-balanced if $1_N^T L = 0$.

The following assumptions are made regarding the dynamics of the agents, the convexity and smoothness of the local objective functions and the connectivity of the digraph.

Assumption 1: For each $i \in \mathcal{N}$, there exist a continuously differentiable function $V_i : \mathbb{R}^{n_i} \times \mathbb{R}^m \to \mathbb{R}_+$, class $\mathcal{K}_\infty$ functions $\varphi_1, \varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$, and positive constants γ and ζ such that

$$\varphi_1(|x_i|) \leq V_i(x_i, 0) \leq \varphi_2(|x_i|), \tag{4}$$

$$\nabla_{x_i}^T V_i(x_i, u_i) f_i(x_i, u_i) \leq -\gamma |g_i(x_i) - u_i|^2, \tag{5}$$

$$|\nabla_{u_i} V_i(x_i, u_i)| \leq \zeta |g_i(x_i) - u_i|, \tag{6}$$

for all $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^m$.

Remark 1: The condition (4) and the continuity of V_i implies that if $V_i(x_i, u_i)$ and u_i are bounded, then x_i is also bounded. The conditions (5) and (6) indicate that the system (1)–(2), augmented with an integrator $\dot{u}_i = r_i$, is output strictly passive with r_i as the input and $\nabla_{u_i} V_i(x_i, u_i)$ as the output [26, p. 236]. When u_i is a constant vector, the conditions (4) and (5) imply that y_i will converge to u_i. It is important to note that if (1)–(2) does not satisfy Assumption 1, but the cascaded system formed by combining (1)–(2) with a controller that has a reference input w_i does satisfy Assumption 1 with u_i replaced by w_i, then our method can still be used to generate the newly introduced reference input w_i; see Section IV for a numerical example. Compared to the assumptions used in [22], [23], [24] that also tackle distributed feedback optimization problems using only relative outputs, Assumption 1 allows for the more complex agent dynamics, e.g., agents with parametric uncertainties, resulting in a more flexible framework.

Assumption 2: For each $i \in \mathcal{N}$, c_i is ω-strongly convex, and ∇c_i is ϑ-Lipschitz. Namely, there exist positive constants ω and ϑ such that

$$(\nabla c_i(\zeta_1) - \nabla c_i(\zeta_2))^T (\zeta_1 - \zeta_2) \geq \omega |\zeta_1 - \zeta_2|^2, \tag{7}$$

$$|\nabla c_i(\zeta_1) - \nabla c_i(\zeta_2)| \leq \vartheta |\zeta_1 - \zeta_2|, \tag{8}$$

for any $i \in \mathcal{N}$ and $\zeta_1, \zeta_2 \in \mathbb{R}^m$.

Assumption 3: The digraph $\mathcal{G}$ is strongly connected and weight-balanced.

Assumptions 2 and 3 are widely used in the study of distributed optimization problem [27], [5] and distributed feedback optimization problem [22], [23], [24]. Note that Assumption 2 guarantees the uniqueness of the solution to the optimization problem $\min_{s\in\mathbb{R}^m} \sum_{j\in\mathcal{N}} c_j(s)$ [28, p. 460]. Define $y_* = \arg\min_{s\in\mathbb{R}^m} \sum_{j\in\mathcal{N}} c_j(s)$. Then, by [28, p. 140], it follows that

$$\sum_{j\in\mathcal{N}} \nabla c_j(y_*) = 0. \tag{9}$$

III. MAIN RESULTS

For the multi-agent system (1)–(2), we design the following distributed feedback optimization algorithm:

$$\dot{u}_i = \varepsilon\Big(-\alpha \nabla c_i(y_i) - \beta \sum_{j\in\mathcal{N}} a_{ij}(y_i - y_j) - q_i\Big), \tag{10}$$

$$\dot{q}_i = \varepsilon\alpha\beta \sum_{j\in\mathcal{N}} a_{ij}(y_i - y_j), \tag{11}$$

for $i \in \mathcal{N}$, where $[u_i^T, q_i^T]^T \in \mathbb{R}^{2m}$ is the state, and $\varepsilon, \alpha, \beta$ are positive constants to be determined later.

Remark 2: The design of algorithm (10)–(11) is inspired by ideas in feedback optimization, distributed optimization, and singular perturbation, where the feedback information used for each agent i is the gradient value $\nabla c_i(y_i)$ at the real-time output y_i and its relative output measurements to neighboring agents. If $y_i \equiv u_i$ for all $i \in \mathscr{N}$ and $\varepsilon = 1$, algorithm (10)–(11) is reduced to the one proposed in [27], where the convergence of u_i to the optimal point y_* is demonstrated. However, for the system (1)–(2) with complex dynamics, e.g., systems featuring nonlinear, uncertain, and other characteristics, the condition $y_i \equiv u_i$ generally does not hold. Thus, the closed-loop multi-agent system, consisting of (1)–(2) and (10)–(11), is essentially an interconnected system, but its stability and convergence properties are not guaranteed. To address this challenge, this paper introduces a constant ε. By defining a new time scale $\tau = \varepsilon t$ [25], the stability and convergence properties of the closed-loop system are ensured.

Remark 3: Following the framework of singular perturbation [25], the transformed system (10)–(11), obtained by the transformation $\tau = \varepsilon t$ and substitution $y_i \equiv u_i$, can be considered as the slow model, while the system (1)–(2), with a constant u_i, can be regarded as the fast model. Since the slow model does not have an isolated equilibrium, and the fast model may not have an exponentially stable equilibrium, the singular perturbation results presented in [25], [26] cannot be directly applied to the convergence analysis of the closed-loop multi-agent system. To solve this issue, Barbalat's lemma will be employed to provide a rigorous convergence analysis for each possible trajectory.

In the distributed feedback optimization algorithm (10)–(11), agent i can measure the relative output $y_i - y_j$ to its neighbor j. The structure of the closed-loop system is depicted in Fig. 1.

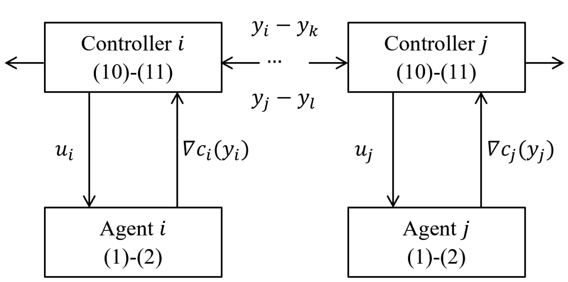

Fig. 1. Structure of the closed-loop multi-agent system.

By defining

$$u = [u_1^T, u_2^T, \ldots, u_N^T]^T, \quad q = [q_1^T, q_2^T, \ldots, q_N^T]^T, \tag{12}$$

$$y = [y_1^T, y_2^T, \ldots, y_N^T]^T, \quad \tilde{c}(y) = \sum_{i \in \mathscr{N}} c_i(y_i), \tag{13}$$

the compact form of the distributed feedback optimization algorithm (10)–(11) is

$$\dot{u} = \varepsilon(-\alpha \nabla \tilde{c}(y) - \beta \overline{L} y - q), \tag{14}$$

$$\dot{q} = \varepsilon \alpha \beta \overline{L} y, \tag{15}$$

where $\overline{L} = L \otimes I_p$.

The main results of this paper are summarized in the following theorem.

Theorem 1: Consider the multi-agent system (1)–(2). Under Assumptions 1, 2 and 3, there exist positive constants α, β, ε, ϕ, σ, ε_1 and ε_2 such that

$$\phi + 1 > 4\vartheta, \quad \rho > 0, \quad \eta_1 > 0, \quad \eta_3 > 0, \quad \eta_4 > 0, \tag{16}$$

where

$$\rho = \alpha^2(\phi+1)\omega + 9\lambda_2 \beta \alpha \phi - 4\alpha^2(\omega\vartheta + (\phi+1)^2), \tag{17}$$

$$\eta_1 = \min\left\{\frac{7}{16}, \frac{\rho}{9}\right\} - \frac{1}{2}\varepsilon_1 |P|^2 \varepsilon^2, \tag{18}$$

$$\eta_2 = \frac{1}{2\varepsilon_1}(\alpha^2\beta^2|L|^2 + 2\alpha^2\vartheta^2 + 2\beta^2|L|^2), \tag{19}$$

$$\eta_3 = \gamma - \frac{1}{2}\varepsilon_2 \zeta^2 - \frac{5}{2\varepsilon_2}\varepsilon^2(\beta^2|L|^2 + \alpha^2) - \sigma\eta_2, \tag{20}$$

$$\eta_4 = \sigma\eta_1 - \frac{5}{2\varepsilon_2}\varepsilon^2 \max\{\alpha^2\vartheta^2 + \beta^2|L|^2, 1\}, \tag{21}$$

and λ_2 is the second smallest eigenvalue of $(L + L^T)/2$. Moreover, for any initial conditions $x_i(0) \in \mathbb{R}^{n_i}$, $u_i(0) \in \mathbb{R}^m$ and $q_i(0) \in \mathbb{R}^m$ satisfying $\sum_{i \in \mathscr{N}} q_i(0) = 0$, $i \in \mathscr{N}$, the optimization objective (3) can be achieved by the distributed feedback optimization algorithm (10)–(11) with any α, β and ε satisfying (16).

Proof: The proof initially demonstrates the existence of the parameters satisfying (16), and subsequently analyzes the attainment of the optimization objective.

The parameters α, β, ε, ϕ, σ, ε_1 and ε_2 can be selected by the following procedure:

1) choose any positive constants α and ε_1;
2) choose positive constants ϕ and ε_2 such that
$$\phi + 1 > 4\vartheta, \qquad \varepsilon_2 < \gamma/(2\zeta^2); \tag{22}$$
3) choose β such that
$$\beta > \frac{\alpha(4\omega\vartheta + 4(\phi+1)^2 - (\phi+1)\omega)}{9\lambda_2\phi}; \tag{23}$$
4) choose $\sigma > 0$ such that $\sigma < \gamma/(4\eta_2)$;
5) choose $\varepsilon > 0$ such that
$$\varepsilon^2 < \min\left\{\frac{2}{\varepsilon_1|P|^2}\min\left\{\frac{7}{16}, \frac{\rho}{9}\right\}, \frac{\varepsilon_2\gamma}{10(\beta^2|L|^2 + \alpha^2)}, \right. \\ \left. \frac{2\sigma\varepsilon_2}{5\max\{\alpha^2\vartheta^2 + \beta^2|L|^2, 1\}}\eta_1\right\}. \tag{24}$$

It can be checked that all the inequalities in (16) are satisfied.

Define $y_0 = 1_N \otimes y_*$. Then, with the state transformation $\bar{u} = u - y_0$ and $\bar{q} = q + \alpha\nabla\tilde{c}(y_0)$, the system (14)–(15) with $\sum_{i \in \mathscr{N}} q_i(0) = 0$ can be rewritten as

$$\begin{aligned}\dot{\bar{u}} = \varepsilon(&-\alpha\delta(\bar{u} + y_0, y_0) - \beta\overline{L}\bar{u} - \bar{q} - \beta\overline{L}e \\ &- \alpha\delta(e + \bar{u} + y_0, \bar{u} + y_0)),\end{aligned} \tag{25}$$

$$\dot{\bar{q}} = \varepsilon(\alpha\beta\overline{L}\bar{u} + \alpha\beta\overline{L}e), \quad \bar{q}(0) = q(0) + \alpha\nabla\tilde{c}(y_0), \tag{26}$$

where the function δ is defined as $\delta(y, y_0) = \nabla\tilde{c}(y) - \nabla\tilde{c}(y_0)$, and $e = y - \bar{u} - y_0$.

For the system (25)–(26), define $Z = [\bar{u}^T U, \bar{q}^T U]^T$, and consider a Lyapunov function candidate

$$V_Z(Z) = \frac{1}{2} Z^T P Z, \tag{27}$$

where $U = [U_1, U_2] \otimes I_m$ is an orthogonal matrix with $U_1 = 1_N/\sqrt{N}$, and

$$P = \begin{bmatrix} \frac{\alpha(\phi+1)}{9} & 0 & 0 & 0 \\ 0 & \alpha(\phi+1)I_{N-1} & 0 & I_{N-1} \\ 0 & 0 & 1 & 0 \\ 0 & I_{N-1} & 0 & \frac{I_{N-1}}{\alpha} \end{bmatrix} \otimes I_m. \tag{28}$$

We can check that P is positive definite. When Assumptions 2 and 3 are satisfied, the time derivative of $V_Z(Z)$ along the solutions of the system (25)–(26) satisfies

$$\begin{aligned} \dot{V}_Z(Z) \leq & -\varepsilon \min\left\{\frac{7}{16}, \frac{\rho}{9}\right\} \left|[\bar{u}^T U, \bar{q}^T (U_2 \otimes I_m)]^T\right|^2 \\ & + \varepsilon Z^T P \begin{bmatrix} \alpha\delta(y,u) + \beta \bar{L} e \\ \alpha\beta\bar{L}e \end{bmatrix}. \end{aligned} \tag{29}$$

By the definition of $\tilde{c}(y)$ and (9), $(U_1^T \otimes I_m)\nabla\tilde{c}(1_N \otimes y_*) = 0$ holds. Thus, (26) and the requirement $\sum_{i\in\mathcal{N}} q_i(0) = 0$ implies $(U_1^T \otimes I_m)\bar{q}(0) = 0$. By Assumption 3, it holds that $1_N^T L = 0$, which, together with (26), leads to

$$(U_1^T \otimes I_m)\dot{\bar{q}} = 0. \tag{30}$$

Then, for all $t \geq 0$, we have

$$(U_1^T \otimes I_m)\bar{q}(t) = 0, \tag{31}$$

and thus, with the definition of the Euclidean norm of vectors, (29) can be rewritten as

$$\begin{aligned} \dot{V}_Z(Z) \leq & -\varepsilon \min\left\{\frac{7}{16}, \frac{\rho}{9}\right\} |Z|^2 \\ & + \varepsilon Z^T P \begin{bmatrix} \alpha\delta(y,u) + \beta \bar{L} e \\ \alpha\beta\bar{L}e \end{bmatrix}. \end{aligned} \tag{32}$$

Using Young's inequality [29], the Lipschitz property of c_i given in Assumption 2 and the definitions of $\delta(y,u)$ and Z, we further have

$$\dot{V}_Z(Z) \leq -\eta_1 |Z|^2 + \eta_2 |e|^2. \tag{33}$$

Now, consider the closed-loop system composed of (1)–(2) and (25)–(26) with the relationship $u = \bar{u} + y_0$. It is clear that the closed-loop system has an unique solution $[x_1^T(t), \ldots, x_N^T(t), Z(t)]^T$ in a domain $D \subseteq [0, \infty)$ for any specific initial condition $x_i(0) \in \mathbb{R}^{n_i}$, $i \in \mathcal{N}$, $\bar{u}_0 \in \mathbb{R}^{Nm}$ and $\bar{q}(0) = q(0) + \alpha\nabla\tilde{c}(y_0)$. Define the following function:

$$W(t) = \sum_{i\in\mathcal{N}} V_i(x_i(t), u_i(t)) + \sigma V_Z(Z(t)) \geq 0. \tag{34}$$

The time derivative of W satisfies

$$\begin{aligned} \dot{W} =& \overline{W}(x_1, \ldots, x_N, Z) \\ :=& \sum_{i\in\mathcal{N}} \dot{V}_i(x_i, u_i) + \sigma \dot{V}_Z(Z) \\ =& \sum_{i\in\mathcal{N}} (\nabla_x^T V_i(x_i,u_i) f_i(x_i,u_i) + \nabla_u^T V_i(x_i,u_i)\dot{u}_i) + \sigma\dot{V}_Z(Z) \\ \leq & -\gamma|e|^2 + \frac{\varepsilon_2}{2}\zeta^2|e|^2 + \frac{1}{2\varepsilon_2}|\dot{u}|^2 + \sigma(\eta_2|e|^2 - \eta_1|Z|^2), \end{aligned} \tag{35}$$

where we have used Young's inequality, Assumption 1, (33) and the definition $e = y - u$ to get the inequality.

By the definition of $\bar{u}$ above (25), we get $\dot{\bar{u}} = \dot{u}$. From (25), Cauchy-Schwarz inequality and the Lipschitz properties of c_is' given in Assumption 2, it follows that

$$\begin{aligned} |\dot{u}|^2 \leq & 5\varepsilon^2(\alpha^2\vartheta^2|\bar{u}|^2 + \beta^2|L|^2|\bar{u}|^2 + |\bar{q}|^2 \\ & + \beta^2|L|^2|e|^2 + \alpha^2|e|^2). \end{aligned} \tag{36}$$

Substituting (36) into (35) leads to

$$\dot{W} \leq -\eta_3|e|^2 - \eta_4|Z|^2, \tag{37}$$

which means that W is decreasing on D. Note that by (34), we have $W \geq 0$. Thus, on D, W is bounded, which, together with (34) and (4) in Assumption 1, implies that x_i, $i \in \mathcal{N}$ and Z are bounded too, i.e., there exists constants $M_x, M_Z > 0$ such that $|x_i| \leq M_x, i \in \mathcal{N}, |Z| \leq M_Z$. Then, $D = [0, \infty)$. Since W is decreasing and is bounded on D, $\lim_{t\to\infty} W(t)$ exists and is finite. It follows from (1)–(2), (25)–(26) and the boundedness of $x_i, i \in \mathcal{N}$ and Z that $\dot{x}_i$ and $\dot{Z}$ are bounded on D, and thus, $x_i, i \in \mathcal{N}$ and Z are uniformly continuous on D. By (35), (25)–(26) and Assumptions 1 and 2, $\overline{W}$ is a continuous funcion of $x_i, i \in \mathcal{N}$ and Z. Thus, $\overline{W}$ is uniformly continuous on $\{[x_1^T, \ldots, x_N^T, Z^T]^T : |x_i| \leq M_x, i \in \mathcal{N}, |Z| \leq M_Z\}$. Then, $\dot{W}$ in (35) is uniformly continuous on D. By Barbalat's lemma [26, p. 323],

$$\lim_{t\to\infty} \dot{W}(t) = 0, \tag{38}$$

which, together with (37) and the positiveness of η_3 and η_4, means

$$\lim_{t\to\infty} e(t) = 0, \quad \lim_{t\to\infty} Z(t) = 0. \tag{39}$$

Then, by the definitions of y, e, $\bar{u}$ and Z, we have

$$\lim_{t\to\infty}(y(t) - y_0) = \lim_{t\to\infty}(e(t) + \bar{u}(t)) = 0. \tag{40}$$

This ends the proof. ■

Remark 4: The requirement of $\sum_{i\in\mathcal{N}} q_i(0) = 0$ on the initial condition of q_i is indispensable for Theorem 1. Indeed, for any initial state $[x_i^T(0), u_i^T(0), q_i^T(0)]^T \in \mathbb{R}^{n_i+2m}$, $i \in \mathcal{N}$, under the assumptions of Theorem 1, $[y^T, u^T, q^T]^T$ will converge to the initial-condition-dependent point $[\hat{y}^T, 1_N^T \otimes \hat{u}_*^T, \hat{q}^T]^T$ satisfying $\hat{y} = 1_N \otimes \hat{u}_*$, $\alpha\sum_{i\in\mathcal{N}} \nabla c_i(\hat{u}_*) = \sum_{i\in\mathcal{N}} q_i(0)$ and $\hat{q} = \alpha\nabla\tilde{c}(1_N \otimes \hat{u}_*)$. The existence and uniqueness of such a point $[\hat{y}^T, 1_N^T \otimes \hat{u}_*^T, \hat{q}^T]^T$ can be derived by repeating the discussion below Assumption 2 for the optimization problem

$$\min_{\hat{u}\in\mathbb{R}^m} \quad \alpha \sum_{i\in\mathcal{N}} c_i(\hat{u}) - \sum_{i\in\mathcal{N}} q_i^T(0)\hat{u}. \tag{41}$$

Thus, to guarantee $\hat{u}_* = y_*$, we need $\sum_{i\in\mathcal{N}} q_i(0) = 0$. The proof of Theorem 1 employs Barbalat's lemma to provide a rigorous convergence analysis for each trajectory starting from the specific initial condition.

Remark 5: Following a similar analysis to that presented in this paper, one can easily extend Theorem 1 to the case with time-varying digraph $G(t) = (\mathcal{N}, \mathcal{E}(t), \mathcal{A}(t))$. Specifically, suppose that the time-varying digraph $G(t)$ satisfies Assumption 3 at any time instant $t \in [0,\infty)$ and the adjacency matrix $\mathcal{A}(t)$ is piecewise constant with the entries $a_{ij}(t) \in \{0,1\}$ on $[0,\infty)$. Then, the number of all possible Laplacian matrices $L(t)$s' is finite. For a Laplacian matrix $L(t)$, define $\lambda_2(t)$ as the second smallest eigenvalue of $(L(t)+L(t))/2$. Then, with Assumptions 1 and 2 satisfied, all the conclusions of Theorem 1 also hold after modifying λ_2 to $\min\{\lambda_2(t), t\in[0,\infty)\}$.

IV. A SIMULATION EXAMPLE

This section aims to validate the efficacy of the proposed method and Remark 4 by simulation.

Consider a multi-agent system with the dynamics of agent $i \in \mathcal{N} = \{1,2,\ldots,6\}$ described by

$$\dot{p}_i = v_i + h_i(p_i)\theta_i, \tag{42}$$
$$y_i = p_i, \tag{43}$$

where $p_i \in \mathbb{R}$ is the state, $y_i \in \mathbb{R}$ is the output, $v_i \in \mathbb{R}$ is the control input, $h_i : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, and $\theta_i \in \mathbb{R}$ is an unknown parameter.

Note that the system (42)–(43) does not satisfy Assumption 1, and therefore, Theorem 1 cannot be directly applied. To overcome this obstacle, we employ the well-known adaptive backstepping method [29] to design a controller v_i for agent $i \in \mathcal{N}$, enabling it to track a reference input u_i. The controller v_i is defined as

$$v_i = -k_i(y_i - u_i) - h_i\hat{\theta}_i, \tag{44}$$
$$\dot{\hat{\theta}}_i = h_i(p_i)(y_i - u_i), \tag{45}$$

where k_i is any positive constant. Then, the system (42)–(45) is in the form of (1)–(2) with the definition

$$x_i = \begin{bmatrix} p_i \\ \theta_i - \hat{\theta}_i \end{bmatrix}, \qquad g_i(x_i) = [1,0]x_i, \tag{46}$$

$$f_i(x_i,u_i) = \begin{bmatrix} -k_i & h_i([1,0]x_i) \\ -h_i([1,0]x_i) & 0 \end{bmatrix} x_i + \begin{bmatrix} k_i \\ h_i([1,0]x_i) \end{bmatrix} u_i. \tag{47}$$

Define

$$V_i(x_i,u_i) = \frac{1}{2}\left|x_i - [1,0]^T u_i\right|^2. \tag{48}$$

Then, direct calculation gives

$$V_i(x_i,0) = \frac{1}{2}|x_i|^2, \tag{49}$$
$$\nabla_{x_i}^T V_i(x_i,u_i) f_i(x_i,u_i) = -k_i|g_i(x_i) - u_i|^2, \tag{50}$$
$$|\nabla_{u_i} V_i(x_i,u_i)| = |g_i(x_i) - u_i|, \tag{51}$$

which means that Assumption 1 is satisfied with $\varphi_1(s) = \varphi_2(s) = s^2/2$, $\gamma = \min\{k_i : i \in \mathcal{N}\}$ and $\zeta = 1$ for $s \in \mathbb{R}_+$.

The local objective functions are chosen as

$$c_1(s) = c_2(s) = 0.1|s-1|^2, \tag{52}$$
$$c_3(s) = c_4(s) = 0.1|s-3|^2 + s, \tag{53}$$
$$c_5(s) = c_6(s) = \ln(e^{-0.1s} + e^{0.3s}) + 0.1s^2, \tag{54}$$

for $s \in \mathbb{R}$. One can check that Assumption 2 is satisfied with $\vartheta = 0.2$ and $\omega = 0.1$.

The information exchange topology is described by a digraph with the positive elements of its adjacency matrix $\mathcal{A} = [a_{ij}]$ defined as $a_{21} = a_{32} = a_{43} = a_{54} = a_{65} = a_{16} = a_{52} = a_{25} = 1$. It can be verified that Assumption 3 is satisfied.

By now, all assumptions of Theorem 1 are proved to be satisfied. Thus, for agent i, design a distributed controller in the form of (10)–(11). In this simulation, choose $h_1(s) = h_2(s) = h_3(s) = 0.1s^2$, $h_4(s) = h_5(s) = h_6(s) = 0.2s^3$, $\theta_1 = \theta_3 = \theta_5 = 1$, $\theta_2 = \theta_4 = \theta_6 = 2$, $k_1 = k_4 = k_6 = 10$, $k_2 = k_3 = k_5 = 20$, for $s \in \mathbb{R}$. Then, according to the design procedure in the proof of Theorem 1, choose

$$\alpha = 0.1, \quad \beta = 0.4232, \quad \varepsilon = 0.0125. \tag{55}$$

When the initial condition is $r(0) = u(0) = [-2,4,2,1,2,1]^T$ and $\hat{\theta}(0) = q(0) = [0,0,0,0,0,0]^T$, Fig. 2 shows that all the outputs of the agents converge to the optimal point -0.4688 of the optimization problem (3). Fig 3 shows that the trajectories of inputs and states of all the agents remain bounded. The simulation results coincide with Theorem 1.

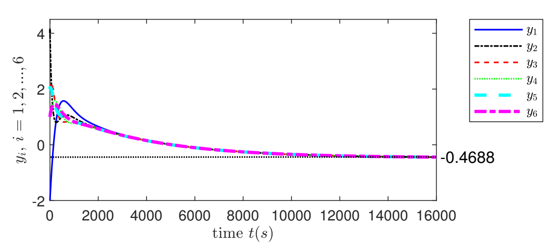

Fig. 2. The trajectories of the outputs of all agents with initial condition $r(0) = u(0) = [-2,4,2,1,2,1]^T$ and $\hat{\theta}(0) = q(0) = [0,0,0,0,0,0]^T$.

To demonstrate the importance of the requirement $\sum_{i\in\mathcal{N}} q_i(0) = 0$ on the initial condition of q_i, we modify the initial condition to the case $q(0) = [0.1,0,0,0,0,0]^T$, which does not satisfy $\sum_{i\in\mathcal{N}} q_i(0) = 0$. As observed in Fig. 4, all the outputs of the agents converge to the point -1.2516, which is the optimal solution of the optimization problem (41), rather than the optimal point -0.4688 of the optimization problem (3). The simulation results are consistent with Remark 4.

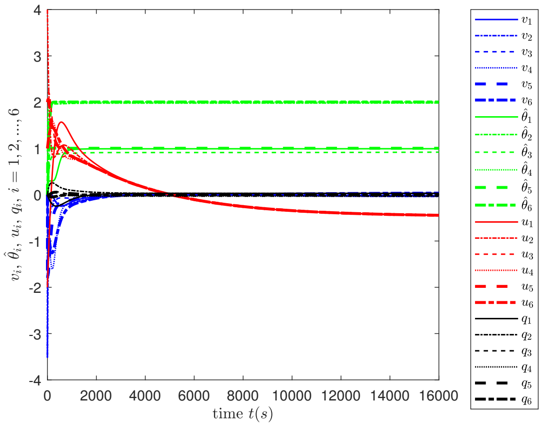

Fig. 3. The trajectories of the inputs and the states of all agents with initial condition $r(0) = u(0) = [-2,4,2,1,2,1]^T$ and $\hat{\theta}(0) = q(0) = [0,0,0,0,0,0]^T$.

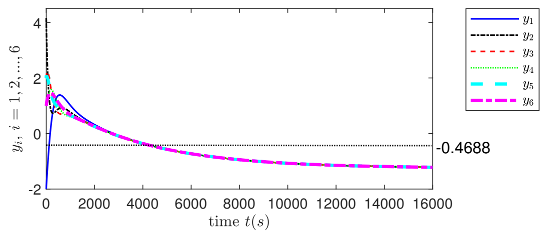

Fig. 4. The trajectories of the outputs of all agents with initial condition $r(0) = u(0) = [-2,4,2,1,2,1]^T$, $\hat{\theta}(0) = [0,0,0,0,0,0]^T$ and $q(0) = [0.1,0,0,0,0,0]^T$.

V. CONCLUSIONS

This paper proposes a novel distributed feedback optimization algorithm for nonlinear multi-agent systems, ensuring that the outputs of all the agents converge to the minimizer of the total objective function. It should be noted that the proposed algorithm only needs the agents to measure the real-time gradient values of the local objective functions and exchange relative outputs with their neighbors. The significance of the initial condition requirement is thoroughly discussed. The results are applicable to both static and time-varying digraphs.

Our future work may be directed at investigating the distributed feedback optimization problem in more general cases, such as where the gradients of the local objective functions are locally Lipschitz rather than globally Lipschitz, the dynamics of each agent is described by a hybrid system, and the digraph is neither weight-balanced nor strongly connected.

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