2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

A Quadratic Approach to Rejection of Amplitude-Bounded Input Disturbances

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***Abstract*— This paper provides a quadratic approach to rejection of amplitude-bounded input disturbances for single-input linear discrete-time systems. Control specification is that a quadratic form decreases along the state trajectories when a quadratic constraint on the state is violated. All the state-feedback controllers that satisfy the specification are parameterized using the solution of the Riccati equation in cheap optimal control. The robustness of the controllers represented by the maximum allowable amplitude of disturbances is not uniform over the state space and proportional to the constrained value. In special cases, the optimal performance is represented using system parameters such as unstable zeros of the plant.**

I. INTRODUCTION

Persistent and fluctuating disturbances are reasonably characterized by the worst-case amplitude. However, analysis and synthesis of control systems based on this characterization rely on computational and complicated methods. This makes it hard to find fundamental relations between system properties and attainable performances. The established ℓ^1 control [1] provides the linear controller that minimizes the worst-case amplification for disturbances. However, controller design reduces to solving infinite dimensional linear program, which is hard to be solved. Also the controller can be dynamic and can have arbitrary high order even in the state-feedback case [2].

There have been many attempts to overcome these difficulties. One direction of study is to construct controller in a static form. In state-feedback case, ℓ^1 performance attainable by linear controller can be attained by static nonlinear controller [3], and this observation leads to controller design algorithms [4], [5]. However, resultant controller is undesirably complicated because control invariant set should be constructed as a polygon with huge number of vertices. Another direction is to construct control invariant set in a simple form. Quadratic invariant set can be represented by a positive definite matrix of the dimension of the state is used to obtain low-order controller [6], [7]. However, controller optimization problem, which requires to solve parameterized LMI, is not guaranteed to be convex.

For these reasons, well-established methods of disturbance rejection control assume disturbances in signal classes that can be more easily dealt with. Such examples are energy bounded signals in H^∞ control, Gaussian stochastic signals in LQG control, and signals generated by linear autonomous dynamics in tracking control.

To make the analysis and synthesis easier, this paper provides a quadratic approach to rejection of amplitude-bounded input disturbances. Control specification is that a quadratic form decreases along the state trajectories when the amplitude of the output exceeds an admissible level. The controllers that satisfy this specification are parameterized by the solution of the Riccati equation in cheap control, LQ optimal control without input weight. The set of all admissible control values is distributed around the control value of cheap control and represents robustness of the system in the state space. Furthermore, the optimal performance is represented using the solution of the Riccati equation. In special cases, it is represented using system parameters such as unstable zeros and a coefficient of the transfer function.

Notations used in this paper are as follows. Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}^n$ be the set of all n-dimensional real vectors, and $\mathbb{R}^{n\times n}$ be the set of all $n \times n$ real matrices. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be stable if the moduli of all its eigenvalues are less than 1. The quadratic form of a positive semidefinite matrix $P \in \mathbb{R}^{n\times n}$ is defined and denoted by $V_P(x) = x^{\mathrm{T}} P x$.

II. Control Under Bounded Input Disturbances

We are concerned with state-feedback control of single-input linear discrete-time systems with bounded input disturbances. Control specification is that a quadratic form decreases along the state trajectories when a quadratic constraint on the state is violated.

A. Control System

Consider the single-input linear discrete-time plant

$$x_{k+1} = Ax_k + Bv_k \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$, where $x_k \in \mathbb{R}^n$ is the state, and $v_k \in \mathbb{R}$ is the input. The state-feedback controller defined as a function $f : \mathbb{R}^n \to \mathbb{R}$ generates its output as

$$u_k = f(x_k). \tag{2}$$

The plant input is determined as

$$v_k = u_k + d_k, \tag{3}$$

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where $d_k \in \mathbb{R}$ is the input disturbance satisfying

$$|d_k| \le \sigma \tag{4}$$

Here $\sigma > 0$ represents the worst-case amplitude of the input disturbance and is assumed to be unknown, that is, f does not depend on σ. For a positive semidefinite matrix $S \in \mathbb{R}^{n\times n}$, the state constraint is given as

$$V_S(x_k) \le \gamma^2\sigma^2. \tag{5}$$

Note that $V_S(x_k)^{1/2}$ represents the magnitude of the state weighted by S, and its ratio to σ is required not to exceed γ. When the plant has the single output

$$y_k = Cx_k \tag{6}$$

with $C \in \mathbb{R}^{1\times n}$, the output constraint

$$|y_k| \le \gamma\sigma$$

can be expressed by letting $S = C^{\mathrm{T}}C$ in (5).

B. Control Specifications

Control specification is that a quadratic form decreases along the state trajectories when the constraint (5) is violated. This specification is described as follows.

Definition 1: Consider the system (1)–(3) and a positive semidefinite matrix $S \in \mathbb{R}^{n\times n}$. A controller f is said to achieve performance γ if there exists some positive semidefinite matrix $P \in \mathbb{R}^{n\times n}$ such that, for any $x_k \in \mathbb{R}^n$, $V_S(x_k) > \gamma^2\sigma^2$ implies $V_P(x_{k+1}) < V_P(x_k)$ for all $d_k \in \mathbb{R}$ satisfying (4).

Note that when P is restricted to be positive definite, the quadratic form V_P acts as a control Lyapunov function outside the admissible region $\Gamma = \{x \in \mathbb{R}^n : V_S(x) \le \gamma^2\sigma^2\}$. The case where P is not positive definite will be concerned to optimize the performance with respect to the output (6), but stability can be guaranteed.

Furthermore, the decay rate of $V_P(x_k)$ is evaluated as follows.

Definition 2: The controller in Definition 1 is said to achieve decay rate $\rho \in (0, 1)$ if, for any $x_k \in \mathbb{R}^n$, $V_S(x_k) > \gamma^2\sigma^2$ implies $V_P(x_{k+1}) < \rho V_P(x_k)$ for all $d_k \in \mathbb{R}$ satisfying (4).

The condition of Definition 1 corresponds to the case $\rho = 1$ of the stronger condition of Definition 2. When P is positive definite, the condition of Definition 2 guarantees that the state returns to the region Γ in finite time steps since $V_S(x) \le \lambda_{\max}(SP^{-1})V_P(x)$.

III. Controller Parameterization

This section provides a parameterization of all controllers that achieve required performance. The parameterization uses the solution of the Riccati equation in cheap control.

Consider the Riccati equation

$$P = A^{\mathrm{T}}PA + Q - A^{\mathrm{T}}PB\left(B^{\mathrm{T}}PB\right)^{-1}B^{\mathrm{T}}PA, \tag{7}$$

where $Q \in \mathbb{R}^{n\times n}$ is positive semidefinite. This equation appears in cheap control, LQ optimal control without input weight, and notable properties of the solution are studied in detail [8], [9]. The stabilizing solution of (7) is the solution P such that, for the state-feedback gain matrix

$$K = -\left(B^{\mathrm{T}}PB\right)^{-1}B^{\mathrm{T}}PA,$$

the closed-loop state matrix

$$F = A + BK$$

is stable. In the special case when $Q > 0$, (7) has a unique positive definite solution, which is stabilizing.

The equation is related to the general Riccati equation

$$\begin{aligned} P = {} & A^{\mathrm{T}}PA + Q \\ & - A^{\mathrm{T}}PB\left(B^{\mathrm{T}}PB + R\right)^{-1}B^{\mathrm{T}}PA \end{aligned} \tag{8}$$

with additional constant $R > 0$. The stabilizing solution of (8) is the solution P such that $F = A + BK$ with $K = -\left(B^{\mathrm{T}}PB + R\right)^{-1}B^{\mathrm{T}}PA$ is stable. Such a solution is unique if it exists. In the case that $Q = C^{\mathrm{T}}C$, the stabilizing solution exists if and only if (A, B) is stabilizable and (C, A) is observable on the unit circle. The stabilizing solution exists as the unique positive semidefinite solution if and only if (A, B) is stabilizable and (C, A) is detectable. Also, the positive definite stabilizing solution exists as the unique positive semidefinite solution if and only if (A, B) is stabilizable and (C, A) is observable. This solution is nondecreasing with respect to R, so tends to a limit as $R \to 0$. The limit P is a solution of (7) when $B^{\mathrm{T}}PB > 0$. The details can be found in, for example, [10], [11].

Using the solution of (7), we can parameterize all the controllers that achieve required performance as follows. A preliminary result of this parameterization is provided in our previous work [12].

Corollary 1: There exists a controller f that achieves performance γ if and only if the Riccati equation (7) holds with

$$Q \ge \frac{B^{\mathrm{T}}PB}{\gamma^2}S,\ B^{\mathrm{T}}PB > 0$$

for some $P \ge 0$. Furthermore, P is the matrix in Definition 1, and all the corresponding controllers are the functions f that satisfy

$$|f(x) - Kx| \le \Delta(x) = \sqrt{\frac{x^{\mathrm{T}}Qx}{B^{\mathrm{T}}PB}}.$$

Proof: The statement is the special case $\rho = 1$ of Theorem 1 below. ■

Corollary 1 shows that the set of all admissible control values are distributed around the optimal control value $u_k = Kx_k$ in cheap control. The robustness of the state can be represented by $\Delta(x_k)$, the half-width of the distribution, is not uniform over the state space. More precisely, for a fixed x_k, the maximum set of all $d_k \in \mathbb{R}$ such that $V_P(x_{k+1}) < V_P(x_k)$ is satisfied by some controller is $[-\Delta(x_k), \Delta(x_k)]$, which is achieved by cheap control. Thus, the maximum disturbance amplitude admissible for x_k is $\Delta(x_k)$, which is proportional to $V_Q(x_k)^{1/2}$. This implies nonuniformity of robustness in the state space.

This observation supports the importance of cheap control in disturbance rejection. Optimality of cheap control is verified for special cases of H^∞ control [13] and ℓ^1 control [2]. A condition under which cheap control achieves disturbance decoupling is also provided [15].

The following slightly extended version describes the dependency of controllers on the guaranteed decay rate ρ of $V_P(x_k)$.

Theorem 1: There exists a controller f that achieves performance γ with decay rate ρ if and only if the Riccati equation

$$\begin{aligned} P &= A_\rho^{\mathrm{T}} P A_\rho + Q \\ &\quad - A_\rho^{\mathrm{T}} P B \left(B^{\mathrm{T}} P B\right)^{-1} B^{\mathrm{T}} P A_\rho,\ A_\rho = A/\rho \end{aligned} \tag{9}$$

holds with

$$\rho^2 Q \geq \frac{B^{\mathrm{T}} P B}{\gamma^2} S,\ B^{\mathrm{T}} P B > 0, \tag{10}$$

for some $P \geq 0$. Furthermore, P is the matrix in Definition 1, and all the corresponding controllers are the functions f that satisfy

$$|f(x) - Kx| \leq \rho\Delta(x),\ \Delta(x) = \sqrt{\frac{x^{\mathrm{T}} Q x}{B^{\mathrm{T}} P B}}.$$

Proof: Let $x_+ = Ax + Bv$ and introduce the quadratic polynomial

$$\begin{aligned} \phi(v) &= \|x_+\|_P^2 - \rho^2 \|x\|_P^2 \\ &= \left(B^{\mathrm{T}} P B\right) v^2 + 2\left(B^{\mathrm{T}} P A x\right) v \\ &\quad + x^{\mathrm{T}} \left(A^{\mathrm{T}} P A - \rho^2 P\right) x. \end{aligned} \tag{11}$$

First, consider the case $B^{\mathrm{T}} P B \neq 0$. The roots of ϕ are

$$Kx \pm \rho\Delta(x),$$

where Q in $\Delta(x)$ is determined by (7). Thus, the set of all u such that $\phi(v) < 0$ with $v = u + d$ for all d satisfying $|d| \leq \sigma$ is

$$U(x) = \{u : |u - Kx| < \rho\Delta(x) - \sigma\}.$$

Therefore, for any controller f, $V(x_{k+1}) < \rho V(x_k)$ is equivalent to $f(x_k) \in U(x_k)$. Furthermore, the existence of f that achieves γ is equivalent to that $V_S(x) \geq \gamma^2\sigma^2$ implies $U(x) \neq \emptyset$. The former is equivalent to $x^{\mathrm{T}}\left(S/\gamma^2\right)x > \sigma^2$, and the latter is equivalent to $\rho\delta(x) > \sigma$, namely $x^{\mathrm{T}}\left(\rho^2 Q/B^{\mathrm{T}} P B\right)x > \sigma^2$. These imply that the existence of f is equivalent to $S/\gamma^2 \leq \rho^2 Q/B^{\mathrm{T}} P B$, or equivalently (10). Finally, consider the case $B^{\mathrm{T}} P B = 0$. Now, $B^{\mathrm{T}} P A = 0$ since $P^{1/2} B = 0$, thus $\phi = x^{\mathrm{T}}(A^{\mathrm{T}} P A - \rho P)x$, hence performance γ is achieved iff $V_S(P) > \gamma^2\sigma^2$ implies $\phi < 0$. Here, $V_S(x) = \alpha^2 B^{\mathrm{T}} S B > \gamma^2\sigma^2$ for $x = \alpha B$ with $\alpha > \gamma\sigma/\sqrt{B^{\mathrm{T}} S B}$, but $\phi = \alpha^2 B^{\mathrm{T}} A^{\mathrm{T}} P A B \geq 0$. This means that γ is not achievable. ■

Theorem 1 describes a trade-off between robustness and decaying speed. The maximum disturbance amplitude admissible for x_k is $\rho\Delta(x_k)$, which is proportional to the decay rate ρ.

We can find a relation of this result to quantized control. The robustness is realized by virtue of freedom in the choice of control values. In quantized control [16], such freedom contributes to reduce the number of control values, and optimal controller is constructed using the strategy of expensive control, LQ optimal control without state weight. An equivalence is shown between quantization effect and uncertainty in control [17].

IV. Performance Analysis

This section represents the optimal attainable performance using the solution of the Riccati equation. In special cases, the performance can be represented by plant parameters.

A. Optimal Performance

The optimal performance, the minimum of γ, is achieved when $Q = S$ and represented by the solution of the corresponding Riccati equation. This means that cheap control is also optimal in the sense of minimizing the admissible bound of state perturbation. The difference to cheap control is that the optimal controller is not unique and has the freedom in choice of control value as we found in the controller parameterization.

Theorem 2: Suppose that (A, B) is stabilizable and (C, A) is detectable. If the Riccati equation (7) with $Q = S$ has a positive semidefinite solution P, then the minimum of γ is

$$\gamma_{\min} = \sqrt{B^{\mathrm{T}} P B} \tag{12}$$

and achieved by the controllers in Corollary 1.

Proof: From Theorem 1, it follows that

$$\begin{aligned} \gamma_{\min}^2 &= \min\left\{\alpha : P \geq A^{\mathrm{T}} P A + \frac{B^{\mathrm{T}} P B}{\alpha} S \right. \\ &\qquad \left. - A^{\mathrm{T}} P B \left(B^{\mathrm{T}} P B\right)^{-1} B^{\mathrm{T}} P A,\ P \geq 0\right\} \\ &= \min\left\{\frac{B^{\mathrm{T}} P B}{\alpha} : P \geq A^{\mathrm{T}} P A + \alpha S \right. \\ &\qquad \left. - A^{\mathrm{T}} P B \left(B^{\mathrm{T}} P B\right)^{-1} B^{\mathrm{T}} P A,\ P \geq 0\right\} \\ &= \min\left\{B^{\mathrm{T}} P B : P \geq A^{\mathrm{T}} P A + S \right. \\ &\qquad \left. - A^{\mathrm{T}} P B \left(B^{\mathrm{T}} P B\right)^{-1} B^{\mathrm{T}} P A,\ P \geq 0\right\} \\ &= \min\left\{B^{\mathrm{T}} P B : P = A^{\mathrm{T}} P A + S \right. \\ &\qquad \left. - A^{\mathrm{T}} P B \left(B^{\mathrm{T}} P B\right)^{-1} B^{\mathrm{T}} P A,\ P \geq 0\right\}. \end{aligned}$$

The last equality is due to the fact that the solution $P \geq 0$ of (7) is nondecreasing with respect to $Q \geq 0$. ■

Theorem 2 related to cheap control is a counterpart to a well-known result [16] related to expensive control.

Therein, the minimum quantization density of control input in quadratic stabilization is represented using the solution of the Riccati equation in expensive control, i.e., (8) with $Q = 0$. The optimal controller is constructed based on the strategy of expensive control.

B. Performance Representation using System Parameters

This section attempts to represent the optimal performance using plant parameters. We evaluate the performance γ with respect to the plant output (6) by letting $S = C^{\mathrm{T}}C$ in (5). We focus on the two special cases:

- Case 1 The plant has no unstable zeros, and its relative degree is 1.
- Case 2 The plant has no stable zeros.

Cases 1 and 2 are the situations where the plant is easy to control and difficult to control, respectively. In both cases, the solution P in Theorem 2 exists.

1) Case 1: In this case, the optimal performance $\gamma_{\min}$ can be represented by the coefficient of the highest order term of the numerator polynomial of the transfer function. It is achieved when $P = C^{\mathrm{T}}C$, which means that the magnitude of the output should be directly reduced.

In the case that the relative degree is 1, i.e., $CB \neq 0$, $P = C^{\mathrm{T}}C$ is obviously a solution of the Riccati equation (7). In the following lemma, the former part is a special case of [8], and the latter part can be seen similarly to the corresponding result for (8) (see, for example, [10]).

Lemma 1: Suppose that $CB \neq 0$ in (7) with $Q = C^{\mathrm{T}}C$, then $P = Q$ is a positive semidefinite solution, which is a stabilizing solution if (C, A) is detectable.

Representation of the solution is given for more general case [8]. We can see that P in Lemma 1 is a solution of (7) since $PB\left(B^{\mathrm{T}}PB\right)^{-1}B^{\mathrm{T}}P = P$.

The state-feedback gain $K = -(CB)^{-1}CA$ corresponding to the solution P in Lemma 1 is used for disturbance decoupling control and sliding mode control for continuous-time systems [14].

The following statement, which is a special case of [8] and can be found in [14], shows that the closed-loop poles are placed at the zeros of the plant and at the origin.

Lemma 2: Suppose that (A, B) is reachable and $CB \neq 0$, then the eigenvalues of F for the solution P in Lemma 1 include all the zeros of the system (A, B, C), and the remaining one is zero.

By the assumption on the relative degree, Lemma 2 can be easily confirmed as follows [8]. Now, without loss of generality, assume that (A, B) is in controller canonical form. Let $C = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \end{bmatrix}$. Then the transfer function of the system (A, B, C) is $G(z) = \phi_o(z)/\Delta_o(z)$ with the open-loop characteristic polynomial $\Delta_o(z) = \det(zI - A)$ and the polynomial

$$\phi_o(z) = c_{n-1}z^{n-1} + c_{n-2}z^{n-2} + \cdots + c_1 z + c_0.$$

Since the relative degree of the system is 1, i.e., $c_{n-1} = CB \neq 0$, it follows that

$$\begin{aligned} F &= \left[I - (CB)^{-1}(BC)\right]A \\ &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & -\frac{c_0}{c_{n-1}} & -\frac{c_1}{c_{n-1}} & \cdots & -\frac{c_{n-2}}{c_{n-1}} \end{bmatrix}, \end{aligned}$$

thus the closed-loop characteristic polynomial is

$$\Delta_c(z) = \det(zI - F) = \frac{1}{c_{n-1}} z\phi_o(z).$$

The following statement is an easy consequence of Lemma 2.

Lemma 3: Suppose that (A, B) is reachable, $CB \neq 0$, and the system (A, B, C) is minimum phase, then the solution P in Lemma 1 is the stabilizing solution.

As a consequence of these facts, we have the following representation of the optimal performance.

Proposition 1: Under all the assumptions of Lemma 1 or those of Lemma 3, we have $\gamma_{\min} = |c_{n-1}|$.

Proof: Using these lemmas, one can obtain the statement directly from Theorem 2. ■

2) Case 2: In this case, the optimal performance $\gamma_{\min}$ can be represented by the product of unstable zeros of the plant. This minimum is achieved for a positive definite P, thus the corresponding V_P acts as a control Lyapunov function outside the admissible region Γ.

The following fact is known as an asymptotic property of the closed-loop poles, the roots of the characteristic polynomial $\Delta_c(z)$, of the optimal control system for vanishing input weight [18].

Lemma 4: Suppose that (A, B) is stabilizable and (C, A) is observable on the unit circle. For the stabilizing solution P of the Riccati equation (8) with $Q = C^{\mathrm{T}}C$, the limits of the roots of $\Delta_c(z)$ as $R \to 0$ include all the stable zeros of the system (A, B, C) and their reciprocals, and the remaining are zero.

This leads to the following statement.

Lemma 5: Suppose that (A, B) is stabilizable, (C, A) is detectable, and the system (A, B, C) has no zeros at 1. Also suppose that the Riccati equation (7) with $Q = C^{\mathrm{T}}C$ has a positive semidefinite solution P. If the system has no unstable zeros, then $B^{\mathrm{T}}PB/c_{n-r}^2 = 1$, otherwise,

$$\frac{B^{\mathrm{T}}PB}{c_{n-r}^2} = \prod_{i:|\nu_i|>1} |\nu_i|^2,$$

where r is the relative degree, $\nu_1, \ldots, \nu_{n-r}$ are the zeros, and c_{n-r} is the coefficient of the highest order term of the numerator of the irreducible transfer function.

Proof: Let $\phi_o(z)$ be the numerator polynomial of the transfer function. According to, for example, [19], it follows that

$$B^{\mathrm{T}}PB \cdot \Delta_c(z^{-1})\Delta_c(z) = \phi_o(z^{-1})\phi_o(z).$$

Substituting $z=1$ and using Lemma 4, we have

$$B^{\mathrm{T}}PB = c_{n-r}^2 \cdot \frac{\prod_{i=1}^{n-r}(1-\nu_i)^2}{\prod_{i=1}^{n}(1-\lambda_i)^2},$$

where $\lambda_i,\ldots,\lambda_n$ are the roots of Δ_o. Thus, if the system has no unstable zeros, then $B^{\mathrm{T}}PB/c_{n-r}^2=1$, otherwise,

$$\begin{aligned}\frac{B^{\mathrm{T}}PB}{c_{n-r}^2} &= \frac{\prod_{i=1}^{n-r}(1-\nu_i)^2}{\prod_{i:|\nu_i|<1}(1-\nu_i)^2\prod_{i:|\nu_i|>1}\left(1-\nu_i^{-1}\right)^2}\\ &= \frac{\prod_{i:|\nu_i|>1}(1-\nu_i)^2}{\prod_{i:|\nu_i|>1}\left(1-\nu_i^{-1}\right)^2} = \prod_{i:|\nu_i|>1}\nu_i^2\\ &= \prod_{i:|\nu_i|>1}|\nu_i|^2.\end{aligned}$$

■

The following statement is obvious by the well-known properties of the Lyapunov equation (see, for example, [10]).

Lemma 6: Suppose that the Riccati equation (7) with $Q=C^{\mathrm{T}}C$ has a stabilizing solution. Then it is positive definite if and only if (C,F) is observable.

Proof: Since (7) is equivalent to $P=F^{\mathrm{T}}PF+Q$, the lemma is a direct consequence of the relation between the solution and the coefficients of the Lyapunov equation. ■

This leads to the following statement.

Lemma 7: Suppose that (A,B) is reachable and (C,A) is detectable. Also suppose that the Riccati equation (7) with $Q=C^{\mathrm{T}}C$ has a stabilizing solution. Then it is positive definite if and only if the system (A,B,C) has no stable zeros.

Proof: The zeros of the system (F,B,C) are the same as those of the system (A,B,C). The poles, i.e., the eigenvalues of F, include all of the stable zeros and the reciprocals of the unstable zeros of the system (A,B,C), and the remaining are zero. Thus, the system (F,B,C) is of minimum order, i.e., pole-zero cancellation occurs iff the system (A,B,C) has a stable zero. Now, (F,B) is reachable since so is (A,B). Hence, the system (F,B,C) is of minimum order iff (C,F) is observable. According to Lemma 6, this is equivalent to that the stabilizing solution is positive definite. These conclude the statement of the lemma. ■

As a consequence of these facts, we have the following representation of the optimal performance.

Proposition 2: Suppose that the system (A,B,C) is of minimum order and has no stable zeros. Also suppose that A does not have eigenvalues on the unit circle. Then we have

$$\gamma_{\min} = |c_{n-r}|\cdot\prod_{i:|\nu_i|>1}|\nu_i|.$$

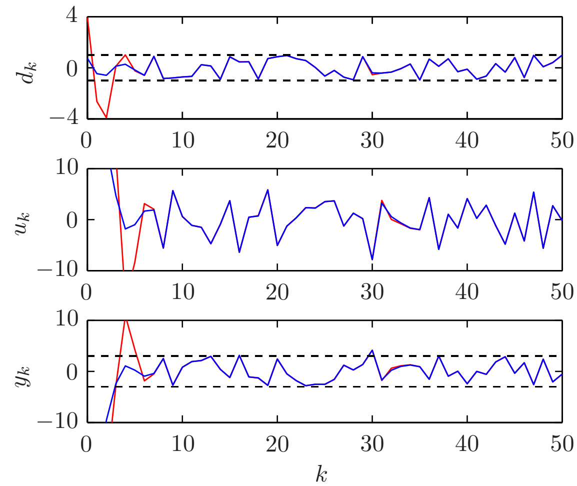

Fig. 1. Time evolutions of the input disturbance, the controller output, and the plant output.

Proof: Without loss of generality, assume that (A,B) is in controller canonical form. Letting p_{ni} be the (n,i) element of P, we see that the closed-loop state matrix corresponding to the Riccati equation (8) with $Q=C^{\mathrm{T}}C$ is

$$F=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\\-\bar{c}_0&-\bar{c}_1&-\bar{c}_2&\cdots&-\bar{c}_{n-1}\end{bmatrix},$$

where

$$(p_{nn}+R)\bar{c}_i=\begin{cases}a_0R, & i=0,\\ a_iR+p_{ni}, & i=1,2,\ldots,n-1.\end{cases}$$

This implies that F converges to a stable matrix as $R\to 0$ since it depends only on the coefficients of the characteristic polynomial $\det(zI-F)=z^n+\bar{c}_{n-1}z^{n-1}+\cdots+\bar{c}_2z^2+\bar{c}_1z+\bar{c}_0$, whose roots tend to the limits in Lemma 4. Using the latter part of Lemma 5 and sufficiency of Lemma 7, one can obtain the statement directly from Theorem 2. ■

In a special case of H^∞ control, the optimal performance is represented as Propositions 1 and 2 [13].

V. Numerical Example

This section provides a numerical example to illustrate the robustness of the control strategy. Consider the case that the plant parameters are

$$A=\begin{bmatrix}0&1\\-5&4\end{bmatrix},\ B=\begin{bmatrix}0\\1\end{bmatrix},\ C=\begin{bmatrix}-3&1\end{bmatrix}.$$

The transfer function of this system is

$$G(z)=\frac{z-3}{z^2-4z+5}.$$

The system has poles at $2\pm j$ and a zero at 3, which are all unstable. According to Proposition 2, the optimal

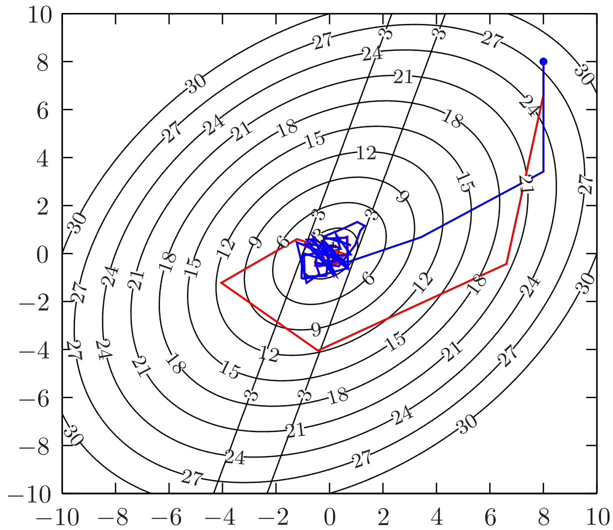

Fig. 2. State trajectories.

performance is $\gamma_{\min} = 3$. The solution of the Riccati equation (1) is

$$P = \begin{bmatrix} 9 & -3 \\ -3 & 9 \end{bmatrix} > 0,$$

and the corresponding state-feedback gain is

$$K = \begin{bmatrix} 5 & -3.6667 \end{bmatrix}.$$

Now, consider the controller $f(x) = Kx$ and let the initial state be $x_0 = [8\ 8]^{\mathrm{T}}$. Under the assumption $\sigma = 1$, each d_k is numerically determined as a realization of a random variable uniformly distributed on $[-\sigma, \sigma]$. Also, we examine the case where d_k is amplified by the 99% of the maximum allowable amplitude $\Delta(x_k)$ when x_k is outside the admissible region $\Gamma = \{x \in \mathbb{R}^2 : |Cx| \leq 3\}$. Fig. 1 shows the time evolutions of the input disturbance, the controller output, and the plant output. The plant output is attracted to the admissible range $[-3, 3]$ bounded by the dashed lines. Although it escapes from this range, it is immediately forced back to this range by virtue of V_P, which acts as a control Lyapunov function outside Γ. Fig. 2 shows the state trajectories. The blue and red lines represent the trajectories for the original and the amplified disturbances, respectively. The ellipses represent the contours of V_P, and the strip bounded by the two lines represents Γ. The state for the amplified disturbance fluctuates largely compared with that for the original disturbance, but the quadratic form of the state is monotonically decreased outside Γ and eventually sustained small as in the case of the original disturbance.

VI. Conclusions

We have provided a quadratic approach to rejection of amplitude-bounded input disturbances by state feedback. Control specification is that a quadratic form decreases along the state trajectories when a quadratic constraint on the state is violated. The optimal performance, the minimum of the constraint level, has been represented using the solution of the Riccati equation in cheap control. The optimal control values are distributed around the control value of cheap control. The robustness represented by the maximum allowable amplitude of disturbances is not uniform in the state space and proportional to the constrained value. For special cases, the optimal performance has been represented using system parameters such as unstable zeros and a coefficient of the transfer function.

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