2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

On Controllability of Temporal Networks

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***Abstract*— Temporality has been recently identified as a useful feature to exploit when controlling a complex network. Empirical evidence has in fact shown that, with respect to their static counterparts, temporal networks (i) are often endowed with larger controllable subspaces and (ii) require less control energy when steered towards an arbitrary target state. However, to date, we lack conditions guaranteeing that the dimension of the controllable subspace of a temporal network is larger than that of its static counterpart. In this work, we consider the case in which a static network is input connected but not controllable. We show that when the structure of the graph underlying the temporal network remains the same throughout each temporal snapshot while the edge weights vary (but stays different from 0), then the temporal network will be completely controllable almost always, even when its static counterpart is not. An upper bound on the number of snapshots needed to achieve controllability is also provided.**

I. INTRODUCTION

Since the work by Liu et al. appeared in 2011 [1], researchers from both the control engineering and the statistical physics community have devoted substantial effort in understanding under which conditions it is possible to control a network. At first, efforts were devoted to understand how to select a set of nodes where input signals should be injected so that all (or a desired target part) of the network becomes controllable [2], [3], [4]. The criticism to this approach was that Boolean ("yes/no") controllability conditions such as that provided by the Kalman test might not be the most suitable way to give a practical assessment of our ability to control a network [5], [6]. Spurred by this consideration, several works in the literature have investigated the relationship between the control effort required to control a network and the ratio between the number of control signals and the number of nodes to be controlled [7], [8]. The problem of determining the location of the minimum energy "driver nodes" has also been investigated extensively [9].

Work supported in part by grants from the Swedish Research Council (grant n. 2020-03701 to C.A.) and by the ELLIIT framework program at Linköping University. This study was carried out within the 2022K8EZBW "Higher-order interactions in social dynamics with application to monetary networks" project – funded by European Union – Next Generation EU within the PRIN 2022 program (D.D. 104 - 02/02/2022 Ministero dell'Università e della Ricerca). This manuscript reflects only the authors' views and opinions and the Ministry cannot be considered responsible for them.

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In most of these studies, the model considered for the network dynamics is linear time-invariant. As is well known from classical control theory, controlling nonlinear systems is in fact by far more complicated than controlling linear ones, and requires substantially different tools [10]. This, together with the broad spectrum of possible behaviors that can occur in nonlinear systems, makes the extension of the network controllability analysis to the nonlinear world extremely daunting and in many cases out of reach.

In between linear and nonlinear, an interesting class of networked systems has recently emerged, the so-called temporal networks [11]. These are essentially time-varying linear systems, in which the dynamics switches between different modes (in our work, between state update matrices $A_1, \dots, A_q$), and the switching between these "snapshots" is not controlled but it occurs according to a time schedule, regardless of the value of the state variables at the nodes. In this respect, the temporal network model considered in this paper differs from most of the switching systems settings investigated in the control literature [12], [13], [14]. It rather resembles the class of temporal networks frequently encountered in the complex networks literature [15]. When it comes to controllability, temporal networks were reported to have "fundamental advantages" [11], in the sense that they seem to reach controllability faster and demand order of magnitude less control energy than their static counterpart. This observation is somewhat counter-intuitive, and relies crucially on the fact that the future snapshots (i.e., the structure of the state matrices $A_1, \dots, A_q$ in our case) are known in advance, and are exploited in designing the controls in the previous steps [16]. Nevertheless, this class of temporal networks remains interesting and worth investigating further.

In this paper we consider a special control problem involving temporal networks. Namely, we assume to be in a case in which a given state/input matrix pair (A_s, B) is not structurally controllable (i.e., generically controllable, for almost all values of the non-zero entries of A_s and B, see [17], [18], [1]) and we show that replacing the "static" A_s with a temporal sequence of state matrices $A_1, \dots, A_q$ in the same structural class of A_s (i.e., $[A_s]_{ij} \neq 0 \iff [A_\ell]_{ij} \neq 0$ for $\ell \in \{1, \dots, q\}$), then controllablity is generically achieved, thus analitically backing the empirical observation made in [19]. To investigate the problem, we use the notion of fixed controllability subspace introduced in [20]. We provide an upper bound on the number of snapshots needed to almost surely achieve controllability in the temporal network: it corresponds to the difference between the dimension of the system and the generic dimension of the controllability subspace of the static system (A_s, B).

Furthermore, we provide a partial characterization of the Lebesgue measure zero set of selections of the edge weights for which the controllable subspace does not grow at each snapshot. Finally, we compound our theoretical results with a few of illustrative examples.

II. Notation and Problem formulation

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the network graph, where $\mathcal{V}$ is the set of n network nodes and $\mathcal{E}$ is the set of edges, i.e., $(i,j) \in \mathcal{E}$ *iff* there exists a directed edge connecting node i to node j in $\mathcal{G}$. We say that in $\mathcal{G}$ there exists a directed path from node i to node j if there exists a sequence of distinct nodes $v_1, \dots, v_\ell$ such that $(v_k, v_{k+1}) \in \mathcal{E}$ for all $k \in \{1, \dots, \ell-1\}$, $v_1 = i$, and $v_\ell = j$. We will denote by e_i the n-dimensional vector whose i-th entry takes the value of 1 and whose other entries are 0. Finally, let A be a square matrix; we will denote by $\lambda_i(A)$ the i-th eigenvalue of the matrix A.

In this work, we study the controllability properties of a temporal network, i.e., a network whose dynamics switches between q different linear modes (hereafter called *snapshots*), according to a preassigned switching schedule. The temporal network can be expressed in matrix form as

$$\dot{x} = A_k x + Bu\,,\ t \in [t_k,\ t_{k+1})\,,\ k \in \{1, \dots, q\}, \quad (1)$$

where $x \in \mathbb{R}^n$ is the stack vector of the state of the n network nodes, the ij-th element a_k^{ij} of each matrix A_k is different from zero *iff* $(j,i) \in \mathcal{E}$, i.e.,

$$A_k \in \mathcal{A} = \{X = (x_{ij}) \in \mathbb{R}^{n\times n} | x_{ij} \neq 0 \text{ iff } (j,i) \in \mathcal{E}\}, \quad (2)$$

where $\mathcal{A}$ denotes the class of structurally equivalent matrices. Note that the value of each nonzero entry, say the ij-th of each matrix A_k, can be interpreted as the weight taken by the edge (j,i) of $\mathcal{G}$ in the time interval $[t_k,\ t_{k+1})$.

In (1), u is the vector of m control inputs injected in the network and the ij-th element b_{ij} of the matrix B is 1 if the j-th input is injected in the i-th node and 0 otherwise. We assume that each of the m control inputs is injected in a different network node, and thus that B is composed of a subset of the columns of the identity matrix. Finally, we define the set of input nodes

$$\mathcal{V}_{\text{in}} := \Big\{ i \in \mathcal{V} \mid \sum_{j=1}^{m} b_{ij} = 1 \Big\} \quad (3)$$

as the subset of $\mathcal{V}$ where input signals are injected.

We define the static counterpart of the temporal network (1) to be the network whose dynamics is fixed (rather than switching) and described by the pair (A_s, B) with $A_s \in \mathcal{A}$, that is to say, a network whose structure is the same as that of all snaphots A_k of the temporal network (1).

As shown in [11], the reachable subspace of the temporal network (1) can be computed as

$$\Omega_t = \mathcal{S}_q + \sum_{k=1}^{q-1} \Big(\prod_{i=q}^{k+1} \Big[e^{A_i \Delta t_i} \Big] \mathcal{S}_k \Big), \quad (4)$$

where $\Delta t_i = t_i - t_{i-1}$ and

$$\mathcal{S}_k = \text{Im}\{K_k := \big[B\ A_k B\ \dots\ A_k^{n-1} B\big]\}. \quad (5)$$

Note that each element of the controllability matrix K_k admits a graphical interpretation. Namely, the ij-th element of the generic block $A_k^\ell B$, say w_{ij}^ℓ, is the sum of the weights of P paths, $\pi_{ij}^\ell(p)$, $p \in \{1, \dots, P\}$, from the j-th node in $\mathcal{V}_{\text{in}}$ to the i-th network node. The p-th path of length ℓ from the j-th input node to the i-th node is a sequence of edges $\{(r_1, r_2), (r_2, r_3), \dots, (r_{\ell-1}, r_\ell)\}$ where node r_1 is the j-th element of $\mathcal{V}_{\text{in}}$ while $r_\ell = i$. The weight of such path is the product of its edge weights, i.e.,

$$\pi_{ij}^\ell(p) = \prod_{h=1}^{\ell-1} a_{r_{h+1}, r_h},$$

and thus $w_{ij}^\ell = \sum_{p=1}^{P} \pi_{ij}^\ell(p)$. While (5) highlights the vectors that generate $\mathcal{S}_k$, providing a basis for Ω_t is nontrivial. We do so in the following Lemma.

Lemma 1: Let $\lambda_\ell(A_j)$ be the ℓ-th of μ_j distinct eigenvalues of A_j and $z_\ell \leq n$ be its multiplicity. Then the controllable subspace Ω_t of the temporal network (1) is the span of the columns of

$$K := [\tilde{K}_1\ \tilde{K}_2\ \dots\ \tilde{K}_q], \quad (6)$$

with

$$\tilde{K}_k = \begin{cases} K_k & \text{if } k = q \\ \prod_{j=q}^{k+1} p_n(A_j) K_k & \text{otherwise,} \end{cases} \quad (7)$$

where

$$p_n(A_j) = \sum_{i=0}^{n-1} \alpha_i(A_j) A_j^i \quad (8)$$

is a matrix polynomial of A_j of degree $n-1$ and where $\alpha_i(A_j)$, $i \in \{0, \dots, n-1\}$ are determined by solving the set of n linearly independent equations obtained by stacking together the n equations

$$\frac{d^z}{d\lambda^z}\Big(e^{\lambda \Delta t_j}\Big)\Big|_{\lambda=\lambda_\ell(A_j)} = \frac{d^z}{d\lambda^z}\Big(\sum_{i=0}^{n-1} \alpha_i(A_j)\lambda^i\Big)\Big|_{\lambda=\lambda_\ell(A_j)} \quad (9)$$

for all $z \in \{0, \dots, z_\ell - 1\}$ and for all $\ell \in \{1, \dots, \mu_j\}$.

Proof: The result is a direct consequence of (4) and of the fact that $e^{A_j \Delta t_j} = \sum_{i=0}^{n-1} \alpha_i(A_j) A_j^i$ with $\alpha_i(A_j)$, $i \in \{0, \dots, n-1\}$ defined as in (9), see e.g. [21]. ■

III. Main Result

We are interested in the case in which the static system $(A_s,\ B)$ fails to be controllable in a structural sense, i.e., the generic dimension $d := |\Omega_s|_g$ of the reachable subspace of the pair (A_s, B) is less than n. To exclude trivial cases, like those in which one or more nodes are inaccessible from the inputs (i.e, a root node of $\mathcal{G}$ is not an input node), a standing assumption in this paper is therefore that the pairs (A_k, B) (as well as (A_s, B)) are irreducible [22], i.e., for all $i \in \mathcal{V}$ either $i \in \mathcal{V}_{\text{in}}$ or in $\mathcal{G}$ there exists a directed path from a node $j \in \mathcal{V}_{\text{in}}$ to i. In fact, for us, the temporal network (1) and its static counterpart have all state matrices in the same class $\mathcal{A}$ and share the same input matrix B, hence they are either both irreducible or both reducible.

The scope of this paper is to investigate the relationship between the generic dimension $|\Omega_t|_g$ of Ω_t, that is the dimension that Ω_t has for almost all the values of the free entries of the matrices A_k, $k \in \{1,\dots,q\}$, and the generic dimension $d := |\Omega_s|_g$ of the reachable subspace of the pair (A_s, B). To do so, let us start by noting that, from [20]

$$\mathcal{S}_k = \mathcal{S}^g + \mathcal{S}_k^r, \tag{10}$$

for all k, where $\mathcal{S}^g$ is *fixed*, in the sense that it is the subspace including all directions of the state space that are in $\mathcal{S}_k$ for almost all the values of the nonzero entries of A_k. Now, we can give Theorem 3 from [20].

Theorem 1: If the pair (A_k, B) is irreducible, then $\mathcal{S}^g$ is generated by unit vectors in the state space defined as follows: the unit vector $e_i \in \mathcal{S}^g$ if and only if enlarging B with e_i the generic dimension of $\mathcal{S}_k$ does not change.

We give two Lemmas on the consequences of Theorem 1.

Lemma 2: If the vector $e_i \in \mathcal{S}_k$, then $e_i \in \mathcal{S}^g$ for almost all values of the nonzero entries in A_k.

Proof: Let us prove the thesis by contradiction and assume that $e_i \in \mathcal{S}_k$ and $e_i \notin \mathcal{S}^g$. Then, from Theorem 1 as $e_i \notin \mathcal{S}^g$ enlarging B with e_i should generically increase the dimension of $\mathcal{S}_k$. However, from (5), for this to be possible we would need $e_i \notin \mathcal{S}_k$ which is a contradiction. ■

Lemma 3: If the pair (A_k, B) is irreducible, then for all k, and for almost all the values of the nonzero entries in the matrices A_k, it is possible to relabel the network nodes and to perform elementary operations on the columns of the matrix K_k that turn it into the matrix

$$F_k = \begin{bmatrix} I_\gamma & 0 & 0 \\ 0 & R_k & 0 \end{bmatrix}, \tag{11}$$

where γ is the generic dimension of $\mathcal{S}^g$, I_γ is the γ dimensional identity matrix, and finally R_k is a $(n-\gamma)\times(d-\gamma)$ submatrix of K_k that is full column rank, whose rows all have at least one nonzero entry, and whose columns have at least two nonzero entries.

Proof: From (2) and (1) the generic rank of K_k is d for all k, and thus there exist elementary column operations on K_k that turn it into the matrix

$$[K_k^d \ 0], \tag{12}$$

where K_k^d is a full column rank $n \times d$ submatrix of K_k (modulo elementary column operations). Moreover, from Theorem 1 we have that there exists a subspace $\mathcal{S}^g$ of $\mathcal{S}_k$ that can be spanned by γ different vectors e_i. From (2), this subspace $\mathcal{S}^g$ is the same for all k. Hence, by relabeling the network nodes so that $e_i \in \mathcal{S}^g$ for all $i \in \{1,\dots,\gamma\}$, and since $e_i \in \mathcal{S}_k$ is equivalent to $e_i \in \mathrm{Im}\{K_k\}$ and thus also $e_i \in \mathrm{Im}\{K_k^d\}$, we have that there exist elementary operations on the columns of K_k^d that turn the matrix in (12) into the matrix F_k in (11) where I_γ is the γ-dimensional identity matrix. Note that these column elementary operations can be designed so that R_k is the block of K_k^d consisting of its last $d-\gamma$ columns and its last $n-\gamma$ rows. Hence, R_k is a submatrix of K_k (modulo elementary column operations) since K_k^d is a submatrix of K_k (modulo elementary column operations). Moreover, R_k is full column rank as K_k^d is full column rank and all its rows contain at least a nonzero entry as the pair (A_k, B) is irreducible. Finally all its columns have at least two nonzero entries as a column of R_k with only one nonzero entry would contradict Lemma 2. ■

Let us give one last preliminary Lemma highlighting the relationship between the structure of K_k and that of $\tilde{K}_k$.

Lemma 4: For all $k \in \{1,\dots,q\}$ if the (r,s)-th entry of K_k is nonzero, then also the (r,s)-th entry of $\tilde{K}_k$ is nonzero for almost all values of the free entries of A_j, $j \in \{k,\dots,q\}$.

Proof: From (7) and (8) we have that $\tilde{K}_k$ is a product of polynomials of order $n-1$ in the matrices A_j, $j \in \{k+1,\dots,q\}$. Hence, by isolating the 0-th order term of each polynomial, we can write $\tilde{K}_k$ as

$$\prod_{j=q}^{k+1} \alpha_0(A_j) K_k + \left(\prod_{j=q}^{k+1} p_n(A_j) - \prod_{j=q}^{k+1} \alpha_0(A_j) \right) K_k. \tag{13}$$

Then, as from (9) the equations whose solutions are the $\alpha_i(A_j)$ for all i and j are linearly independent and all the known terms are nonzero, we will have that $\alpha_0(A_j) = 0$ at most for a set of Lebesgue measure zero of the nonzero entries of A_j. Hence, we have just proven that $\prod_{j=q}^{k+1} \alpha_0(A_j) K_k$ in (13) is a matrix whose structure is the same as that of K_k for almost all values of the nonzero entries of A_j, $j \in \{k+1,\dots,q\}$. Now, for the element (r,s) of $\tilde{K}_k$ to be zero when the corresponding element of K_k is nonzero, we must have that the matrix equality

$$\prod_{j=q}^{k+1} p_n(A_j) K_k = 0$$

is fulfilled for its (r,s)-th element. Again, as this equality defines a proper variety in the parameter space of A_j, $j \in \{k,\dots,q\}$, then the thesis follows. ■

Theorem 2: If $q > n-d$ and $\mathcal{A}$ is such that the pair (A_k, B) is irreducible for all k, then $|\Omega_t|_g = n$.

Proof: The statement is trivial when $d = n$ as $A_k \in \mathcal{A}$ for all k and as $A_s \in \mathcal{A}$. Hence, if $d = n$ both the temporal network and its static counterpart are completely controllable. Let us thus focus on the case $d < n$, and prove that when $q > n-d$, the matrix K in (6) is full rank. To start with, from (6), (7), and Lemma 3, the matrix K can be recast through elementary operations on its columns as

$$\begin{bmatrix} \tilde{K}_1 & \tilde{K}_2 & \dots & \tilde{K}_{q-1} & F_q \end{bmatrix}. \tag{14}$$

Then, we can exploit the block $[I_\gamma \ 0^{\mathrm{T}}]^{\mathrm{T}}$ in F_q to perform elementary operations that turn (14) into

$$\begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 0 & I_\gamma & 0 \\ * & \tilde{R}_1 & \dots & * & \tilde{R}_{q-2} & * & \tilde{R}_{q-1} & 0 & R_q \end{bmatrix}. \tag{15}$$

where with $*$ we denote blocks of dimension $(n-\gamma)\times\gamma$ that are irrelevant for this proof, while the blocks $\tilde{R}_k$, $k \in \{1,\dots,q-1\}$ share the same dimensions of R_q, i.e., $(n-\gamma)\times(d-\gamma)$. Then let us define the $(n-\gamma)\times q(d-\gamma)$ submatrix of (15)

$$R := [\tilde{R}_1 \ \tilde{R}_2 \ \dots \ \tilde{R}_{q-1} \ \tilde{R}_q], \tag{16}$$

with $\tilde{R}_q = R_q$. Indeed, from (15), to prove our thesis it is sufficient to show that if $q > n-d$ then $\rho_g(R) = n - \gamma$ where $\rho_g(R)$ is the rank that the matrix R takes for almost all values of the nonzero entries of the matrices $\{A_1, A_2, \ldots, A_q\}$. To do so, let us consider an integer k such that $1 \leq k \leq n-d < q$, and the submatrix of R defined $\forall\, j \in \{1, \ldots, k\}$ as

$$[\tilde{R}_{q-(k-1)} \ldots \tilde{R}_{q-(k-j)} \ldots \tilde{R}_q] \tag{17}$$

and assume that its generic rank is $d-\gamma+(k-1)$. In (17), relabel the network nodes so that the first $d-\gamma+(k-1)$ rows form a square matrix, say $\hat{R}_{q-(k-1)}$, with full generic rank. Then, consider a column $\tilde{c}_k$ of $\tilde{R}_{q-k}$ such that its $(d-\gamma+k)$-th entry, $\tilde{c}_k(d-\gamma+k)$, is nonzero. Note that, as the pair (A_k, B) is irreducible, Lemma 3 and 4 ensure such a column exists. As $\hat{R}_{q-(k-1)}$ is generically full rank, there generically exists a unique θ_k such that

$$\hat{R}_{q-(k-1)}\theta_k = [\tilde{c}_k(1) \ \ldots \ \tilde{c}_k(d-\gamma+k-1)]^T.$$

Define the matrix

$$\hat{R}_{q-k} = \begin{bmatrix} \hat{R}_{q-(k-1)} & [\tilde{c}_k(1) \ \ldots \ \tilde{c}_k(d-\gamma+k-1)]^T \\ \hat{r}_{k-1} & \tilde{c}_k(d-\gamma+k) \end{bmatrix} \tag{18}$$

where $\hat{r}_{k-1}$ is a row vector obtained by taking, among the elements of the $(d-\gamma+k)$-th row of the matrix in (17), only the elements corresponding to the columns that are selected in $\hat{R}_{q-(k-1)}$. Indeed, $\hat{R}_{q-k}$ in (18) is full rank unless $\hat{r}_{k-1}\theta_k = \tilde{c}_k(d-\gamma+k)$, that is, for almost all values of the nonzero entries of A_k except a set of Lebesgue measure zero. As the generic rank of $\tilde{R}_q$ is $d-\gamma$, then indeed we have proved by recursion that for all k such that $1 \leq k \leq n-d+1 \leq q$ the generic rank of (17) is $d-\gamma+k-1$. Hence, for $k = n-d+1$ we have that (17) and thus R in (16) is generically full rank, which proves our thesis. ■

Corollary 1: If a temporal network satisfying the hypotheses of Theorem 2 achieves complete controllability in q snapshots, then any temporal network obtained by reordering the snapshots $A_1, \ldots, A_q$ also achieves complete controllability in q snapshots.

Proof: As the rank of a matrix, R in (16) in this case, is independent of the order of its columns, the thesis follows. ■

In the following Remark we give a geometric interpretation of the proof of Theorem 2.

Remark 1: The controllable subspace of a linear system described by a pair (A, B) (and thus also of a linear network) is the smallest A-invariant subspace including the range of the matrix B. Hence, the controllable subspace is the span of a set of right eigenvectors of the matrix A. Among this basis, according to [20] we can distinguish a set of γ elements that span the fixed space $\mathcal{S}^g$, a space that does not vary with the free entries of A. When $\mathcal{S}^g$ is smaller than the controllable subspace, then it is completed by adding to the basis of $\mathcal{S}^g$ another set of $d-\gamma$ eigenvectors that span a non-generic subspace of the network state-space. These $d-\gamma$ elements form a basis of a subspace that is not generic (although its size is generic) and that is "tilted" as the free entries of A are modified. When concatenating in a temporal network matrices A_k whose structure is the same but whose entries vary, we will have that, according to Lemma 4, no rank is lost because of the "rotations" due to the left multiplication with exponentials in (4), and that at least a direction is added to the controllable space at each snapshot because of the tilting in A_k, see proof of Theorem 2.

A. Investigating the (zero measure) set of non-controllable temporal networks

From Theorem 2, almost all choices of $A_1, \ldots, A_q \in \mathcal{A}$ achieve $|\Omega_t|_g = n$ in at most $n-d+1$ steps. Characterizing the zero-measure set of matrices for which Theorem 2 fails and the reachability subspace does not reach a generic rank n is a challenging problem. The following Theorem provides an explicit class of matrices that will never increase the dimension of the reachable subspace Ω_t, regardless of the number of snapshots q.

Theorem 3: If $A_1 \in \mathcal{A}$ and for all $k \in \{2, \ldots, q\}$

$$A_k = \alpha_\ell A_\ell + BL_\ell \qquad \ell \in \{1, \ldots, k-1\}, \tag{19}$$

where $\alpha_\ell \in \mathbb{R} \setminus \{0\}$, and $L_\ell \in \mathbb{R}^{m \times n}$ is s.t. $\alpha_\ell A_\ell + BL_\ell \in \mathcal{A}$, then $\mathcal{S}_k = \mathcal{S}_\ell$, and thus $|\Omega_t|_g \leq |\Omega_s|_g$. Moreover, if $A_s \in \text{span}\{A_1, \ldots, A_q\} \cap \mathcal{A}$ we have $\Omega_s = \mathcal{S}_i$, $\forall i \in \{1, \ldots, k\}$, and $\Omega_t \subseteq \Omega_s$.

Proof: From the Cayley-Hamilton theorem we have that $A_\ell \mathcal{S}_\ell \subseteq \mathcal{S}_\ell$ and that $\alpha_\ell \mathcal{S}_\ell = \mathcal{S}_\ell$. Therefore from (19) we have $A_k \mathcal{S}_\ell \subseteq \alpha_\ell A_\ell \mathcal{S}_\ell + \text{Im}\{B\} \subseteq \alpha_\ell \mathcal{S}_\ell + \text{Im}\{B\} = \mathcal{S}_\ell$. Thus, $\forall k \in \{2, \ldots, q\}$, $\ell \in \{1, \ldots, k-1\}$, we get that $\mathcal{S}_\ell$ is A_k-invariant and it includes $\text{Im}\{B\}$, and so it includes $\mathcal{S}_k$, which is the smallest subspace with those properties. Conversely, we note that if (19) holds then $A_\ell = \tilde{\alpha}_k A_k + B\tilde{L}_k$, with $\tilde{\alpha}_k = \alpha_\ell^{-1}$ and $\tilde{L}_k = -\tilde{\alpha}_k L_\ell$. We obtain $A_\ell \mathcal{S}_k \subseteq \mathcal{S}_k$ and so $\mathcal{S}_k$ is A_ℓ-invariant and it includes $\text{Im}\{B\}$, and so it includes $\mathcal{S}_\ell$. Therefore $\mathcal{S}_k \subseteq \mathcal{S}_\ell \subseteq \mathcal{S}_k$ which implies $\mathcal{S}_k = \mathcal{S}_\ell$, and since it holds $\forall\, k \in \{2, \ldots, q\}$, $\ell \in \{1, \ldots, k-1\}$ we have

$$\mathcal{S}_k = \mathcal{S}_i \quad i \in \{1, \ldots, k\}. \tag{20}$$

To prove that $|\Omega_s|_g \geq |\Omega_t|_g$, we first observe that

$$\sum_{k=1}^{q} \mathcal{S}_k = \mathcal{S}_i, \quad 1 \leq i \leq q. \tag{21}$$

We then note that if a subspace $\mathcal{S}_\ell$ is A_k-invariant it is also e^{A_k}-invariant[1], i.e., $\forall k \in \{1, \ldots, q\}$, $\ell \in \{1, \ldots, k-1\}$, we

[1]Given $A_k^i \in \mathbb{R}^{n \times n}$, we have from the Cayley-Hamilton theorem that $A_k^n \in \text{span}\{A_k, \ldots, A_k^{n-1}\}$, $\forall k \in \{1, \ldots, q\}$, therefore $A_k^i \in \text{span}\{A_k, \ldots, A_k^{n-1}\}$, $\forall i \geq n$. Thus, it holds $\forall\, i \in \mathbb{N}$ and $\sum_{i=0}^{\infty} \mu_i A_k^i \in \text{span}\{A_k, \ldots, A_k^{n-1}\}$, $\forall\, \mu_i \in \mathbb{R} \setminus \{0\}$. If $A_k \mathcal{S}_\ell := \mathcal{P}_1 \subseteq \mathcal{S}_\ell$, then $A_k^2 \mathcal{S}_\ell = A_k \mathcal{P}_1 := \mathcal{P}_2 \subseteq \mathcal{S}_\ell$, therefore $A_k^i \mathcal{S}_\ell := \mathcal{P}_i \subseteq \mathcal{S}_\ell$, which proves that A_k-invariance implies A_k^i-invariance $\forall\, i \in \mathbb{N}$. If $\mathcal{S}_\ell$ is invariant for $\{A_k, \ldots, A_k^{n-1}\}$, it is also invariant for every linear combination of these matrices, because we can always rewrite such linear combination as a polynomial in A_ℓ, $p(A_\ell)$. Indeed $\mathcal{S}_\ell = \{p(A_\ell)\,\text{Im}\{B\} | \deg(p(A_\ell)) = n-1\}$ and if (19) holds, then $\sum_{i=0}^{n-1} \mu_i A_k^i = p(A_\ell)$.

have $e^{A_k\Delta t_k}\mathcal{S}_\ell = \sum_{i=0}^{\infty}\frac{A_k^i\Delta t_k^i}{i!}\mathcal{S}_\ell = \sum_{i=0}^{n-1}\frac{A_k^i\Delta t_k^i}{i!}\mathcal{S}_\ell \subseteq \mathcal{S}_\ell$. Hence, we observe that $\forall k \in \{1,\ldots,q\}$:

$$\prod_{i=q}^{k+1}\Big[e^{A_i\Delta t_i}\Big]\mathcal{S}_k = \prod_{i=q}^{k+2}\Big[e^{A_i\Delta t_i}\Big]e^{A_{k+1}\Delta t_{k+1}}\mathcal{S}_k \subseteq \prod_{i=q}^{k+2}\Big[e^{A_i\Delta t_i}\Big]\mathcal{S}_k \subseteq \ldots \subseteq e^{A_q\Delta t_q}\mathcal{S}_k \subseteq \mathcal{S}_k. \tag{22}$$

Therefore, from (20) and (21), substituting (22) in (4) yields

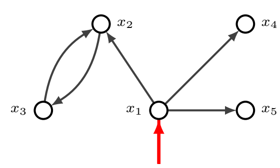
$$\Omega_t = \mathcal{S}_q + \sum_{k=1}^{q-1}\Big(\prod_{i=q}^{k+1}\Big[e^{A_i\Delta t_i}\Big]\mathcal{S}_k\Big) \subseteq \mathcal{S}_q + \sum_{k=1}^{q-1}\mathcal{S}_k = \mathcal{S}_i. \tag{23}$$

Since A_s, $A_i \in \mathcal{A}$ we have that $|\Omega_s|_g = |\mathcal{S}_i|_g = d$, and from (23) we obtain $|\Omega_s|_g \geq |\Omega_t|_g$. If we further assume that $A_s \in \operatorname{span}\{A_1,\ldots,A_q\}\cap\mathcal{A}$, then there exist β_k, $\hat{\beta}_k \in \mathbb{R}\setminus\{0\}$, and $\hat{L}_k \in \mathbb{R}^{m\times n}$, $k \in \{1,\ldots,q\}$ such that $A_s = \sum_{j=1}^{q}\beta_j A_j = \hat{\beta}_i A_i + B\hat{L}_i$, $i \in \{1,\ldots,k\}$. Then, the same lines of argument showing that $\mathcal{S}_i = \mathcal{S}_k$ for all (k,i) smaller than q imply that $\Omega_s = \mathcal{S}_i \supseteq \Omega_t$, completing the proof. ■

IV. Numerical Examples

We start this numerical section by giving an example that illustrates the main result of Theorem 2.

Example 1. Let us consider the following network



whose adjacency matrix for each snapshot is

$$A_k = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ a_{21}(k) & 0 & a_{23}(k) & 0 & 0 \\ 0 & a_{32}(k) & 0 & 0 & 0 \\ a_{41}(k) & 0 & 0 & 0 & 0 \\ a_{51}(k) & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{A},$$

and where the red arrow entering node 1 means that the only control input is injected in node 1, i.e., $B = e_1$. Consequently, we have that the structure of the controllability matrices K_k for each snapshot is

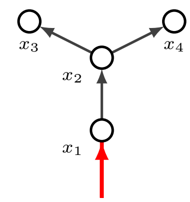
$$\begin{bmatrix} * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & 0 \\ 0 & 0 & * & 0 & * \\ 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \end{bmatrix},$$

where the symbol $*$ denotes a nonzero entry whose value as a function of the entries of A_k is omitted for brevity. From the structure of the first three rows of K_k, it is evident that $\mathcal{S}^g = \operatorname{Im}\{e_1,\ e_2,\ e_3\}$ and thus there exists a set of elementary column operations such that K_k can be turned into

$$F_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a_{41}(k) & 0 \\ 0 & 0 & 0 & a_{51}(k) & 0 \end{bmatrix},$$

where $[a_{41}\ a_{51}]^T = R_k$. As from the structure of F_k it is $d = 4$, from Theorem 2, in $q = 2 > n - d = 1$ snapshots we almost always achieve $\Omega_t = \mathbb{R}^n$. ■

Example 2. In this example we exploit the same network structure with two different parametrizations to (i) provide an instance of the class of matrices introduced in Theorem 3, and (ii) show that this class does not completely characterize the set of Lebesgue measure zero of parameter selections for which the controllable subspace does not grow at each snapshot. We consider the following network



where the red arrow indicates that $B = e_1$ and where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_{21}(1) & 0 & 0 & 0 \\ 0 & a_{32}(1) & 0 & 0 \\ 0 & a_{42}(1) & 0 & 0 \end{bmatrix} \in \mathcal{A}.$$

A_2 is selected as

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \beta a_{21}(1) & 0 & 0 & 0 \\ 0 & \mu a_{32}(1) & 0 & 0 \\ 0 & \mu a_{42}(1) & 0 & 0 \end{bmatrix} \in \mathcal{A},$$

where β and μ are two nonzero scalars. By exploiting the nilpotency of A_2, we get the following exponential matrix

$$e^{A_2\Delta t_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta a_{21}(1)\Delta t_2 & 1 & 0 & 0 \\ \mu\beta a_{21}(1)a_{32}(1)\frac{\Delta t_2^2}{2} & \mu a_{32}(1)\Delta t_2 & 1 & 0 \\ \mu\beta a_{21}(1)a_{42}(1)\frac{\Delta t_2^2}{2} & \mu a_{42}(1)\Delta t_2 & 0 & 1 \end{bmatrix}.$$

The controllability matrix of the pair (A_1, B) is given by

$$K_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{21}(1) & 0 & 0 \\ 0 & 0 & a_{21}(1)a_{32}(1) & 0 \\ 0 & 0 & a_{21}(1)a_{42}(1) & 0 \end{bmatrix}.$$

It is easy to verify that all the columns of $\tilde{K}_1 = e^{A_2\Delta t_2}K_1$ can be written as linear combinations of the columns of

$$K_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta a_{21}(1) & 0 & 0 \\ 0 & 0 & \mu\beta a_{21}(1)a_{32}(1) & 0 \\ 0 & 0 & \mu\beta a_{21}(1)a_{42}(1) & 0 \end{bmatrix}.$$

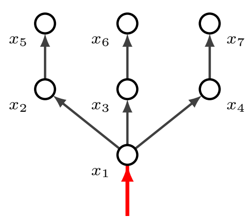
Indeed, for this temporal network we have that $\gamma = 2$ and

$$R = \begin{bmatrix} a_{21}(1)a_{32}(1) & \mu\beta a_{21}(1)a_{32}(1) \\ a_{21}(1)a_{42}(1) & \mu\beta a_{21}(1)a_{42}(1) \end{bmatrix}.$$

Hence, regardless of the choice of β and μ, the rank of K is 3 and thus the temporal network is not controllable. When $\beta = \mu$ we have $A_2 = \beta A_1$ that fulfills the assumption of Theorem 3 given in (19) with $\alpha_1 = \beta \in \mathbb{R} \setminus \{0\}$ and $L_1 = 0$, while setting $\beta \neq \mu$ defines an uncontrollable temporal network that is not characterized by Theorem 3. ■

In Theorem 2 we provide an upper bound on the number of snapshots required to achieve complete controllability. However, the tightness of this bound depends on the network structure. The number of leaves and the maximal length of the paths of the disjoint trees rooted in the input nodes that span the network are two structural features known to affect the controllability properties of static networks [17], [23]. Example 3 shows that these two structural features play a role also in the temporal case, as they may determine the number of snaphshots required to achieve $\Omega_t = \mathbb{R}^n$.

Example 3. Let us consider a tree network of 7 nodes with 3 leaves and branches of length 2, with control input in the root node 1.



The adjacency matrix of the k-th snapshot is such that $a_{ij}(k) \neq 0$ for all pairs $(i, j) \in \{(2, 1), (3, 1), (4, 1), (5, 2), (6, 3), (7, 4)\}$ and zero otherwise. Exploiting the graphical mapping in [24] it is possible to show that the generic dimension of the controllable subspace of this network is $d = 3$. Therefore from Theorem 2 we should achieve complete controllability in at most $q = n - d + 1 = 5$. However, for almost all values of the nonzero entries of the matrices A_k, $k \in \{1, 2, 3\}$, controllability will be reached in $q = 3$ snapshots. To see this, consider that for this temporal network $\gamma = 1$ and

$$R_q = \begin{bmatrix} a_{21}(q) & 0 \\ a_{31}(q) & 0 \\ a_{41}(q) & 0 \\ 0 & a_{21}(q)a_{52}(q) \\ 0 & a_{31}(q)a_{63}(q) \\ 0 & a_{41}(q)a_{74}(q) \end{bmatrix}.$$

For $q = 3$ we have that $R = [\tilde{R}_1 \ \tilde{R}_2 \ R_3]$ will be a 6×6 full rank matrix for almost all values of $a_{ij}(k)$, $k \in \{1, 2, 3\}$. ■

V. Conclusions

Motivated by [11], in this paper we studied controllability of a class of temporal networks for which the edge weights vary in time while the structural graph $\mathcal{G}$, describing the topology of the connections, is time invariant. We found that this class of temporal networks almost always reaches controllability provided that in $\mathcal{G}$ there exists a directed path connecting at least an input node to every other network node, thus proving the empirical observation made in [19]. Furthermore, we provided a partial characterization of the Lebesgue measure zero set of selections of the edge weights for which temporality is not advantageous for controllability in this setting. Future work will be devoted to extend our results to temporal networks whose structure varies in time.

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