2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

A Dynamic Model of Network Formation: Network Participation Game and Network Sharing Game

Ahmed Luqman and Hassan Jaleel

Abstract— **We introduce a dynamic model for network participation and resource-sharing problems grounded in non-cooperative game theory. Within a social network, individuals must decide whether to join cooperative activities or share resources based on anticipated benefits versus incurred costs. We cast these problems as non-cooperative games and comprehensively characterize the Nash equilibria in these settings. Furthermore, we introduce Log-Linear Learning (LLL) as a potential decision strategy for the participants and analyze the long-term dynamics of this approach within the framework. We perform extensive simulations on random networks to empirically validate our research findings. These simulations provide compelling evidence that within our proposed framework, user engagement in network participation and sharing dilemmas closely aligns with the well-established concepts of k-core and (r, s)-core within network structures.**

I. Introduction

The exchange of information and resources within a social network depends on the willingness of users to share. These resources may take the form of information, such as job openings, stock prices, or product reviews, or they can be physical resources like sensor suites for weather data in agricultural planning or location-sharing for accurate traffic congestion updates on cellphones. Consequently, when individuals choose not to contribute their available resources, even if they have many connections in the underlying social network, it can hinder the progression of a specific phenomenon. Thus, within the framework of a given social network, an important challenge is to identify users who are inclined to engage in cooperative activities ([1], [2], [3], and [4]).

We propose a game-theoretic framework for network formation within network participation and sharing games. In participation games, users face the choice of joining a cooperative activity, which involves incurring certain costs but also reaping associated benefits. Similarly, in sharing games, each player possesses a set of personal resources and must decide whether to share these resources with their friends or neighbors in a social network. When mutual resource sharing occurs among friends with different resources, it can benefit all parties involved. Various models for user engagement have been proposed in the literature on social network analysis, behavioral psychology, and economics, based on empirical evidence and experimental data. An important aspect of these models is the concept of reciprocity as discussed in [5], [6], [7], and [8]. The principle of reciprocity dictates that users only participate in any activity as long as their expected benefits outweigh the cost of participation.

A. Luqman and H. Jaleel are with the Intelligent Machines & Sociotechnical Systems (iMaSS) Lab, Department of Electrical Engineering, Syed Babar Ali School of Science & Engineering at LUMS, Lahore, Pakistan. Emails: hassan.jaleel@lums.edu.pk, 24100041@lums.edu.pk

Based on reciprocity, the following model has been widely adopted for the network participation problem. A user decides to participate in a network activity if a minimum number, say k, of his friends are also participating. He opts out of the activity if the number of participating friends drops below k. This model leads to the concept of k-core of the network as a measure of user engagement (see, for example, [1], [8], [9], [10]). Here, the k-core is a graph theoretic notion and refers to the maximal sub-graph of a graph in which each vertex has at least k neighbors. Similarly, in the network sharing problem as presented in [4], there exists a total of r resources in the network, and each user is randomly assigned s of these resources, where s is less than r. Then, the users agree to share their resources if they can access all the r resources through their friends. This model leads to (r, s)-core as a solution concept for network-sharing games.

The analysis that establishes k-core and (r, s)-core as a measure of user engagement in [1] and [4], and the other related literature assumes that all the players are initially participating in the cooperative/sharing activity. Then, the players who do not satisfy the k-neighbor criteria or do not get access to all the r resources from their participating friends decide to opt out. This user attrition can lead to a cascade of users leaving the activity in steps. The iterative process converges to the k-core or (r, s)-core of the network, in which all the users satisfy the desired criteria. However, this approach cannot explain the formation of a participation/sharing network from initial conditions in which the number of initial participants is significantly less than the k-core or (r, s)-core of the network.

There also exists an extensive body of literature on network formation in which the primary research question is to figure out the type of networks that emerge when a group of myopic and self-interested agents has to decide whether to participate in a cooperative activity or not (see e.g., [11], [12], [13], [14]). However, these works focus on edge formation so that users can select a specific subset of neighbors with whom they wish to collaborate. This problem differs from the user participation problem, in which all the neighbors of a participating user benefit.

Our objective in this work is to generalize the network participation and sharing problem setups by proposing a dynamic network formation approach that can lead to a participation/sharing network from any initial condition. In particular, we formulate these problems as non-cooperative games where users are modeled as self-interested rational players with a utility function. The players decide between participating in a network activity or opting out by maximizing their local utility functions. We provide a complete characterization of the Nash equilibria set for both games. Then, we propose a dynamic model for players' decisions in which players select their actions using Log-Linear Learning as defined in [15] and [16].

Log-linear learning (LLL) is a version of noisy best-response dynamics that assumes the players have bounded rationality. In bounded rationality, players select the best response actions with a high probability but select suboptimal actions with a small but non-zero probability. This behavior based on bounded rationality has been explored in a variety of theoretical and empirical settings as in [17], [18], [19], and [20]. After setting up the game, we prove for the network participation game that all the best response paths are acyclic and stochastically stable joint action profiles under LLL will belong to the set of Nash equilibria.

II. Setup

We consider a social network of n players represented by a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{1,2,\ldots,n\}$ is the set of players and $\mathcal{E}$ is the edge set. An edge $(i,j)\in\mathcal{E}$ implies that players i and j can interact and benefit from each other. Let $\mathcal{N}(i)$ be the neighborhood set of player i, i.e.,

$$\mathcal{N}(i)=\{j\in\mathcal{V}\mid (i,j)\in\mathcal{E}\},$$

and $\mathcal{N}[i]=\mathcal{N}(i)\cup\{i\}$ be the closed neighborhood of i. Each player has a set of actions A_i and $\mathcal{A}=A_1,A_2,\ldots,A_n$ is the set of joint action profiles. Each element σ in $\mathcal{A}$ is an n-tuple $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ such that $\sigma_i\in A_i$ is an action of player i. We use the notation $\sigma=(\sigma_i,\sigma_{-i})$ to represent actions from the perspective of player i where σ_{-i} is an $(n-1)$-tuple representing the actions of all the players other than i. Similarly, for any set $S\subset V$, $\sigma=(\sigma_S,\sigma_{V\setminus S})$ is a decomposition of action profile between players in the sets S and $V\setminus S$ respectively.

A player has a utility function defined over the set of joint action profiles, i.e., $U_i\ :\mathcal{A}\to\mathbb{R}$. Given an action profile σ_{-i}, player i prefers action σ_i over σ_i' if and only if $U_i(\sigma_i,\sigma_{-i})>U_i(\sigma_i',\sigma_{-i})$. The set of best responses of player i to σ_{-i} is

$$B_i(\sigma_{-i})=\{\sigma_i^*\in A_i\mid U_i(\sigma_i^*,\sigma_{-i})\geq U_i(\sigma_i,\sigma_{-i})\,\forall\,\sigma_i\in A_i\}.$$

Let

$$d_H(\sigma,\sigma')=|\{j\in V\mid \sigma_j\neq\sigma_j'\}|$$

be the number of players whose actions differ in profiles σ and σ'. A sequence of joint action profiles $\mathcal{P}=(\sigma^0,\sigma^1,\cdots,\sigma^{l-1})$, $\sigma^p\in\mathcal{A}$ for all $p\in\{0,1,\cdots,l-1\}$, is a best-response path if $d_H(\sigma^p,\sigma^{p+1})=1$ for each consecutive pair of profiles and σ_i^{p+1} belongs to $B_i(\sigma_{-i}^p)$ for the updating player i. Thus, at each step of the best response path, only one player updates his action, and the updated action is the best response to the actions of other players. For the path $\mathcal{P}$, we define

$$\mathcal{P}(\sigma^p,\sigma^{p+1})=\{i\in\mathcal{V}\mid \sigma_i^p\neq\sigma_i^{p+1}\},$$

i.e., $\mathcal{P}(\sigma^p,\sigma^{p+1})$ is the index of the player updating his action from σ^p to σ^{p+1}.

Given an action profile, players update their actions based on some strategic rule. A variety of learning rules have been proposed and analyzed in the literature. We assume that players' action selection strategy can be modeled by Log-Linear Learning (LLL), which is a version of noisy best response dynamics (see e.g., [15] and [21]). Log-linear learning is an asynchronous learning rule in which, at each decision time, only one player updates his action. It induces a Markov chain on the $\mathcal{A}$, and the induced Markov chain is ergodic and reversible with a unique stationary distribution μ_T^{LLL}. An action profile σ is *stochastically stable* if and only if $\lim_{T\to 0}\mu_T^{\mathrm{LLL}}(\sigma)>0$, where T is a noise parameter for selecting noisy actions.

III. Network Participation Game-k neighbor setup

Consider a social network represented by a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{1,2,\cdots,n\}$ is the set of players, and $\mathcal{E}$ is the set of connections among the players. Suppose the players are allowed to participate in some collaborative activity. Each agent will have to pay some cost to participate in that activity. Let the cost of participation be k. Each player also anticipates receiving some benefit from participating in this activity, and the benefit depends on the number of neighbors of the player who also decide to participate. To formulate this problem as a game, each player has two actions in his action set $A_i=\{1,0\}$, where the actions 1 and 0 represent participating or not participating in the activity. Given an action profile σ in $\mathcal{A}$. We define two sets

$$V_\sigma=\{j\in\mathcal{V}\mid \sigma_j=1\},$$

i.e., V_σ is the set of players participating in the network in action profile σ. From a player's perspective, we define

$$\mathcal{N}_i^p(\sigma)=\{j\in\mathcal{N}(i)\mid \sigma_j=1\}.$$

where $\mathcal{N}_i^p(\sigma)$ is the set of neighbors of i that are already participating in the network. As before $\mathcal{N}_i^p[\sigma]=\mathcal{N}_i^p(\sigma)\cup\{i\}$ be the *closed neighborhood of participating players of* i.

A key challenge in formulating the network participation problem as a non-cooperative game is to propose a utility function that should support the empirical evidence and experimental observations regarding players' participation decisions. It should also result in a game setup that can be analyzed in detail. The utility function of a player that we propose in this work is

$$U_i(\sigma_i,\sigma_{-i})=\sigma_i\left(\frac{1}{\mathcal{N}(i)}(\mathcal{N}_i^p(\sigma)-k)+\frac{\alpha}{\mathcal{N}(i)}\mathbf{1}_i(\sigma)\right).\tag{1}$$

A player's utility is zero for not participating in the cooperative activity. For $\sigma_i=1$, the utility has two terms. The first term depends on the number of his neighbors participating in the network, represented by $\mathcal{N}_i^p(\sigma)$. A player receives a positive utility if $\mathcal{N}_i^p(\sigma)$ is greater than k and a negative utility if $\mathcal{N}_i^p(\sigma) < k$. The second term is an indicator function.

$$\mathbf{1}_i(\sigma) = \begin{cases} 1 & |\mathcal{N}_i^p(\sigma)| = k, \\ 0 & \text{otherwise}, \end{cases}$$

and $\alpha \in (0, 1)$. The objective of the second term is to break the tie when the number of participating neighbors of a player is exactly equal to k.

The utility function is normalized by $\mathcal{N}(i)$, which results in a utility function that is asymmetric about the real axis depending on the relative values of k and $\mathcal{N}(i)$. This asymmetry helps in modeling the phenomenon that for players with the number of neighbors significantly higher than k, the cost of joining the network when $\mathcal{N}_i^p(\sigma)$ is less than k is small as compared to the players with a small neighborhood set. This relative difference in cost implies that players with a large number of neighbors will be more willing to take risks, and the probability of selecting noisy actions in LLL will be higher than those with a small number of neighbors. Thus, well-connected players in the network are more likely to join early than players with few neighbors.

A. Analysis

Next, we analyze the network participation game for which the player utilities are defined in (1), and players use noisy best response, modeled by LLL, as their decision strategy. We will use Nash equilibrium and stochastic stability as our solution concept to analyze network formation behavior for this setup.

Proposition 1: For the network participation game with utility function defined in (1), an action profile σ^* is a Nash equilibrium if either of the following conditions is satisfied.

1) $\sigma_i^* = 0$ for all $i \in \mathcal{V}$.
2) For all i in V_{σ^*}, $d(i, V_{\sigma^*}) \geq k$ and for all j in $V \setminus V_{\sigma^*}$, $d(j, V_{\sigma^*}) < k$.

Proof: To prove the first part of the proposition, suppose the joint action profile is $\sigma = (0, 0 \ldots, 0)$. Then, the utility of all the players is zero. At each iteration, a random player, say i, is selected to update his action under LLL. Then,

$$U_i(0, \sigma_{-i}) = 0 \text{ and } U_i(1, \sigma_{-i}) = -k/|\mathcal{N}(i)|.$$

Thus, each player prefers no participation over participation if the current joint profile is $(0, \ldots, 0)$.

For condition 2), the joint action profile σ is represented as $\sigma = (\sigma_{V_{\sigma^*}}, \sigma_{V \setminus V_{\sigma^*}})$. For each player in V_{σ^*}, the condition $d(i, V_{\sigma^*}) \geq k$ implies that player i has at least k neighbors who are participating in the activity. Therefore, $U_i(1, \sigma_{-i}^*) > U_i(0, \sigma_{-i}^*)$. The strict inequality is enforced by the second term in the utility action that adds a value of α to the utility function when the number of participating neighbors is exactly equal to k. Similarly, for all players $j \in V \setminus V_{\sigma^*}$, the condition $d(j, V_{\sigma^*}) < k$ implies that the number of participating neighbors is less than k and therefore $U_i(0, \sigma_{-i}^*) < U_i(1, \sigma_{-i}^*)$. ∎

Proposition 2: For the network participation game with utility function defined in (1), all best response paths are acyclic.

Proof: Suppose the above statement is not true and there exists a best response path $\mathcal{P} = (\sigma^1, \sigma^2, \cdots, \sigma^l, \sigma^{l+1}$ that is a cycle, i.e., $\sigma^1 = \sigma^{l+1}$. For a path to be a cycle, every player that transitions from 0 to 1 must transition back to 1 and every player that transitions from 1 to 0 must transition back from 0 to 1. Thus, we can only have a cycle with an even number of transitions, which implies that l must be an even number with $\sigma^{l+1} = \sigma^1$.

Let G be a global function on the set of joint action profiles and is defined as

$$G(\sigma) = \sum_{i=1}^{n} \hat{U}_i(\sigma_i, \sigma_{-i}),$$

where

$$\hat{U}_i(\sigma_i, \sigma_{-i}) = \mathcal{N}(i) U_i(\sigma_i, \sigma_{-i}) - \alpha \mathbf{1}_i(\sigma)\sigma_i. \tag{2}$$

If the path $\mathcal{P}$ is a cycle, then the following equality must hold

$$\sum_{q=1}^{l} \left[G(\sigma^{q+1}) - G(\sigma^q)\right] = 0.$$

Since every transition must be reversed in a cyclic path. Suppose

$$\mathcal{P}(\sigma^{q_1}, \sigma^{q_1+1}) = \mathcal{P}(\sigma^{q_2}, \sigma^{q_2+1}) = i,$$

i.e., player i updates the actions in transitions $(\sigma^{q_1}, \sigma^{q_1+1})$ and $(\sigma^{q_2}, \sigma^{q_2+1})$. Without loss of generality, we assume that i transitions from 0 to 1 in σ^{q_1} to σ^{q_1+1} transition and from 1 to 0 in some later transition σ^{q_2} to σ^{q_2}, where $1 < q_1 < q_2 < l$. Then,

$$G(\sigma^{q_1+1}) - G(\sigma^{q_1}) = [\hat{U}_i(1, \sigma_{-i}^{q_1}) - \hat{U}_i(0, \sigma_{-i}^{q_1}] + \sum_{j \in \mathcal{N}_i^p(\sigma^{q_1})} \left[\hat{U}_j(\sigma_j^{q_1}, 1, \sigma_{-\{i,j\}}^{q_1}) - \hat{U}_j(\sigma_j^{q_1}, 0, \sigma_{-\{i,j\}}^{q_1})\right] + \sum_{j \notin \mathcal{N}_i^p[\sigma^{q_1}]} \left[\hat{U}_j(\sigma^{q_1}) - \hat{U}_j(\sigma^{q_1})\right].$$

Here $\hat{U}_j(\sigma_j^{q_1}, 1, \sigma_{-\{i,j\}}^{q_1})$ is the updated utility of player j when $\sigma_j^{q_1}$ is the action of j, $\sigma_i^{q_1} = 1$ is the action of player i and $\sigma_{-\{i,j\}}^{q_1}$ is the joint action profile of actions of players other than i and j.

The last term in the above equation is equal to zero since the utility of the players that are not in the participating neighborhood of player i remains the same from σ^{q_1} to σ^{q_1+1}. Player i transitions from 0 to 1 as his best response if the $|\mathcal{N}_i^p(\sigma^{q_1})| \geq k$. Thus,

$$[\hat{U}_i(1, \sigma_{-i}^{q_1}) - \hat{U}_i(0, \sigma_{-i}^{q_1})] = |\mathcal{N}_i^p(\sigma^{q_1})| - k.$$

The second term in the expression for $G(\sigma^{q_1}) - G(\sigma^{q_1+1})$ is the impact of player i's participation on the utility of his participating neighbors and is equal to $\mathcal{N}_i^p(\sigma^{q_1})$ since the utility of each participating neighbor of i is increased by a factor of $1/\mathcal{N}(j)$. Thus,

$$G(\sigma^{q_1}) - G(\sigma^{q_1+1}) = 2\mathcal{N}_i^p(\sigma^{q_1}) - k.$$

When player i transitions from 1 to 0 at some later step of the cycle, say $\sigma^{q_2} = (1, \sigma^{q_2}_{-i})$ and $\sigma^{q_2+1} = (0, \sigma^{q_2}_{-i})$, then

$$G(\sigma^{q_2}) - G(\sigma^{q_2+1}) = 2\mathcal{N}^p_i(\sigma^{q_2}) - k.$$

Since a transition from 1 to 0 is the best response only if the number of participating neighbors is less than k, we get $\mathcal{N}^p(\sigma^{q_2}) < k \le \mathcal{N}^p(\sigma^{q_1})$. Therefore,

$$[G(\sigma^{q_1}) - G(\sigma^{q_1+1})] - [G(\sigma^{q_2}) - G(\sigma^{q_2+1})] > 0$$

Thus, given the best response cycle, we can divide the entire path into a pair of transitions of individual players from 1 to 0 and then from 0 to 1. The corresponding difference in the global function $G(\cdot)$ for these transition pairs is always greater than zero. Therefore,

$$\sum_{p=1}^{l-1} \left[G(\sigma^{p+1}) - G(\sigma^p)\right] > 0$$

for any best response cycle, which is a contradiction. Thus, the best response path cannot be a cycle for our network participation game with the utility function defined in (2). ■

Proposition 3: Consider the network participation game with the utility function defined in (1). If all the players adhere to Log-Linear Learning for decision-making, then the stochastically stable joint action profiles belong to the set of Nash equilibria.

Proof: The proof is a direct consequence of Props. 1 and 2. Prop. 1 implies that all the Nash equilibria of the game are strict, i.e. if $\sigma^* \in \mathcal{A}$ is a Nash equilibrium then $U_i(\sigma^*_i, \sigma^*_{-i})$ is strictly greater than $U_i(\sigma_i, \sigma^*_{-i})$ for any $\sigma_i \in A_i$. Prop 2 establishes that the best response paths are acyclic. For a finite number of players with a finite number of actions, $|\mathcal{A}|$ is finite, and therefore, every best response path should have a finite length. Thus, every best response path has to terminate, and it cannot terminate to any profile other than a Nash equilibrium. The Markov chain induced by LLL follows a best response path with high probability and Nash equilibria are the only absorbing states of the Markov chain if $T = 0$. Thus, in the limiting case of $T \to 0$, the stochastically stable profiles for which $\mu^{\mathrm{LLL}}_T > 0$ will belong to the set of Nash equilibria, which concludes the proof. ■

IV. Network Participation Game: Assorted Resources Setup

We consider a generalization of the network participation game as presented in [4]. In the generalized setup, each player has a set of personal resources, which can be certain physical sensors, information, or expertise for performing certain tasks. A player can share his resources with his immediate neighbors in the network. Let $\mathcal{R} = \{0, 1, \ldots, r-1\}$ be the set of resources available in the network. Each node is assigned a subset of s resources where $s \le r$. This setup can be represented as a labeled graph in which each node is assigned a label, where the label of anode is the set of resources assigned to that player.

In this game setup with assorted resources, $\mathcal{V} = \{1, 2, \ldots, n\}$ is the set of players, and each player can interact with a subset of other players. Each player has a set of actions $A_i = \{1, 0\}$, where $\sigma_i = 1$ and $\sigma_i = 0$ imply that player i is participating or not participating in the social activity. Given an action profile $\sigma = (\sigma_i, \sigma_{-i})$ in $\mathcal{A}$, where σ_i is the action of player i and σ_{-i} is the joint action profile of all the players other than i, we define

$$L_i(\sigma_{-i}) = \bigcup_{j \in \mathcal{N}^p_i[\sigma]} l(j).$$

where $\mathcal{N}^p_i[\sigma]$ is the closed neighborhood of i and comprises i and all the neighbors of i that are participating in the sharing network in profile σ_{-i} Thus, $L_i(\sigma_{-i})$ is the set of resources that i can access either directly or through its immediate neighbors who are participating. We propose the following utility function for each player

$$U_i(\sigma_i, \sigma_{-i}) = \frac{\sigma_i}{r}\left(|L_i(\sigma_{-i})| - r + \alpha \mathbf{1}(\sigma)\right), \tag{3}$$

where

$$\mathbf{1}_i(\sigma) = \begin{cases} 1 & |L_i(\sigma_{-i})| = r, \\ 0 & \text{otherwise}, \end{cases}$$

Notice that we have redefined $\mathbf{1}_i$ for notational convenience and it will be obvious from the situation which definition is applicable.

Player i receives zero utility when there is no participation, i.e., $\sigma_i = 0$. If the player decides to participate, the resulting utility, as defined in (3), is negative if the number of resources that i can access in the closed neighborhood of participating players, $\mathcal{N}^p_i(\sigma)$, is less than r. However, if i has access to all the r resources, then U_i equals α/r for some positive α.

A. Analysis

We have set up a non-cooperative game for the network sharing game with assorted resources by defining a utility function in (3). Again, we consider Nash equilibrium to be our primary solution concept for analyzing network behavior. The analysis approach will be similar to the k neighbor setup approach.

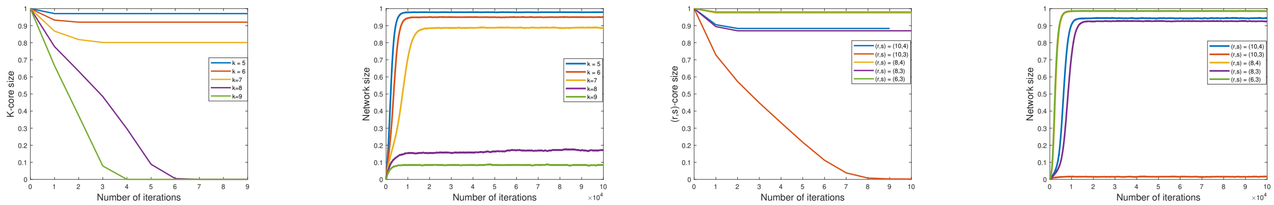
Proposition 4: For the network sharing game with utility function defined in (3), an action profile σ^* is a Nash equilibrium if either of the following conditions is satisfied.

1) $\sigma^*_i = 0$ for all $i \in V$.
2) For all i in V_{σ^*}, $|L_i(\sigma^*_{-i})| = r$ and for all j in $V \setminus V_{\sigma^*}$, $|L_i(\sigma^*_{-i})| < r$, where $L_i(\sigma^*_{-i})$ is the set of resources that player i can access in the closed neighborhood of participating players.

Proof: We can verify the above statements by applying the definition of Nash equilibrium. Consider condition 1) and let $\sigma^* = (0, 0 \ldots, 0)$, i.e., no player participates in the sharing network. Suppose player i is randomly selected to update the action. Then, i's utilities for both actions will be the following.

$$U_i(0, \sigma^*_{-i}) = 0 \text{ and } U_i(1, \sigma^*_{-i}) = (s - r)/r.$$

Thus, i will prefer no participation over participation if σ^* is all zero.



(a) k-core size (b) Participation Network size with LLL (c) (r, s)-core size (d) Sharing network size with LLL

Fig. 1. Simulation Results for Network Participation and Sharing Games

Let $\sigma^* = (\sigma^*_{V_{\sigma^*}}, \sigma^*_{V \setminus V_{\sigma^*}})$ be an action profile that satisfies condition 2). Suppose player i is randomly selected to update the action under LLL. Then, if i belongs to V_{σ^*}, condition 2) implies that $L_i(\sigma^*_{-i}) = r$, i.e., i has access to all the r resources in the closed neighborhood of participating players. Therefore,

$$U_i(1, \sigma^*_{-i}) = \alpha/r \text{ and } U_i(0, \sigma^*_{-i}) = 0,$$

and player i will prefer to keep participating in the sharing network over not participating. Suppose $i \in V \setminus V_{\sigma^*}$ and condition 2) implies that $\sigma^*_i = 0$ and $L_i(\sigma^*_{-i})| < r$. When i will receive an opportunity to update his action, his utility for both the actions will be

$$U_i(0, \sigma^*_{-i}) = 0 \text{ and } U_i(1, \sigma^*_{-i}) = (s - r).$$

Therefore, player i's best response will be $\sigma_i = 0$.

Finally, we argue that any action profile that does not satisfy conditions 1) or 2) is not a Nash equilibrium. Suppose there exists a non-zero such profile $\hat{\sigma}$ that does not satisfy condition 2) but is a Nash equilibrium. Not satisfying 2) implies that either there exists a player with $\hat{\sigma}_i = 1$ but $|L_i(\hat{\sigma}_{-i})| < r$ or there exists a player with $\hat{\sigma}_i = 0$ but $|L_i(\hat{\sigma}_{-i})| = r$. In both of these scenarios, the best response of that player will be to switch the current action, and hence $\hat{\sigma}$ cannot be Nash equilibrium. ∎

V. Simulation

We performed extensive simulations in Matlab to validate our results. We modeled the underlying social network with Erdös-Rényi (ER) graphs. In an ER graph, an edge exists between any node pair with probability p. We considered ER graphs with $n = 1000$ players and link formation probability $p = 0.010$. Thus, each player had ten neighbors on average. We generated 100 ER graphs and verified our results on all the randomly generated networks. Then, we computed the average performance over all the hundred networks.

A. Simulation results for the Network Participation Game:

For the network participation game, we selected $T = 0.3$ and $\alpha = 0.5$, where T was the noise parameter in LLL and α was a parameter in the utility function defined in (1). We simulated networks with $k = \{5, 6, 7, 8, 9\}$. The results are presented in Fig. 1. In Fig. 1(a), we present the results of the standard k-core selection algorithm as described in [1] in which we started with all the players participating in the network, and then we iteratively removed the players who did not satisfy the k neighbor criteria. The x-axis in this plot corresponds to the number of iterations, and the y-axis corresponds to the expected fraction of players that belong to the k-core of the network, averaged over one hundred randomly generated networks. As can be seen from the figure, the k-core of the networks was empty for $k = 8$ and $k = 9$, and a significant fraction of players were included in the k-core for $k < 8$.

k	5	6	7	8	9
% error	0.22	0.28	2.28	1.3	0

TABLE I

Performance comparison for Network Participation Game

In Fig. 1(b), we present the results of our proposed approach for the network participation problem. In this plot, the x-axis represents the number of iterations of LLL, and the y-axis represents the network size equal to the fraction of players participating. For our results, we started with the initial condition with no participating player. However, under LLL, the fraction of players who decide to participate in the network starts to increase and reach a steady state after around twenty thousand iterations. From these results, we can immediately observe a direct relation between the sizes of the k-core and network sizes under our proposed scheme for various values of k.

We compared each player's action in the k-core formulation and our proposed formulation for performance comparison. Because of the noise in LLL, players have a non-zero probability of switching their actions. However, choosing a suboptimal action is a rare event. Therefore, we considered the actions of all the players when LLL reached a steady state. In particular, we considered the actions of all the players in the final thirty thousand iterations of LLL. From Fig. 1(b). we can observe that the state of the network is the steady state for the final thirty thousand iterations. Then, we imposed a constraint: if a player decided to participate in the network for over 95% of these thirty thousand iterations, we considered that player a network member. Otherwise, we assumed that the player's participation was a rare event and the player was not considered a network member.

The results of this analysis are presented in Table I. For the cases of $k \in \{5, 6, 7, 8, 9\}$, the expected error is less

k	(10,4)	(10,3)	(8,4)	(8,3)	(6,4)
% error	0.53	0.24	0.156	0.77	0.08

TABLE II
PERFORMANCE COMPARISON FOR NETWORK PARTICIPATION GAME

than 2.5%, which is an extremely tight bound considering the stochastic nature of players' decision strategy. Based on these results, we can declare with high confidence that in the network participation game that we presented in Section III where the objective of the players was to maximize their utility defined in (1) using LLL, the global state of network converges to the k-core of the network. Thus, we have addressed the challenge of k-core formation as a non-cooperative game.

B. Simulation results for the Network Sharing Game:

We performed a similar analysis for the Network Sharing Game, and the results are presented in Fig. 1. As before, the underlying social network is an ER graph with $n = 1000$ and $p = 0.01$, which resulted in an expected number of ten neighbors per player. The simulations were performed for (r, s) in $\{(10, 4), (10, 3), (8, 4), (8, 3), (6, 3)\}$. We selected the noise parameter $T = 0.15$ and $\alpha = 0.5$ for (3). The results were averaged over one hundred randomly generated ER networks. In Fig. 1(c), we present the results for the expected fraction of players included in the (r, s) core of ER networks. The only case in which (r, s)-core was empty was $(r, s) = (10, 3)$. The percentage of players included in the (r, s)-core for all the other simulated scenarios was over 80%. In Fig. 1(d), we present the results for our proposed scheme for the network sharing game. The y-axis represents the network size regarding the expected fraction of players sharing their resources over the network with LLL as their decision strategy. We can observe that the results are very similar.

We compared the network performance under (r, s)-core and the size of the sharing network under our proposed scheme. For this comparison, we again compared the actions of all players in the (r, s)-core and our proposed scheme in the steady state, which is the final thirty thousand iterations described above. The results of this comparison are presented in Table II, and it can be seen that the errors in all the scenarios are less than 1%. Thus, the players' actions under LLL were the same as in the (r, s)-core setup, which validates our proposed approach for the network sharing game as presented in Section IV.

VI. CONCLUSIONS

We introduced a game theoretic model addressing user participation and sharing issues pivotal in social network analysis. Previous approaches assumed full participation from the start or focused on edge-level network formation, requiring users to decide their connections. We formulated these setups as non-cooperative games with bounded rationality. We provided utility functions and fully characterized Nash equilibria. Analyzing best response paths for network participation, we demonstrated that Log-Linear Learning (LLL) dynamics converged to Nash equilibria. We validated our approach through extensive simulations, illustrating the evolution from zero user engagement to k-cores and (r, s)-cores in randomly generated ER networks.

REFERENCES

[1] K. Bhawalkar, J. Kleinberg, K. Lewi, T. Roughgarden, and A. Sharma, "Preventing unraveling in social networks: the anchored k-core problem," *SIAM Journal on Discrete Mathematics*, vol. 29, no. 3, pp. 1452–1475, 2015.

[2] F. Zhang, Y. Zhang, L. Qin, W. Zhang, and X. Lin, "When engagement meets similarity: efficient (k, r)-core computation on social networks," *arXiv preprint arXiv:1611.03254*, 2016.

[3] A. Falk and M. Kosfeld, "It's all about connections: Evidence on network formation," *Review of Network Economics*, vol. 11, no. 3, 2012.

[4] W. Abbas, A. Laszka, M. Shabbir, and X. Koutsoukos, "Graph-theoretic approach for increasing participation in networks with assorted resources," *IEEE Transactions on Network Science and Engineering*, vol. 7, no. 3, pp. 930–946, 2019.

[5] M. A. Nowak and K. Sigmund, "Evolution of indirect reciprocity," *Nature*, vol. 437, no. 7063, pp. 1291–1298, 2005.

[6] J. Surma, "Social exchange in online social networks. the reciprocity phenomenon on facebook," *Computer Communications*, vol. 73, pp. 342–346, 2016.

[7] S. Wu, A. Das Sarma, A. Fabrikant, S. Lattanzi, and A. Tomkins, "Arrival and departure dynamics in social networks," in *Proceedings of the sixth ACM international conference on Web search and data mining*, 2013, pp. 233–242.

[8] F. D. Malliaros and M. Vazirgiannis, "To stay or not to stay: modeling engagement dynamics in social graphs," in *Proceedings of the 22nd ACM international conference on Information & Knowledge Management*, 2013, pp. 469–478.

[9] S. Medya, T. Ma, A. Silva, and A. Singh, "K-core minimization: A game theoretic approach," *arXiv preprint arXiv:1901.02166*, 2019.

[10] Y. Peng, Y. Zhang, W. Zhang, X. Lin, and L. Qin, "Efficient probabilistic k-core computation on uncertain graphs," in *2018 IEEE 34th International Conference on Data Engineering (ICDE)*. IEEE, 2018, pp. 1192–1203.

[11] M. O. Jackson and A. Wolinsky, *A strategic model of social and economic networks*. Springer, 2003.

[12] V. Bala and S. Goyal, "A noncooperative model of network formation," *Econometrica*, vol. 68, no. 5, pp. 1181–1229, 2000.

[13] A. Watts, "A dynamic model of network formation," *Games and Economic Behavior*, vol. 34, no. 2, pp. 331–341, 2001.

[14] S. A. Boorman, "A combinatiorial optimization model for transmission of job information through contact networks," *The bell journal of economics*, pp. 216–249, 1975.

[15] L. E. Blume, "The statistical mechanics of strategic interaction," *Games and economic behavior*, vol. 5, no. 3, pp. 387–424, 1993.

[16] A. Montanari and A. Saberi, "The spread of innovations in social networks," *Proceedings of the National Academy of Sciences*, vol. 107, no. 47, pp. 20 196–20 201, 2010.

[17] H. P. Young, "The dynamics of social innovation," *Proceedings of the National Academy of Sciences*, vol. 108, no. supplement_4, pp. 21 285–21 291, 2011.

[18] M. Kandori, G. J. Mailath, and R. Rob, "Learning, mutation, and long run equilibria in games," *Econometrica*, vol. 61, p. 29, 01 1993.

[19] G. E. Kreindler and H. P. Young, "Rapid innovation diffusion in social networks," *Proceedings of the National Academy of Sciences*, vol. 111, no. Supplement 3, pp. 10 881–10 888, 2014.

[20] H. P. Young, "The evolution of conventions," *Econometrica: Journal of the Econometric Society*, pp. 57–84, 1993.

[21] C. Alós-Ferrer and N. Netzer, "The logit-response dynamics," *Games and Economic Behavior*, vol. 68, no. 2, pp. 413–427, 2010.