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# A Unified Framework on Global Stability and Lyapunov Dimension of Lur'e Systems

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***Abstract*— The equivalence between local and global characteristics of Lur'e systems is investigated. Historically, such problems date back to *Vyshnegradskii's conjecture* on Watt governors and *Eden's conjecture* on Lorenz attractors. In the present paper, we develop a unified framework for stability and dimension analyses. This is motivated by the recent works on hidden oscillations and their relations with absolute stability theory. We combine an energy perspective in Lyapunov analysis and a linearization approach in contraction analysis to study global stability and Lyapunov dimension. Notably, we allow a gap between the storage function and the Lyapunov function by utilizing what we call a *differential Lyapunov function of the Leonov form*. Our framework is also less conservative in the sense that the exact global stability condition and the exact Lyapunov dimension can be characterized. The effectiveness of our method is demonstrated through the Lorenz attractor.**

## I. Introduction

Global stability of Lur'e systems has long been studied in the control community (see Fig. 1). In particular, the classical conjectures by Aizerman and Kalman in the absolute stability problem have led to the developments of the celebrated Popov and circle criteria. Today, global stability of Lur'e systems can be analyzed via the so-called IQC theory [1]. On the other hand, *hidden oscillations* have recently attracted attention in connection with counterexamples to the Aizerman and Kalman conjectures [2], [3]. In these works, it was shown that a hidden attractor may not be detected by the describing function method, which is a well-known approximate method. Moreover, the IQC theory is difficult to apply in the analysis of periodic and chaotic oscillations. These facts motivated us to develop a unified framework for studying stable and unstable feedback systems. In this paper, we consider global stability and Lyapunov dimension of Lur'e systems. The latter one is related to the existence or absence of oscillations.

Historically, global stability of control systems was first investigated by Vyshnegradskii. According to the recent survey [4], in the paper published in 1877, Vyshnegradskii studied a mathematical model of Watt governors and derived a sufficient condition for stability of the linearized system. Besides, it was conjectured that the same condition is also sufficient for convergence of all solutions to the equilibrium. This problem was positively solved by Andronov and Mayer

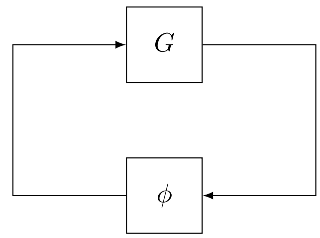


Fig. 1. Lur'e system

around 1945 (see [4] for the list of references). In general circumstances, we need to justify such a linearization approach for global stability analysis. Even if some additional requirements are imposed as in the Aizerman, Kalman, and Markus–Yamabe conjectures, local stability does not imply global stability. Recently, since the landmark paper [5], contraction analysis has attracted much attention. Namely, global stability is guaranteed if the symmetrized Jacobian matrix of a vector field is uniformly Hurwitz. As shown in [6], contraction analysis can also be studied by the Lyapunov function method. However, the differential Lyapunov framework in that paper is based on a different idea from the Barbashin–Krasovskii theorem and LaSalle's invariance principle. Thus, there is a limited perspective on the relation between Lyapunov analysis and contraction analysis.

As regards unstable systems, after the discovery of chaotic phenomena by Ueda and Lorenz, strange attractors became an active research area. To calculate the fractal dimension of a strange attractor, Kaplan and Yorke first introduced the concept of *Lyapunov dimension* by some formula involving Lyapunov exponents [7]. However, the definition by Kaplan and Yorke cannot be used to estimate the attractor dimension. At almost the same time, an upper estimate of the Hausdorff dimension was obtained by Douady and Oesterlé based on the so-called *singular value functions* [8]. On the other hand, Eden suggested a local estimation of Lyapunov dimension [9], [10], [11], [12]. In particular, Eden conjectured that the Lyapunov dimension of the Lorenz attractor coincides with the *local Lyapunov dimension* at one of the equilibria. If this conjecture is true, then it is not difficult to obtain the exact value of Lyapunov dimension. The validity of the conjecture for the Lorenz attractor was proved by Leonov in [13].

Here, we focus on the analogy between Vyshnegradskii's conjecture on the equivalence between local and global stability and Eden's conjecture on the equivalence between local and global Lyapunov dimensions. An interesting fact is that both conjectures are devoted to the local dynamics at

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an equilibrium even in the presence of nonlinearity. Thus, it is meaningful to develop a unified framework on global stability and Lyapunov dimension for feedback systems in the Lur'e form, which have an equilibrium at the origin. In this paper, we derive conditions for which local characteristics at the origin (local stability or local Lyapunov dimension) coincide with global characteristics of invariant sets (global stability or global Lyapunov dimension). Note that hidden oscillations can be analyzed by Lyapunov dimension compared with the describing function method.

The paper contributions are summarized as follows. We investigate global stability and Lyapunov dimension of Lur'e systems in a unified manner. Our results are largely inspired by the recent works on Lur'e systems in [14], [15]. However, our method combines an energy perspective in Lyapunov analysis and a linearization approach in contraction analysis. This idea comes from Leonov's method in dimension theory [16]. First, we derive sufficient conditions for global stability in terms of the existence of a dissipated energy-like function and a constraint on the nonlinearity. It is shown that these conditions are also necessary for global stability in some sense. Notably, we allow a certain gap between the storage function for the LTI system and the Lyapunov function for the closed-loop system. Some connection with the differential Lyapunov framework developed in [6] is also discussed. In particular, we introduce what we call a *differential Lyapunov function of the Leonov form*. Then, we naturally extends the stability result to estimation of Lyapunov dimension. We also provide conditions for which the Eden conjecture is valid and those for which there is no sustained oscillation.

*Notations:* Let $A$ be a real $n \times n$ matrix. We denote by $\sigma_1(A), \dots, \sigma_n(A)$ the singular values of $A$ arranged in decreasing order. For a real number $d \in [0, n]$, we define

$$\omega_d(A) := \begin{cases} 1 & \text{if } d = 0, \\ \sigma_1(A) \cdots \sigma_j(A) \sigma_{j+1}(A)^s & \text{otherwise}, \end{cases}$$

where $j \in \{0, 1, \dots, n-1\}$ and $s \in (0, 1]$ are chosen such that $d = j + s$. The function $\omega_d$ is called the *singular value function* of order $d$.

## II. Preliminaries

This section provides some preliminary results and useful facts from the theory of dynamical systems.

### A. Dynamical Systems and Contraction Analysis

Consider an autonomous differential equation

$$\dot{x} = f(x), \tag{1}$$

where $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is a $C^1$-function. Assume that for every initial state $x_0 \in \mathbb{R}^n$, a unique solution $\phi(t, x_0)$ to (1) exists on $\mathbb{R}_+$. Under this hypothesis, the one-parameter transformation defined by $\varphi^t(x_0) := \phi(t, x_0)$ satisfies the semigroup property:

$$\varphi^0(x_0) = x_0, \quad x_0 \in \mathbb{R}^n;$$
$$\varphi^{t+s}(x_0) = \varphi^t(\varphi^s(x_0)), \quad t, s \in \mathbb{R}_+, \quad x_0 \in \mathbb{R}^n.$$

Hence, the equation (1) defines the (semi)flow $\{\varphi^t\}_{t \in \mathbb{R}_+}$.

The linearization of the nonlinear system (1) along the trajectory starting from $x$ is described by

$$\delta\dot{x} = J(\varphi^t(x))\delta x, \tag{2}$$

where $J(x) := \mathrm{D}f(x)$ denotes the Jacobian matrix of $f$ at $x$. Note that $\mathrm{D}\varphi^t(x)$ is a fundamental matrix of (2) and satisfies the (multiplicative) cocycle property:

$$\mathrm{D}\varphi^{t+s}(x) = \mathrm{D}\varphi^t(\varphi^s(x))\mathrm{D}\varphi^s(x), \ t, s \in \mathbb{R}_+, \ x \in \mathbb{R}^n.$$

Thus, the variational equation (2) defines a linear cocycle.

Consider the product of the two systems (1) and (2):

$$\begin{cases} \dot{x} = f(x), \\ \dot{\delta x} = J(x)\delta x. \end{cases}$$

In the differential Lyapunov framework proposed in [6], the product system mentioned above is analyzed instead of the original nonlinear system. In recent years, a similar framework was generalized to systems with inputs and outputs in [17], where analogues of storage functions and supply rates were introduced. Such frameworks were further studied together with the Nyquist criterion and the KYP lemma [14], performance analysis with general incremental notions [18], and Hausdorff dimension estimates of interconnected systems [19]. The following proposition is a natural consequence of contraction analysis [5], [20].

*Proposition 1:* Suppose that there exist constants $C \geq 1$ and $\lambda > 0$ such that

$$\|\mathrm{D}\varphi^t(x)\| \leq C\mathrm{e}^{-\lambda t}$$

for all $t \in \mathbb{R}_+$ and all $x \in \mathbb{R}^n$. Then,

$$\|\varphi^t(x_1) - \varphi^t(x_0)\| \leq C\mathrm{e}^{-\lambda t}\|x_1 - x_0\|$$

for all $t \in \mathbb{R}_+$ and all $x_0, x_1 \in \mathbb{R}^n$.

The condition in Proposition 1 means that the largest Lyapunov exponent is uniformly negative among all trajectories. The concept of Lyapunov dimension to be introduced can be regarded a natural generalization of the result mentioned above. In contrast to searching a Lyapunov function, it is reasonable to consider contraction analysis since the stability condition can be derived in terms of linear matrix inequalities or logarithmic norms (see [21]). However, such a condition is usually conservative, and how to reduce conservativeness of contraction analysis is an essential problem.

### B. Lyapunov Dimension

We introduce the definitions of local and global Lyapunov dimensions. Our definitions are based on those in [11], but we follow the terminologies in the recent book [22]. In particular, Eden used the term Douady–Oesterlé dimension instead of Lyapunov dimension in [11]. We refer to [22, Chap. 6] for relations between various definitions of Lyapunov dimension in the literature. In what follows, let $\mathcal{K}$ denote a compact invariant set of the flow $\{\varphi^t\}_{t \in \mathbb{R}_+}$. We note that the Lyapunov dimension is an invariant of the dynamical system [23], and its value is independent of the geometric complexity of $\mathcal{K}$. The following two definitions are adopted from [11].

*Definition 1:* The *local Lyapunov dimension* of the flow $\{\varphi^t\}_{t\in\mathbb{R}_+}$ at $x$ is the number

$$\begin{aligned}&\dim_L(\{\varphi^t\}_{t\in\mathbb{R}_+}, x)\\ &\quad := \inf\left\{d \in (0,n] : \limsup_{t\to\infty} \frac{1}{t}\ln \omega_d(\mathrm{D}\varphi^t(x)) < 0\right\}.\end{aligned}$$

*Definition 2:* The *global Lyapunov dimension* of the flow $\{\varphi^t\}_{t\in\mathbb{R}_+}$ with respect to $\mathcal{K}$ is the number

$$\begin{aligned}&\overline{\dim}_L(\{\varphi^t\}_{t\in\mathbb{R}_+}, \mathcal{K})\\ &\quad := \inf\left\{d \in (0,n] : \lim_{t\to\infty} \frac{1}{t}\ln \sup_{x\in\mathcal{K}} \omega_d(\mathrm{D}\varphi^t(x)) < 0\right\}.\end{aligned}$$

Remark that the limit in the latter definition exists because $\mathcal{K}$ is an invariant set. The following relations between local and global Lyapunov dimensions play a fundamental role in this paper [11, Prop. 5.3].

*Lemma 1:* Let $\mathcal{K}$ be a compact invariant set. Then, the following statements are true:

1) $\dim_L(\{\varphi^t\}_{t\in\mathbb{R}_+}, x) \le \overline{\dim}_L(\{\varphi^t\}_{t\in\mathbb{R}_+}, \mathcal{K})$ for all $x \in \mathcal{K}$.
2) There exists a point $x^* \in \mathcal{K}$ such that

$$\dim_L(\{\varphi^t\}_{t\in\mathbb{R}_+}, x^*) = \overline{\dim}_L(\{\varphi^t\}_{t\in\mathbb{R}_+}, \mathcal{K}).$$

The above lemma indicates that there is a critical point at which the local Lyapunov dimension coincides with the global Lyapunov dimension. Such a point is not necessarily an equilibrium and is in general unknown. The expectation that the local Lyapunov dimension at an unstable equilibrium coincides with the global Lyapunov dimension with respect to the global attractor is known as the *Eden conjecture* [9]. This conjecture is valid for some well-known models (see [22, Chap. 6]). If the Eden conjecture is valid, then it is fairly easy to calculate the exact Lyapunov dimension.

### C. Leonov's Method

Before presenting our results, we recall Leonov's method in dimension theory [16]. In this paper, we investigate global stability and Lyapunov dimension of Lur'e systems through Leonov's method.

*Proposition 2 (Leonov [24]):* Let $\mathcal{K}$ be a compact invariant set. For a positive-definite matrix $P$, let $\lambda_1(x) \ge \cdots \ge \lambda_n(x)$ be the $n$ solutions to the characteristic equation

$$\det(J(x)^\mathsf{T} P + PJ(x) - 2\lambda(x)P) = 0.$$

Suppose that there exist a $C^1$-function $v\colon \mathbb{R}^n \to \mathbb{R}$ and a constant $d \in (0,n]$ such that

$$\lambda_1(x) + \cdots + \lambda_{\lfloor d\rfloor}(x) + (d - \lfloor d\rfloor)\lambda_{\lfloor d\rfloor+1}(x) + \dot{v}(x) < 0$$

for all $x \in \mathcal{K}$. Then, $\overline{\dim}_L(\{\varphi^t\}_{t\in\mathbb{R}_+}, \mathcal{K}) < d$.

It is a noteworthy fact that Leonov's method is closely related to LaSalle's invariance principle. Note that we can assume that $\dot{v}(x) \le 0$ for all $x \in \mathbb{R}^n$. This means that by multiplying $v$ with a large constant, the term $\dot{v}(x)$ can be made arbitrarily small as long as it is strictly negative. As a result, the inequality in Proposition 2 is satisfied whenever $\dot{v}(x)$ is strictly negative. On the other hand, the function $v$ can be considered as a Lyapunov-like function in LaSalle's invariance principle. Thus, every solution converges to the region where $\dot{v}(x) = 0$.

## III. Main Results

In this section, we investigate global stability and Lyapunov dimension for the Lur'e system in Fig. 1. Particularly, our framework is based on a dissipated energy-like function and contraction analysis.

### A. Problem Formulation

In the state-space formulation, the $G$-block in Fig. 1 is represented by

$$\begin{cases}\dot{x} = Ax + Bu,\\ y = Cx,\end{cases} \tag{3}$$

where $A$, $B$, and $C$ are $n\times n$, $n\times m$, and $p\times n$ matrices, respectively. The $\phi$-block in Fig. 1 describes nonlinear feedback of the form

$$u = \phi(y), \tag{4}$$

where $\phi\colon \mathbb{R}^p \to \mathbb{R}^m$ is a $C^1$-function such that $\phi(0) = 0$ and $\mathrm{D}\phi(0) = 0$. The last assumptions are not restrictive if the Lur'e system has at least one equilibrium. Further, we assume that the $G$-block in Fig. 1 is $M$-dissipative, i.e., there exists a positive-definite matrix $P$ such that

$$\begin{bmatrix}A^\mathsf{T}P + PA & PB\\ B^\mathsf{T}P & 0\end{bmatrix} - \begin{bmatrix}C & 0\\ 0 & I\end{bmatrix}^\mathsf{T} M \begin{bmatrix}C & 0\\ 0 & I\end{bmatrix} \preceq 0, \tag{5}$$

where $M$ is a given symmetric matrix. In what follows, we employ the following notation:

$$\Phi(x) := C^\mathsf{T}\begin{bmatrix}I\\ \mathrm{D}\phi(Cx)\end{bmatrix}^\mathsf{T} M \begin{bmatrix}I\\ \mathrm{D}\phi(Cx)\end{bmatrix} C. \tag{6}$$

Following [14], we now consider the *prolonged system*

$$\begin{aligned}\begin{bmatrix}\dot{x}\\ \delta\dot{x}\end{bmatrix} &= (A\oplus A)\begin{bmatrix}x\\ \delta x\end{bmatrix} + (B\oplus B)\begin{bmatrix}u\\ \delta u\end{bmatrix},\\ \begin{bmatrix}y\\ \delta y\end{bmatrix} &= (C\oplus C)\begin{bmatrix}x\\ \delta x\end{bmatrix},\end{aligned}$$

where $\oplus$ is the direct sum. In the terminology of differential dissipativity theory [17], the above assumption indicates that the $G$-block is differentially dissipative with the storage

$$S(x,\delta x) = \delta x^\mathsf{T} P \delta x \tag{7}$$

and the supply rate

$$s(u,\delta u, y, \delta y) = \begin{bmatrix}\delta y\\ \delta u\end{bmatrix}^\mathsf{T} M \begin{bmatrix}\delta y\\ \delta u\end{bmatrix}.$$

In other words, the following differential dissipation inequality holds: $\dot{S}(x,\delta x) \le s(u,\delta u,y,\delta y)$. Note that the associated feedback is given by

$$\begin{bmatrix}u\\ \delta u\end{bmatrix} = \begin{bmatrix}\phi(y)\\ \mathrm{D}\phi(y)\delta y\end{bmatrix}.$$

Then, we have the following relation:

$$\delta x^{\mathsf{T}}\Phi(x)\delta x = \begin{bmatrix}\delta y\\ \delta u\end{bmatrix}^{\mathsf{T}} M \begin{bmatrix}\delta y\\ \delta u\end{bmatrix}. \tag{8}$$

To analyze stability and dimension, we provide an appropriate constraint in terms of $\Phi(x)$ defined in (6).

### B. Global Stability of Lur'e Systems

First, we identify global stability conditions for the Lur'e system in Fig. 1. The following result is related to both the classical absolute stability theory and the recent contraction theory.

*Theorem 1:* Suppose that the following conditions hold:

1) There exists a bounded $C^1$-function $v\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\dot{v}(x) \equiv \langle \nabla v(x), Ax + B\phi(Cx)\rangle \le 0$$

for all $x \in \mathbb{R}^n$.

2) There exists a constant $\varepsilon > 0$ such that

$$\Phi(x) \preceq -\varepsilon I$$

for all $x \in \mathbb{R}^n$ with $\dot{v}(x) = 0$.

Then, the origin is globally asymptotically stable.

Our result can be interpreted within the differential Lyapunov framework [6]. In particular, we employ the following differential Lyapunov function:

$$V(x, \delta x) = p(x)^2 \delta x^{\mathsf{T}} P \delta x, \tag{9}$$

where the positive-definite matrix $P$ is given in (5) and the positive function $p$ is given by the formula $v(x) = 2\ln p(x)$. We call the function (9) a *differential Lyapunov function of the Leonov form* because the original idea was proposed by Leonov [24]. In our framework, we can admit a certain gap between the storage function (7) for the LTI system and the Lyapunov function (9) for the closed-loop system. Note that the derivative of $V$ is given by

$$\begin{aligned}\dot{V}(x, \delta x) &= 2p(x)^2 \delta x^{\mathsf{T}} P \delta \dot{x} + \dot{v}(x) p(x)^2 \delta x^{\mathsf{T}} P \delta x\\ &= 2p(x)^2 \delta x^{\mathsf{T}} P \delta \dot{x} + \dot{v}(x) V(x, \delta x).\end{aligned}$$

Hence, the term $\dot{v}(x)$ partly serves as the decay rate of the differential Lyapunov function. Recall that the function $v$ is not necessarily a Lyapunov function. The introduction of such a dissipated energy-like function is, however, useful to define a nonconstant contraction metric [5].

The existence of $v$ in Theorem 1 is not only sufficient but also necessary for global stability in some sense. To confirm this, we now assume that the origin is globally asymptotically stable. From the converse Lyapunov theorem, there exists a positive-definite function $v$ such that $\dot{v}$ is a negative-definite function. In that case, $\dot{v}(x) = 0$ implies $x = 0$. Thus, since $\mathrm{D}\phi(0) = 0$, the condition 2) reads as follows:

$$\begin{bmatrix}C\\ 0\end{bmatrix}^{\mathsf{T}} M \begin{bmatrix}C\\ 0\end{bmatrix} \prec 0.$$

This is just a constraint on the supply rate for the LTI system. It follows that for such a dissipative LTI system, the two conditions in Theorem 1 are necessary for global stability.

In the following, we state some technical remarks.

*Remark 1:* How to find $v$ in the condition 1) is out of the scope of this paper. However, the IQC theory would be helpful to construct a function $v$ such that the condition 1) is valid for some class of nonlinearities. Notice that $v$ is not necessarily a positive-definite function and that we can always choose $v$ as a constant function so that the condition 1) is satisfied. Also, the assumption on the boundedness of $v$ can be removed if the system has a bounded absorbing set.

*Remark 2:* The condition 2) implies that

$$\Phi(x) + \dot{v}(x) P \preceq -\varepsilon I \tag{10}$$

for all $x \in \mathbb{R}^n$. Note that the two inequalities are equivalent if $\Phi$ is bounded. The reason is because $v$ can be multiplied by an arbitrarily large constant. Furthermore, it is not difficult to remove positive definiteness of $P$ to investigate dominance analysis as studied in [14]. In that case, the inertia of $P$ plays a fundamental role.

*Remark 3:* Assume that the pair $(A, B)$ is controllable. By the KYP lemma, the feasibility of the linear matrix inequality (5) is equivalent to that the frequency-domain inequality

$$\begin{bmatrix}G(\mathrm{j}\omega)\\ I\end{bmatrix}^{*} M \begin{bmatrix}G(\mathrm{j}\omega)\\ I\end{bmatrix} \succeq 0$$

holds for all $\omega \in \mathbb{R}$. Because of the relation (8), the condition 2) can be regarded as the input/output constraint

$$\begin{bmatrix}\delta y\\ \delta u\end{bmatrix}^{\mathsf{T}} M \begin{bmatrix}\delta y\\ \delta u\end{bmatrix} \preceq -\varepsilon \|\delta x\|^2.$$

Compared with the usual input/output stability theory, the above inequality need not hold for all possible input/output pairs $(\delta u, \delta y)$ produced on the entire state space. We only need to verify the condition for all input/output pairs $(\delta u, \delta y)$ produced on the region where $\dot{v}(x) = 0$.

### C. Lyapunov Dimension of Lur'e Systems

In the previous subsection, we have focused on a globally asymptotically stable equilibrium of the Lur'e system. Next, we investigate an unstable equilibrium which dominates the global dynamics. The following result is based on Leonov's method in Proposition 2 and is able to characterize nonlinearities for dimension estimates in a less conservative manner.

*Theorem 2:* Let $\lambda_1(x) \ge \cdots \ge \lambda_n(x)$ be the $n$ solutions to the characteristic equation

$$\det(\Phi(x) - \lambda(x) P) = 0,$$

where $P$ is a positive-definite matrix in (5). Suppose that the following conditions hold:

1) There exists a $C^1$-function $v\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\dot{v}(x) \equiv \langle \nabla v(x), Ax + B\phi(Cx)\rangle \le 0$$

for all $x \in \mathbb{R}^n$.

2) There exists a constant $d \in (0, n]$ such that

$$\lambda_1(x) + \cdots + \lambda_{\lfloor d \rfloor}(x) + (d - \lfloor d \rfloor)\lambda_{\lfloor d \rfloor + 1}(x) < 0$$

for all $x \in \mathbb{R}^n$ with $\dot{v}(x) = 0$.

Then, we have

$$\overline{\dim}_L(\{\varphi^t\}_{t \in \mathbb{R}_+}, \mathcal{K}) < d$$

for every compact invariant set $\mathcal{K}$.

Notice that the condition 1) in Theorem 2 is the same as the previous one. Hence, the classical absolute stability theory is still useful to construct such a function $v$. Because the system under study is unstable, the function $v$ may take negative values. However, it is useful to consider $v$ as just a dissipated energy-like function.

It is worth noting that the inequality in the condition 2) needs to hold only on the region where $v(x)$ is stationary. In this sense, the use of a dissipated energy-like function is advantageous compared with the recent related works in [14], [15], where the nonlinearity of a Lur'e system is constrained on the entire space. From the theoretical point of view, a more general result can be found in [25], where the metric tensor $P$ in (5) is a function of $x$. However, it is difficult to construct such a general metric. The merit of our method is that we can employ the separated conditions, each of which is easier to verify than other results.

The aforementioned result has provided an upper estimate of the global Lyapunov dimension. Our next examination is the validity of the Eden conjecture, which states that the local Lyapunov dimension at an equilibrium dominates the global Lyapunov dimension. Because the linearization at the origin induces the linearized system $\dot{x} = Ax$, the local Lyapunov dimension at the origin is given by

$$d_0 := \inf \left\{ d \in (0, n] : \lim_{t \to \infty} \frac{1}{t} \ln \omega_d(\mathrm{e}^{tA}) < 0 \right\}.$$

The following result specifies a class of nonlinearities for which the Eden conjecture is valid.

*Corollary 1:* Let $d_0$ be the local Lyapunov dimension at the origin. If all conditions in Theorem 2 are satisfied for any $d \in (d_0, n]$, then we have

$$\overline{\dim}_L(\{\varphi^t\}_{t \in \mathbb{R}_+}, \mathcal{K}) = d_0.$$

for every compact invariant set $\mathcal{K}$ containing the origin.

Corollary 1 provides a way to calculate the exact Lyapunov dimension because $d_0$ is determined only by $A$. In the next section, we show that the exact Lyapunov dimension of the Lorenz system can be obtained based on our framework. Compared with Theorem 2, the above result requires the task to find an appropriate function $v$ depending on $d$. Whether the existence of such a function $v$ is necessary for the validity of the Eden conjecture is an open problem. Nevertheless, it is necessary for the case where $d_0 = 0$ in the sense explained in the previous subsection.

We finally provide sufficient conditions for the absence of closed orbits. The following result is important because the describing function method is not sufficient for the analysis of oscillations (see [2], [3]).

*Corollary 2:* If all conditions in Theorem 2 are satisfied for $d = 2$, then every bounded solution converges to an equilibrium.

## IV. Application to Lorenz Systems

We verify that the existing result on the exact Lyapunov dimension formula of Lorenz systems [13] can be obtained from our framework. Fortunately, we can choose the same matrices $P$ and $M$ as well as the same function $v$ as in the meritorious paper [13] (see also [26]).

Consider the following Lorenz system:

$$\begin{cases} \dot{x} = -\sigma x + \sigma y, \\ \dot{y} = rx - y - xz, \\ \dot{z} = -bz + xy, \end{cases}$$

where $r$, $b$, and $\sigma$ are positive parameters. As in [13], we assume that $\sigma + 1 \geq b \geq 2$ and

$$r\sigma^2(4 - b) + 2\sigma(b - 1)(2\sigma - 3b) > b(b - 1)^2.$$

This system can be written in the form (3) with

$$A = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = I.$$

The feedback nonlinearity (4) and its derivative are given by

$$\phi(x, y, z) = \begin{bmatrix} 0 \\ xz \\ xy \end{bmatrix}, \quad \mathrm{D}\phi(x, y, z) = \begin{bmatrix} 0 & 0 & 0 \\ z & 0 & x \\ y & x & 0 \end{bmatrix}.$$

Following [13], we let

$$P := \begin{bmatrix} a^{-2} + \sigma^{-2}(b-1)^2 & -\sigma^{-1}(b-1) & 0 \\ -\sigma^{-1}(b-1) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $a := \sigma / \sqrt{r\sigma + (b-1)(\sigma - b)}$. By setting

$$M := \begin{bmatrix} A^\mathsf{T} P + PA & PB \\ B^\mathsf{T} P & 0 \end{bmatrix},$$

we can clearly observe that the linear matrix inequality (5) is satisfied. The problem is to find a $C^1$-function $v$ such that the conditions in Corollary 1 (Theorem 2) hold. Note that

$$\begin{aligned} \Phi(x) &= \begin{bmatrix} I \\ \mathrm{D}\phi(x, y, z) \end{bmatrix}^\mathsf{T} M \begin{bmatrix} I \\ \mathrm{D}\phi(x, y, z) \end{bmatrix} \\ &= [A + B\mathrm{D}\phi(x, y, z)]^\mathsf{T} P + P[A + B\mathrm{D}\phi(x, y, z)] \\ &= J(x, y, z)^\mathsf{T} P + PJ(x, y, z), \end{aligned}$$

where $J(x, y, z)$ is the Jacobian matrix associated with the closed-loop system. Thus, the numbers $\lambda_1(x), \ldots, \lambda_n(x)$ in Theorem 2 are the same as those in Proposition 2. As a result, we can employ the function $v$ obtained in [13] to satisfy the conditions in Corollary 1. Therefore, we obtain

$$\overline{\dim}_L(\{\varphi^t\}_{t \in \mathbb{R}_+}, \mathcal{K}) = 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + 4r\sigma}}.$$

In Fig. 2, we illustrate the Lorenz attractor with parameters $r = 28$, $b = 8/3$, and $\sigma = 10$. In general, calculating

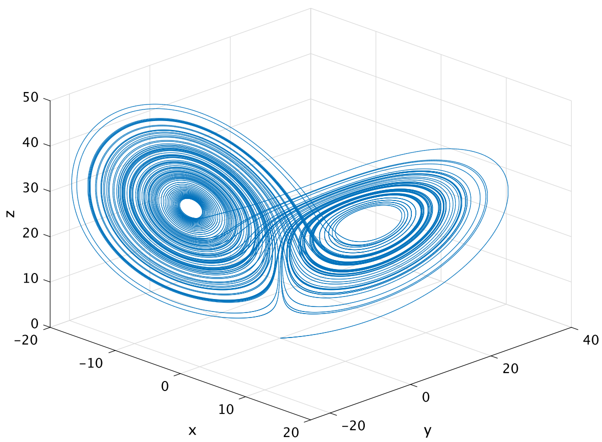


Fig. 2. Lorenz attractor

the exact Hausdorff dimension of such a strange attractor is hopeless, and the best known estimate is the exact Lyapunov dimension. In the above case, we have

$$\overline{\dim}_L(\{\varphi^t\}_{t\in\mathbb{R}_+}, \mathcal{K}) = 2.401.$$

The Hausdorff dimension of the Lorenz attractor is known to be about $2.062$, which is a numerically derived value. It may be difficult to reduce this gap. Furthermore, a lower bound of the Hausdorff dimension is generally harder to obtain than an upper bound.

## V. Conclusions and Future Work

We have revisited the classical works by Vyshnegradskii and Eden, which came from different contexts. By focusing on the analogy of these two problems, we have developed a unified framework on global stability and Lyapunov dimension for Lur'e systems. In Theorem 1, we have studied global stability of Lur'e systems by combining an energy perspective in Lyapunov analysis and a linearization approach in contraction analysis. This result has been generalized to dimension analysis in Theorem 2. The validity of the Eden conjecture and the absence of oscillations have been studied in Corollary 1 and Corollary 2, respectively. Finally, we have verified that our result is applicable to the Lorenz system to obtain the exact Lyapunov dimension formula due to Leonov. In future work, it would be interesting to consider dimension and entropy concepts for control systems [27] (see also [28]). In particular, both Lyapunov functions and Lyapunov exponents may play important roles in the study of dimension-like characteristics; the latter one has been well studied in the literature. Also, analysis of discontinuity is important as Vyshnegradskii's model of governors contained discontinuous nonlinearity.

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