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Sparsity-Constrained Linear Quadratic Regulation Problem: Greedy Approach with Performance Guarantee

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Abstract— **We study a linear quadratic regulation problem with a constraint where the control input can be nonzero only at a limited number of times. Given that this constraint leads to a combinational optimization problem, we adopt a greedy method to find a suboptimal solution. To quantify the performance of the greedy algorithm, we employ two metrics that reflect the submodularity level of the objective function: The submodularity ratio and curvature. We first present an explicit form of the optimal control input that is amenable to evaluating these metrics. Subsequently, we establish bounds on the submodularity ratio and curvature, which enable us to offer a practical performance guarantee for the greedy algorithm. The effectiveness of our guarantee is further demonstrated through numerical simulations.**

I. INTRODUCTION

A signal is termed sparse if most of its values are exactly zero. In control systems, a sparse control input means the input takes zero—the controller can remain turned off for the majority of the operational duration. The design of sparse control signals has attracted much research attention due to its energy-saving potential. For example, in the control of railcars, leveraging inertia to move without any external driving force is an effective strategy to reduce power consumption. Moreover, sparse control plays an important role in networked control systems by minimizing network usage. Such a reduction is crucial for battery-powered devices and facilitates efficient network sharing among multiple nodes. The seminal paper [1] has introduced a framework termed maximum hands-off control: In this approach, the control input remains at zero for as long as possible under a constraint on the terminal state. A number of subsequent studies [2]–[5] building on this foundation have followed.

In the above literature, the authors have focused on maximizing sparsity with limited attention to control performance. As a result, maximum hands-off control may lead to undesirable transient phenomena. The present paper aims to establish a methodology that takes into account both transient performance and input sparsity. Several papers have studied optimal control with a limited number of control actions. In [6], the authors have examined a threshold-based control strategy that is optimal when the control objectives ignore the energy of the control input. For the case of including the quadratic form of the input, a convex relaxation approach [7] has been proposed. In addition, a suboptimal method in which the control inputs are applied at the beginning of the control horizon [8] has also been investigated.

Aside from these approaches, the greedy algorithm has been recognized as a practical approach to determine the actuation timings [9]. In addition to its simplicity in implementation, a notable feature of the greedy algorithm is that it offers theoretical performance guarantees. While such guarantees were traditionally associated with submodular objective functions [10], recent studies have been extending these guarantees to non-submodular functions [11], which are prevalent in control problems [12]–[16].

This paper addresses the design of the optimal control input that minimizes the standard quadratic cost subject to a sparsity constraint. A greedy algorithm is employed to obtain an approximate solution. In line with [11], we investigate the degree of submodularity of the objective function through the submodularity ratio [17] and generalized curvature [11]. To evaluate these metrics, we structure the objective function by solving a least squares problem, while the dynamic programming approach has been used in [7]–[9]. Numerical examples demonstrate that our result provides a tighter guarantee than those found in [9].

The rest of the paper is organized as follows. In Section II, we formulate the considered problem as a cardinality-constrained optimization problem. Next, we provide preliminary results on the set function maximization in Section III. We then present the optimal control input for the LQR problem in Section IV and provide a performance guarantee for the greedy algorithm in Section V. We illustrate our performance guarantee in Section VI. Finally, we conclude our results in Section VII. Due to space limitation, the technical proofs are omitted and we refer to [18].

Notation: We denote the set of real numbers and nonnegative integers as $\mathbb{R}$ and $\mathbb{Z}_+$, respectively. The $n \times n$ identity matrix is denoted by I_n, and $\operatorname{diag}\{d_1, \dots, d_n\}$ represents a diagonal matrix where the diagonal elements are d_is. For a matrix A, $[A]_{i,j}$ means the (i, j) element of A. The spectral norm of A is denoted by $\|A\|$. For a symmetric matrix $S \in \mathbb{R}^{n\times n}$, its eigenvalues arranged in descending order are given by $\lambda_1(S) \geq \cdots \geq \lambda_n(S)$. The symbol $\otimes$ denotes the Kronecker product. For a finite set $\mathscr{X}$, $|\mathscr{X}|$ and $2^{\mathscr{X}}$ denote the cardinality and the power set of $\mathscr{X}$, respectively.

II. PROBLEM FORMULATION

We investigate a discrete-time feedback system depicted in Fig. 1. The plant is modeled as a linear time-invariant

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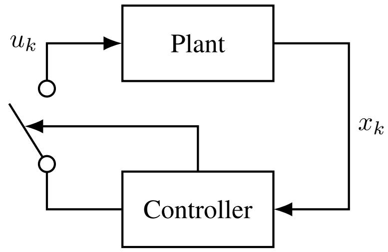

Fig. 1. Feedback system

system given by

$$x_{k+1} = Ax_k + Bu_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are the state and the control input, respectively. At each time step $k \in \mathbb{Z}_+$, the controller observes the state x_k and determines the control input u_k. Notably, the control input is a sparse signal, taking non-zero values at most d times within an N-step control interval $\mathscr{T} \coloneqq \{0, 1, \ldots, N-1\}$.

Let $\mathscr{S} \subseteq \mathscr{T}$ be the set of time instants at which the control input is allowed to be a nonzero value. The objective of control is formally stated as the following minimization problem.

$$\begin{aligned} \underset{u_0,\ldots,u_{N-1},\mathscr{S}}{\text{minimize}} \quad & x_N^\top Q_N x_N + \sum_{k=0}^{N-1} x_k^\top Q_k x_k + u_k^\top R_k u_k, \\ \text{subject to} \quad & u_k = 0 \quad (\forall k \notin \mathscr{S}), \\ & |\mathscr{S}| \le d. \end{aligned} \tag{2}$$

Here, $Q_k \succeq 0$ for all $k \in \mathscr{T} \cup \{N\}$, and $R_k \succ 0$ for all $k \in \mathscr{T}$. Problem (2) entails a joint design of the control inputs $u_0, \ldots, u_{N-1}$ and the actuation timing $\mathscr{S}$. Once a timing set $\mathscr{S}$ is established, the optimal control inputs are determined by a solution of the standard LQR problem. We define the optimal LQ cost as

$$\begin{aligned} J(\mathscr{S}) = \min_{u_0(\mathscr{S}),\ldots,u_{N-1}(\mathscr{S})} \Big[& x_N^\top Q_N x_N \\ & + \sum_{k=0}^{N-1} \left(x_k^\top Q_k x_k + u_k(\mathscr{S})^\top R_k u_k(\mathscr{S}) \right) \Big], \end{aligned} \tag{3}$$

where we use the notation $u_k(\mathscr{S})$ to emphasize that the control inputs must satisfy the sparsity constraint: $u_k = 0$ when $k \notin \mathscr{S}$. Consequently, the original problem (2) can be transformed into the subsequent combinatorial optimization problem:

$$\begin{aligned} \underset{\mathscr{S} \subseteq \mathscr{T}}{\text{minimize}} \quad & J(\mathscr{S}), \\ \text{subject to} \quad & |\mathscr{S}| \le d. \end{aligned} \tag{4}$$

Given that an efficient method for this problem has yet to be established, we adopt the greedy algorithm as an approximation technique.

Algorithm 1 The greedy algorithm for Problem (5).

Input: Finite discrete set $\mathscr{T}$, set function f, integer d
$\mathscr{S}_0 \leftarrow \emptyset$
for $i = 1, \cdots, d$ **do**
$\omega^* \leftarrow \underset{\omega \in \mathscr{T} \setminus \mathscr{S}_{i-1}}{\arg\max} \; f(\mathscr{S}_{i-1} \cup \{\omega\}) - f(\mathscr{S}_{i-1})$
$\mathscr{S}_i \leftarrow \mathscr{S}_{i-1} \cup \{\omega^*\}$
end for
Output: $\mathscr{S}^g \leftarrow \mathscr{S}_d$

III. Greedy Algorithm and (Non-)Submodular Function Maximization

In this section, we provide preliminary results for the optimization of set functions. Let us consider the following problem:

$$\begin{aligned} \underset{\mathscr{S} \subseteq \mathscr{T}}{\text{maximize}} \quad & f(\mathscr{S}), \\ \text{subject to} \quad & |\mathscr{S}| \le d, \end{aligned} \tag{5}$$

where $f \colon 2^{\mathscr{T}} \to \mathbb{R}$ is a set function. The brute-force search over the feasible solutions becomes quickly intractable even for moderately sized problems.

The greedy algorithm, as shown in Algorithm 1, is one of the most common approximation methods for the aforementioned problem. Algorithm 1 can yield a solution in polynomial time, which often performs well empirically. Moreover, it is important to note that bounds exist on the deviation of greedy solutions from the optimal.

To describe a performance guarantee for the greedy algorithm, we now introduce fundamental notions associated with a set function.

Definition 1: A set function $f \colon 2^{\mathscr{T}} \to \mathbb{R}$ is *monotone nondecreasing* if for all subsets $\mathscr{A}, \mathscr{B}$ that satisfy $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{T}$, it holds that

$$f(\mathscr{A}) \le f(\mathscr{B}).$$

Let us denote the marginal gain of a set $\Omega \subseteq \mathscr{T}$ with respect to a set $\mathscr{S} \subseteq \mathscr{T}$ by

$$\rho_\Omega(\mathscr{S}) \coloneqq f(\mathscr{S} \cup \Omega) - f(\mathscr{S}).$$

Definition 2 ([10]): A set function $f \colon 2^{\mathscr{T}} \to \mathbb{R}$ is *submodular* if for all subsets $\mathscr{S}_1, \mathscr{S}_2 \subseteq \mathscr{T}$ ($\mathscr{S}_1 \subseteq \mathscr{S}_2$) and all elements $\omega \notin \mathscr{S}_2$, it holds that

$$\rho_{\{\omega\}}(\mathscr{S}_1) \ge \rho_{\{\omega\}}(\mathscr{S}_2).$$

We use the following measures to quantify how close a non-submodular function is to being submodular.

Definition 3 ([17]): The *submodularity ratio* of a non-negative set function f is the largest scalar γ such that

$$\sum_{\omega \in \Omega \setminus \mathscr{S}} \rho_{\{\omega\}}(\mathscr{S}) \ge \gamma \rho_\Omega(\mathscr{S}), \quad \forall \Omega, \mathscr{S} \subseteq \mathscr{T}. \tag{6}$$

Definition 4 ([11]): The *curvature* of a nonnegative set function f is the smallest scalar α such that

$$\rho_{\{j\}}(\mathscr{S}\backslash\{j\}\cup\Omega) \geq (1-\alpha)\rho_{\{j\}}(\mathscr{S}\backslash\{j\}), \quad \forall\Omega,\mathscr{S}\subseteq\mathscr{T}, \quad \forall j\in\mathscr{S}\backslash\Omega. \tag{7}$$

For a nondecreasing set function, it holds that $\gamma\in[0,1]$ and $\alpha\in[0,1]$ [11].

Let $\mathscr{S}^g$ be the solution to Problem (5) obtained by Algorithm 1, and $\mathscr{S}^*$ be the optimal solution. With γ and α, the greedy algorithm provides an approximation guarantee for Problem (5).

Proposition 1 ([11]): Let f be a monotone nondecreasing set function with submodularity $\gamma\in[0,1]$ and curvature $\alpha\in[0,1]$. Then, Algorithm 1 enjoys the following approximation guarantee for solving Problem (5):

$$f(\mathscr{S}^g)-f(\emptyset)\geq\frac{1}{\alpha}\left(1-e^{-\alpha\gamma}\right)(f(\mathscr{S}^*)-f(\emptyset)). \tag{8}$$

Proposition 1 generalizes a well-known result [10] available for submodular functions to non-submodular functions. For the case of $\gamma=1$ and $\alpha=1$, i.e., when f is submodular, the coefficient in (8) is equal to $1-e^{-1}$, which corresponds to the classical approximation factor given in [10].

Note that computing γ and α directly by Definitions 3 and 4 is intractable because the constraints in (6) and (7) are combinational. Therefore, we consider deriving bounds on γ and α and using them to establish a guarantee. In the following section, we derive an explicit form of the optimal control input for Problem (2) and rewrite the objective function $J(\mathscr{S})$ to evaluate its submodularity ratio and curvature.

Remark 1: Recently, Harshaw *et al.* [19] have shown that no polynomial algorithm can achieve a better performance guarantee than (8) for a nonnegative non-submodular function maximization with a cardinality constraint. Thus, we use Proposition 1 as an approximation guarantee for the greedy algorithm.

IV. Explicit form of the optimal control input and the optimal cost

We here present an explicit form of the optimal control in preparation for deriving our main result. Let us define $\tilde{S}(\mathscr{S})\in\mathbb{R}^{|\mathscr{S}|\times N}$ as the matrix created by removing the ith rows that hold $u_{i-1}=0$ from I_N for all $i=1,\dots,N$. For a given $\mathscr{S}$, let $t_1<t_2<\cdots<t_{|\mathscr{S}|}$ be the elements of $\mathscr{S}$ in ascending order. Then, $\tilde{S}$ is formally defined as follows:

$$\left[\tilde{S}(\mathscr{S})\right]_{i,j}=\begin{cases}1 & j-1=t_i,\\ 0 & \text{otherwise.}\end{cases}$$

The number of rows of $\tilde{S}(\mathscr{S})$ implies how many control inputs are allowed to be nonzero. Using $\tilde{S}(\mathscr{S})$, we define the matrix $S(\mathscr{S})\in\mathbb{R}^{|\mathscr{S}|m\times Nm}$ as follows:

$$S(\mathscr{S}):=\tilde{S}(\mathscr{S})\otimes I_m. \tag{9}$$

In addition, let $I_N^{(i)}$ be the $N\times N$ matrix where $[I_N^{(i)}]_{i,i}=1$ and the other elements are zero. For simplicity of notation, we write $\tilde{S}(\mathscr{S})$ and $S(\mathscr{S})$ as $\tilde{S}$ and S respectively in the subsequent text.

From (9) and properties of the Kronecker product, the following lemma holds.

Lemma 1: Given a set $\mathscr{S}\subseteq\mathscr{T}$, it holds that

$$S^\top S=\sum_{i\in\mathscr{S}}\left(I_N^{(i+1)}\otimes I_m\right).$$

Let $U:=[u_0(\mathscr{S})^\top,\dots,u_{N-1}(\mathscr{S})^\top]^\top$ be a control input associated with a feasible set $\mathscr{S}$ satisfying the constraint in Problem (2). Then, the possibly nonzero inputs in U are given as $SU=[u_{t_1}^\top,\dots,u_{t_{|\mathscr{S}|}}^\top]^\top$. Let also $U^*=[u_0^*(\mathscr{S})^\top,\dots,u_{N-1}^*(\mathscr{S})^\top]^\top$ be the optimal control giving (3).

The following proposition shows the possibly nonzero values of the optimal input SU^* and the optimal cost $J(\mathscr{S})$ associated with $\mathscr{S}$.

Proposition 2: Given a set $\mathscr{S}\subseteq\mathscr{T}$, it holds that

$$SU^*=-(S\bar{R}S^\top+S\bar{B}\Phi^\top\bar{Q}\Phi\bar{B}S^\top)^{-1}S\bar{B}^\top\Phi^\top\bar{Q}\Psi x_0, \tag{10}$$

where

$$\bar{Q}=\operatorname{diag}\{Q_1,\cdots,Q_N\},\quad \bar{R}=\operatorname{diag}\{R_0,\cdots,R_{N-1}\},$$
$$\Phi=\begin{bmatrix} I_n & 0 & \cdots & 0\\ A & I_n & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ A^{N-1} & A^{N-2} & \cdots & I_n\end{bmatrix},\quad \Psi=\begin{bmatrix}A\\ A^2\\ \vdots\\ A^N\end{bmatrix},$$
$$\bar{B}=I_N\otimes B.$$

Furthermore, the optimal cost $J(\mathscr{S})$ is given by

$$J(\mathscr{S})=\operatorname{tr}\left[L(I_{Nn}+K(\mathscr{S}))^{-1}\right]+c, \tag{11}$$

where

$$L=\bar{Q}^{1/2}\Psi x_0x_0^\top\Psi^\top\bar{Q}^{1/2},$$
$$K(\mathscr{S})=\bar{Q}^{1/2}\Phi\bar{B}S^\top S\bar{R}^{-1}S^\top S\bar{B}^\top\Phi^\top\bar{Q}^{1/2},$$
$$c=x_0^\top Q_0x_0.$$

The optimal control input (10) can be derived by solving a least-squares problem with respect to SU. The corresponding cost (11) is obtained by (10) and the Woodbury matrix identity.

Remark 2: We emphasize that within the trace operator of the optimal cost expression (11), a positive semidefinite matrix L is present. This inclusion introduces a technical complexity when assessing the submodularity ratio and curvature of J. It is noted that in the existing work [12], which also utilizes Proposition 1, the objective function is described solely in terms of the inverse of a positive definite matrix.

Finally, we give an important property of $K(\mathscr{S})$ which is used to derive the main result shown in Section V.

Lemma 2: For any given $\mathscr{S}\subseteq\mathscr{T}$ and any $\omega\in\mathscr{T}\backslash\mathscr{S}$, it holds that

$$K(\mathscr{S}\cup\{\omega\})=K(\mathscr{S})+K(\{\omega\}).$$

The proof is followed by Lemma 1.

V. Performance guarantee of the greedy algorithm

We are now ready to present our main result; a performance guarantee for the greedy algorithm to the optimal control with a sparsity constraint. To apply Proposition 1 to Problem (4), we structure the problem as the following maximization:

$$\begin{aligned} \underset{\mathscr{S} \subseteq \mathscr{T}}{\text{maximize}} \quad & f(\mathscr{S}) := -J(\mathscr{S}) + J(\emptyset), \\ \text{subject to} \quad & |\mathscr{S}| \leq d. \end{aligned} \tag{12}$$

Note that the sign of the objective function is flipped and $f(\emptyset) = 0$. We seek to find a guarantee of the greedy solution to Problem (12) via Proposition 1. As already noted in Section III, it is computationally difficult to find the exact values of γ and α. Therefore, our goal is to bound them with computationally feasible values.

Let us define $\underline{\gamma}$ and $\overline{\alpha}$ as follows:

$$\underline{\gamma} := \frac{\min_{\omega \in \mathscr{T}} \operatorname{tr}[LK(\{\omega\})] \{\min_{\omega \in \mathscr{T}} \lambda_n[I_{Nn} + K(\{\omega\})]\}^2}{\max_{\omega \in \mathscr{T}} \operatorname{tr}[LK(\{\omega\})] \{\lambda_1[I_{Nn} + K(\mathscr{T})]\}^2},$$

$$\overline{\alpha} := 1 - \underline{\gamma}.$$

Remark 3: The values $\underline{\gamma}$ and $\overline{\alpha}$ are well defined if $\max_{\omega \in \mathscr{T}} \operatorname{tr}[LK(\{\omega\})] \neq 0$. Since all eigenvalues of $LK(\omega)$ are nonnegative for every $\omega \in \mathscr{T}$, this condition is typically met except for uninteresting cases such as $Ax_0 = 0$. A detailed analysis of this point will be the subject of a future study.

The above values represent bounds on γ and α of $f(\mathscr{S})$ in Problem (12).

Theorem 1: Suppose that there exists $\omega \in \mathscr{T}$ such that $\operatorname{tr}[LK(\{\omega\})] \neq 0$. It holds that the set function $f(\mathscr{S})$ defined in (12) is monotone nondecreasing. In addition, its submodularity ratio γ and curvature α are bounded by

$$1 \geq \gamma \geq \underline{\gamma} \geq 0, \quad 0 \leq \alpha \leq \overline{\alpha} \leq 1.$$

Let $\mathscr{S}^g$ be the solution to Problem (12) obtained by Algorithm 1, and let $\mathscr{S}^*$ be the optimal solution. From Theorem 1 and Proposition 1, a performance guarantee of the greedy solution to the problem can be obtained as follows.

Corollary 1: Consider the function $f(\mathscr{S})$ in Problem (12) and suppose that the assumption in Theorem 1 is satisfied. The following inequality holds:

$$f(\mathscr{S}^g) \geq \frac{1}{\overline{\alpha}} \left(1 - e^{-\overline{\alpha}\underline{\gamma}}\right) f(\mathscr{S}^*).$$

Corollary 1 provides a feasible approximation factor because $\underline{\gamma}$ and $\overline{\alpha}$ can be computed in polynomial time. We use this corollary to provide the approximation guarantee with a numerical example in the next section.

VI. Numerical examples and comparison with the existing results

In this section, we validate the effectiveness of the greedy solution to Problem (2) through a numerical example. Furthermore, the conservativeness of the guarantee given in Corollary 1 is discussed.

A. Control performance of the greedy solution

Consider a mass-spring system consisting of two masses and three springs. Let the masses be $m_1 = 1$ kg and $m_2 = 2$ kg, and let the spring constants be $k_1 = k_2 = k_3 = 1$ N/m. This system is expressed by the following continuous-time system:

$$\dot{x}_c(t) = A_c x_c(t) + B_c u_c(t),$$

where $x_c \in \mathbb{R}^4$ is the state vector consisting of the positions and velocities of m_1 and m_2, and $u_c \in \mathbb{R}^2$ is the forces applied to the masses. The matrices A_c and B_c are given as

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2+k_3}{m_2} & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}.$$

This continuous-time system is discretized using zero-order hold with a sample time of 0.1 s, resulting in the system as in (1). Suppose that the discrete-time initial state is $x_0 = [1, 0, 1, 0]^\top$. The objective function is set as $N = 100$, $Q_k = I_4$ and $R_k = I_2$ for all k.

We investigate three methods to determine timing sets $\mathscr{S}$ and evaluate the corresponding costs $J(\mathscr{S})$: The first method is Algorithm 1—the greedy algorithm. The second method is to randomly select time steps $\mathscr{S}$ subject to the sparsity constraint and then compute the optimal control according to Proposition 2. Among 1000 trials, the input with the lowest cost is chosen. The last one is adopted from [8], where the control inputs are applied during the first d time steps. We solve the optimal control problem (4) with different values of $d \in [10, 100]$ for each method.

Fig. 2 compares the three cases by plotting the values of $J(\mathscr{S})$ against the maximum number of control actions d. We see that the greedy algorithm achieves lower costs than the random policy and [8], especially when d is less than 30. Fig. 3 and Fig. 4 demonstrate the control inputs and the state trajectories for the case of $d = 20$, respectively. The greedy algorithm provides better transient performance than the other two methods while maintaining the control input sparse.

It should be noted that the feedforward control is employed in this simulation: The control inputs for the entire horizon are computed at the initial time depending on x_0. In practical systems, the input can be updated at each time instance based on the latest observation and the number of times that the input has been applied.

B. Performance guarantee of the greedy algorithm

We now illustrate how the established performance guarantee varies depending on the target system. In the following, we call $f(\mathscr{S}^g)/f(\mathscr{S}^*)$ the *approximation ratio* of Algorithm 1 for Problem (12). Since both $\underline{\gamma}$ and $\overline{\alpha}$ contain the powers of A, it is expected that singular values of A characterize the approximation ratio. Therefore, we examine the ratio in relation to the maximum singular value, which is the spectral norm of A.

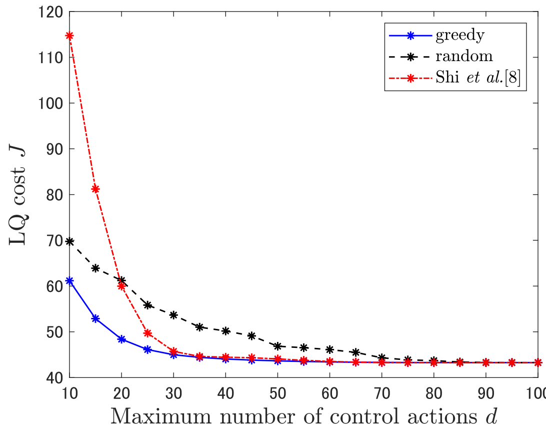

Fig. 2. LQ cost defined by (3) versus the maximum number of control actions d.

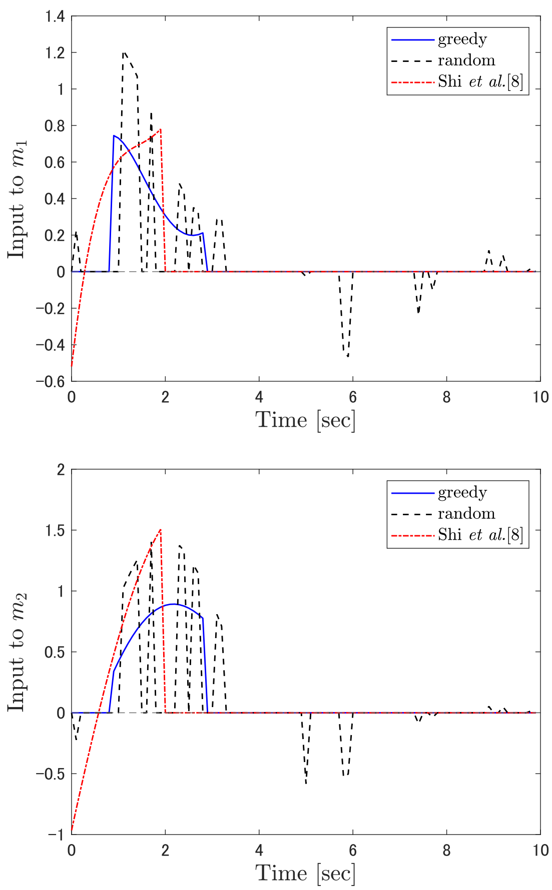

Fig. 3. Control inputs for the case where $d = 20$.

Consider the system (1) with $n = 2$ and the control horizon of $N = 5$. Suppose that $n = m$ and $B = 0.1I_n$. Furthermore, we assume that $Q_k = 0.1I_n$ for all $k \in \mathscr{T} \cup \{N\}$, $R_0 = 10I_n$, and $R_k = 10/k^2 I_n$ for all $k \in \mathscr{T}\backslash\{0\}$. The matrix A is randomly chosen as $A = \text{diag}\{a_1, a_2\}$ where $|a_1|, |a_2| \leq 1.5$. The initial state x_0 is also randomly chosen so that all elements are in $[-10, 10]$. Fig. 5 shows the

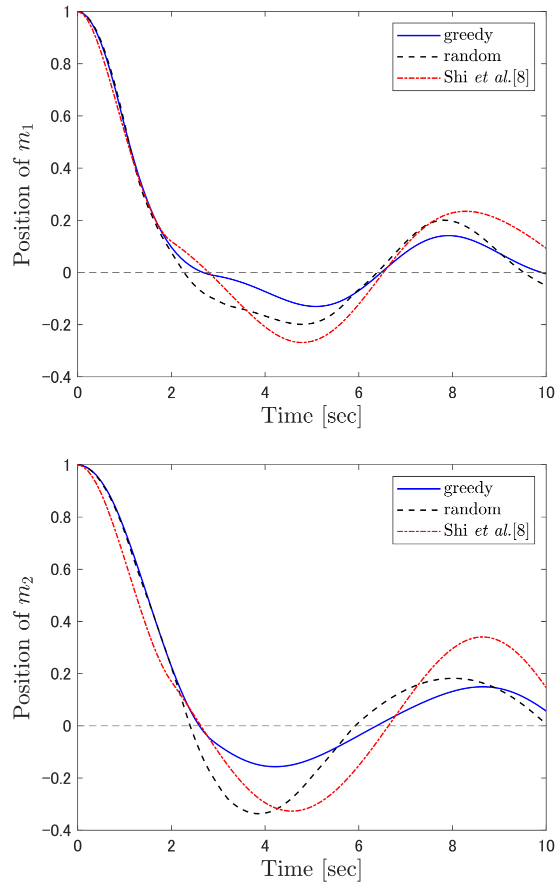

Fig. 4. State trajectories corresponding to the control input in Fig. 3.

lower bound on the approximation ratio given by Corollary 1 in relation to $\|A\|$. The solid line depicts the mean over 1000 realizations, while the shaded area represents the standard deviation. It is observed that the approximation ratio archives about 0.4 when $\|A\|$ is around 1. On the other hand, the bound is markedly small when $\|A\|$ is closed to 0.1; the system is highly stable in those cases. For the case where $\|A\| > 1$, the spectral radius of A can be greater than 1, that is, a system is unstable. The instability results in the magnification of L and $K(\mathscr{S})$, and thus $\max_{\omega \in \mathscr{T}} \text{tr}[LK(\{\omega\})]$ becomes much greater than $\min_{\omega \in \mathscr{T}} \text{tr}[LK(\{\omega\})]$. Consequently, the approximation ratio tends to decrease as $\|A\|$ increases.

C. Conservativeness of the established guarantee

Finally, we analyze the conservativeness of Theorem 1. The authors in [9] have explored an actuator scheduling problem, aiming to select a subset of the actuators to apply the control inputs at each time in order to minimize the control cost. A linear deterministic system with a Gaussian initial state is considered. The problem is formulated as a matroid-constrained optimization problem, and a performance guarantee of the greedy approach is derived using the concept of α-supermodularity [20]. It is worth noting that [9] assumes that A is full rank, which is required for applying the results of [21].

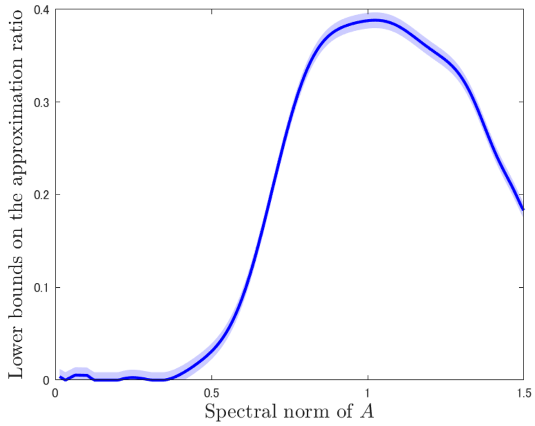

Fig. 5. Lower bounds on the approximation ratio versus the spectral norm of A. The blue solid line represents the mean for 1000 simulations, and the shadowed region around the line visualizes standard deviations from the mean.

Theorem 1 can be modified to the case where x_0 is a random vector. In such a case, the optimal cost in Proposition 2 becomes the expectation of $J(\mathscr{S})$, which is characterized by the covariance of x_0. With these modifications, it is possible to compare the approximation ratio in Corollary 1 with the result presented in [9]. Suppose the same scenario as in Section VI-B except here $x_0 \sim \mathcal{N}(0, I_n)$. According to Corollary 1, we have that $f(\mathscr{S}^g)/f(\mathscr{S}^*) \leq 0.264$ on average over 1000 trials, whereas the result in [9] yields $f(\mathscr{S}^g)/f(\mathscr{S}^*) \leq 0.089$. This comparison suggests that our result is less conservative.

VII. Conclusion

In this paper, we have addressed the sparsity-constrained LQR problem. To evaluate the approximation guarantee of the greedy solution, we have initially derived the explicit optimal control input by solving the least squares problem. With the form of the optimal control, bounds on the submodularity ratio and curvature of the quadratic cost function have been derived. Those bounds have been utilized to establish a theoretical performance guarantee of the greedy solution. Through numerical simulations, we have illustrated the effectiveness of the greedy solution and the performance guarantee.

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